Int. J. Contemp. Math. Sci., Vol. 3, 2008, no. 21, 1031 - 1040

On the Vanishing of Cohomology of Divisors on Nonsingular Rational Surfaces

Dedicated to Gioia Bellighieri

Mustapha Lahyane

Instituto de Física y Matemáticas (IFM)
Universidad Michoacana de San Nicolás de Hidalgo
Edificio C-3, Ciudad Universitaria
Avenida Francisco J. Mujica s/n, Colonia Felicitas del Rio
C. P. 58040 Morelia. Michoacán, México
lahyane@ifm.umich.mx

and

Centro de Investigación en Matemáticas, A. C. (CIMAT)
Jalisco s/n. Mineral de Valenciana, Apdo. Postal 402
C.P. 36240, Guanajuato. Gto., México
lahyane@cimat.mx

Gioia Failla

Department of Mathematics
University of Messina, C.da Sant'Agata, Salita Sperone
98166-Messina, Italy
gfailla@dipmat.unime.it

Israel Moreno Mejía

Instituto de Matemáticas
Universidad Nacional Autónoma de México
Área de la investigación científica, Circuito Exterior, Ciudad Universitaria Coyoacán
C.P. 04510 México D.F., México
israel@matem.unam.mx

Abstract

We prove the vanishing of the first cohomology group of any numerically effective divisor on generalized rational surfaces of type (n, m), n and m being some nonnegative integers. In particular, we cover such property in the classical cases of blowing up the projective plane at points (possibly infinitely near) which are either all on an integral conic or all on a line. As a consequence, the dimension of the complete linear system of any divisor on a generalized rational surface of type (n, m) is computed.

Mathematics Subject Classification 2000: Primary 14J26, Secondary 14F17, 14F05

Keywords: Rational surfaces, Numerical equivalence, Néron-Severi group, Anticanonical divisor, Blowing up, Points in general position, Points in good position, Monoid of effective divisor classes, Complete linear systems, Numerically effective divisors, Specialization, Points infinitely near

1 Introduction

In this paper we deal with the following linear algebra question:

What is the dimension of the vector space $H^1(Z, \mathcal{O}_Z(D))$ over the field k? Where $H^1(Z, \mathcal{O}_Z(D))$ is the first cohomology group of the invertible sheaf $\mathcal{O}_Z(D)$ associated to the divisor D on a given nonsingular projective surface Z whose ground field k is assumed to be algebraically closed of arbitrary characteristic.

The Kodaira vanishing theorem states that the first cohomology group of the invertible sheaf associated to a given anti-ample divisor D on Z vanishes. Here D is anti-ample means, according to Nakai-Moisheson numerical criterion, that its self-intersection D^2 and the intersection number of D with any nonzero effective divisor on Z are larger than zero and less than zero respectively. This result was generalized by David Mumford to any divisor which is big and anti-numerically effective. Here a divisor is said to be big if its self-intersection is larger than zero and it is said to be anti-numerically effective if its intersection number with any effective divisor is less than or equal to zero.

It seems that in general, little is known, even in the case when the surface Z is rational, that is, when Z is birationally isomorphic to the projective plane. Even more, there is no complete answer even in the nice case where the rational surface Z is generic, except a conjectural one given in the 1986 Harbourne Conjecture (see [6, Remark (I.6.2). The first and second conjectures, p. 102]), equivalently reformulated by André Hirschowitz in 1989 (see [13]). See also [12].

This conjecture states that every divisor on a generic rational surface has the natural cohomology. In particular, this conjecture (if true) implies the vanishing of the first cohomology of any effective and numerically effective divisor on a given generic rational surface. Here a divisor D is said to be numerically effective if $-D$ is anti-numerically effective.

The main purpose of this paper is to give a partial answer to the question mentioned above for some rational surfaces Z with $K_Z^2 < 0$, here K_Z denotes a canonical divisor on Z (the case where $K_Z^2 \geq 0$ is already known, see for instance the 1996 Harbourne work [9], in particular Theorem 8 page 732 when $K_Z^2 > 0$ and Corollary 10 page 733 when $K_Z^2 = 0$). That is, to construct rational surfaces X for which the first cohomology group of the invertible sheaf $\mathcal{O}_X(C)$ vanishes for any numerically effective divisor C on X, see Theorem 1.2 (in the literature, a divisor which enjoys this property is called either non superabundant or nonspecial). Furthermore, as a consequence, we compute the dimension of global sections of any divisor on such surface X, see Corollary 1.3 in the case when the divisor is numerically effective and see Corollary 1.4 when the divisor is not assumed to be numerically effective.

These surfaces X are generalizations of the surfaces $S_{(n,m)}$ of type (n, m) studied in [2] and also some surfaces studied by Eduard Looijenga in [18]. The surface $X = \tilde{S}_{(m,n)}$, where m and n are nonnegative integers such that either $mn = 0$ or $mn - 4n - m < 0$, that we will deal with is constructed as follows. Take a degenerate cubic which consists of a line L on the projective plane and an integral projective conic C (e.g., L and C are such that they meet each other transversally). Then take n points, say $P_1, \ldots, P_n$, on L and m points, say $Q_1, \ldots, Q_m$, on C with the assumption that the set of these $(n+m)$ points should not contain the intersection set of the line L and the conic C. $\tilde{S}_{(m,n)}$ is then nothing but the surface obtained by blowing up the projective plane with center the zero dimensional closed subscheme $\{P_1, \ldots, P_n, Q_1, \ldots, Q_m\}$. Here we allow P_i (respectively Q_i) to be infinitely near P_j (respectively Q_j) if i is greater than j.

The following lemma determine the numerically effective divisors which are orthogonal to $K_{\tilde{S}_{(m,n)}}$, its proof is postponed to Section 3. Here $K_{\tilde{S}_{(m,n)}}$ denotes a canonical divisor on $\tilde{S}_{(m,n)}$. We recall that a divisor on $\tilde{S}_{(m,n)}$ is numerically effective if it meets every integral curve on $\tilde{S}_{(m,n)}$ nonnegatively.

Lemma 1.1. *With the same notation as above, there is no nonzero numerically effective divisor D on $\tilde{S}_{(m,n)}$ satisfying the equality $K_{\tilde{S}_{(m,n)}}.D = 0$, $K_{\tilde{S}_{(m,n)}}$ being a canonical divisor on $\tilde{S}_{(m,n)}$.*

Recall that for a divisor C on a smooth projective surface Y which is defined over a ground field k, and for every integer i $(i = 0, 1, 2)$, $h^i(Y, \mathcal{O}_Y(C))$ denotes the dimension over k of the i^{th} cohomology group $H^i(Y, \mathcal{O}_Y(C))$ of the natural sheaf $O_Y(C)$ associated to the divisor C. Here we study the vanishing problem of the first cohomology group of an arbitrary numerically effective divisor.

Theorem 1.2. *With the same notation as above, if D is a numerically effective divisor on $\tilde{S}_{(m,n)}$, then*

$$h^1(\tilde{S}_{(m,n)}, \mathcal{O}_{\tilde{S}_{(m,n)}}(C)) = 0,$$

$\mathcal{O}_{\tilde{S}_{(m,n)}}(D)$ being an invertible sheaf associated to the divisor D.

Proof: Apply [10, Theorem III.1, page 1197] and Lemma 1.1.

A straightforward consequence of Theorem 1.2 gives the dimension of the complete linear system of any numerically effective divisor on $\tilde{S}_{(m,n)}$.

Corollary 1.3. *With notation as above. Let D be a numerically effective divisor on $\tilde{S}_{(m,n)}$. Then*

$$h^0(\tilde{S}_{(m,n)}, \mathcal{O}_{\tilde{S}_{(m,n)}}(D)) - 1 = \frac{1}{2}(D^2 - D.K_{\tilde{S}_{(m,n)}}),$$

$\mathcal{O}_{\tilde{S}_{(m,n)}}(D)$ being an invertible sheaf associated to the divisor D, and $K_{\tilde{S}_{(m,n)}}$ being a canonical divisor on $\tilde{S}_{(m,n)}$.

Proof: Apply Theorem 1.2 and the below Lemmas 2.1 and 2.3.

Here, we compute the dimension of global sections of any divisor on $\tilde{S}_{(m,n)}$.

Corollary 1.4. *With notation as above. Let D be a given divisor on $\tilde{S}_{(m,n)}$. Then $h^0(\tilde{S}_{(m,n)}, \mathcal{O}_{\tilde{S}_{(m,n)}}(D))$ is either less than two, or equals to*

$$1 + \frac{1}{2}(M^2 - M.K_{\tilde{S}_{(m,n)}}),$$

where $\mathcal{O}_{\tilde{S}_{(m,n)}}(D)$ is an invertible sheaf associated to the divisor D, M is the mobile part of the complete linear system $|D|$ and $K_{\tilde{S}_{(m,n)}}$ being a canonical divisor on $\tilde{S}_{(m,n)}$.

Proof: Apply Corollary 1.3 to the numerically effective divisor M if the integer $h^0(\tilde{S}_{(m,n)}, \mathcal{O}_{\tilde{S}_{(m,n)}}(D))$ is larger than or equal to two.

Remark 1.5. *Points in the projective plane with the property that the surface obtained by blowing up these points satisfies the conclusion of Theorem 1.2 for every numerically effective divisor are called good points. See the works by Brian Harbourne in [6], [7] and [9].*

Remark 1.6. *For related problems regarding the finite generation of the monoid of effective divisor classes and the irreducibility of -1-classes on rational surfaces, one may see [21], [22], [20], [5], [19], [8], [14], [15], [16], [17], [2], [3] and [4].*

2 Background on Rational Surfaces

Let Z be a smooth projective rational surface defined over an algebraically closed field of arbitrary characteristic. A canonical divisor on Z, respectively the Néron-Severi group of Z will be denoted by K_Z and $NS(Z)$ respectively. There is an intersection form on $NS(Z)$ induced by the intersection of divisors on Z, it will be denoted by a dot, that is, for x and y in $NS(Z)$, $x.y$ is the intersection number of x and y (see [11] and [1]).

The following result known as the Riemann-Roch theorem for smooth projective rational surfaces is stated using the Serre duality.

Lemma 2.1. *Let D be a divisor on a smooth projective rational surface Z having an algebraically closed field of arbitrary characteristic as a ground field. Then the following equality holds:*

$$h^0(Z, \mathcal{O}_Z(D)) - h^1(Z, \mathcal{O}_Z(D)) + h^0(Z, \mathcal{O}_Z(K_Z - D)) = 1 + \frac{1}{2}(D^2 - D.K_Z).$$

$\mathcal{O}_Z(D)$ being an invertible sheaf associated canonically to the divisor D.

Here we recall some standard results, see [10] and [11]. A divisor class x modulo numerical equivalence on a smooth projective rational surface Z is effective respectively numerically effective, nef in short, if an element of x is an effective, respectively numerically effective, divisor on Z. Here a divisor D on Z is nef if $D.C \geq 0$ for every integral curve C on Z. Now, we start with some properties which follow from a successive iterations of blowing up closed points of a smooth projective rational surface.

Lemma 2.2. *Let $\pi^\star : NS(X) \to NS(Y)$ be the natural group homomorphism on Néron-Severi groups induced by a given birational morphism $\pi : Y \to X$ of smooth projective rational surfaces. Then $\pi^\star$ is an injective intersection-form preserving map of free abelian groups of finite rank. Furthermore, it preserves the dimensions of cohomology groups, the effective divisor classes and the numerically effective divisor classes.*

Proof: See [10, Lemma II.1, page 1193].

Lemma 2.3. *Let x be an element of the Néron-Severi group $NS(X)$ of a smooth projective rational surface X. The effectiveness or the the nefness of x implies the noneffectiveness of $k_X - x$, where k_X denotes the element of $NS(X)$ which contains a canonical divisor on X. Moreover, the nefness of x implies also that the self-intersection of x is greater than or equal to zero.*

Proof: See [10, Lemma II.2, page 1193].

Finally, we recall the classical Bézout Theorem:

Lemma 2.4. *Two projective plane curves de degree μ and ν defined over an algebraically closed field intersect in exactly $\mu\nu$ points, unless the curves have a common component.*

3 Proof of Lemma 1.1

In this section, we give a proof of the result stated in the Lemma 1.1 of section one. To do so, we need to give explicitly the lattice structure of $Pic(\tilde{S}_{(m,n)})$.

Firstly, the integral basis

$$(\mathcal{E}_0, -\mathcal{E}_1^L, \dots, -\mathcal{E}_n^L, -\mathcal{E}_1^C, \dots, -\mathcal{E}_m^C),$$

is defined by:

- $\mathcal{E}_0$ is the class of a line on the projective plane which do not pass through any of the assigned points $P_1, \dots, P_n, Q_1, \dots, Q_m$ in consideration.
- $\mathcal{E}_i^L$ is the class of the exceptional divisor corresponding to the i^{th} point blown-up P_i for every $i = 1, \dots, n$,
- $\mathcal{E}_j^C$ is the class of the exceptional divisor corresponding to the j^{th} point blown-up Q_j for every $j = 1, \dots, m$.

The class of a divisor on $\tilde{S}_{(m,n)}$ will be represented by the $(1 + n + m)-$tuple

$$(a; b_1^L, \dots, b_n^L; b_1^C, \dots, b_m^C),$$

Secondly, the intersection form on $Pic(\tilde{S}_{(m,n)})$ is given by:

- $\mathcal{E}_0^2 = 1 = -(\mathcal{E}_i^L)^2 = -(\mathcal{E}_j^C)^2$ for every $i = 1, \dots, n$ and every $j = 1, \dots, m$;
- $\mathcal{E}_i^L.\mathcal{E}_{i'}^L = 0$ for every $i, i' = 1, \dots, n$, with $i \neq i'$;
- $\mathcal{E}_j^C.\mathcal{E}_{j'}^C = 0$ for every $j, j' = 1, \dots, m$, with $j \neq j'$;
- $\mathcal{E}_i^L.\mathcal{E}_j^C = 0$ for every $i = 1, \dots, n$ and for every $j = 1, \dots, m$;
- $\mathcal{E}_0.\mathcal{E}_i^L = \mathcal{E}_0.\mathcal{E}_j^C = 0$ for every $i = 1, \dots, n$ and for every $j = 1, \dots, m$.

Remark 3.1. *We observe that if the class $(a; b_1^L, \dots, b_n^L; b_1^C, \dots, b_m^C)$ is effective, then it represents the class of a projective curve of degree a and having at least multiplicity $b_1^L, \dots, b_n^L$ (respectively, $b_1^C, \dots, b_m^C$). Also we note that by assumption that the classes $\mathcal{E}_i^L$, $\mathcal{E}_j^C$ need not be the classes of smooth rational curves on $\tilde{S}_{(m,n)}$ and that every numerically effective divisor on $\tilde{S}_{(m,n)}$ is effective.*

Let D be a numerically effective divisor on $\tilde{S}_{(m,n)}$ such that $D.K_{\tilde{S}_{(m,n)}} = 0$. We would like to prove that D is the zero divisor. For let $(a; b_1^L, \dots, b_n^L; b_1^C, \dots, b_m^C)$ be the $(1+n+m)-$tuple representing the class of D in the Néron-Severi group $NS(\tilde{S}_{(m,n)})$ relatively to the integral basis $(\mathcal{E}_0, -\mathcal{E}_1^L, \dots, -\mathcal{E}_n^L, -\mathcal{E}_1^C, \dots, -\mathcal{E}_m^C)$.

From the fact that the divisor D is zero if and only the integer a vanishes, let us prove the vanishing of a. Since $D.K_{\tilde{S}_{(m,n)}} = 0$, one may obtain the following equality:

$$(a - b_1^L - \dots - b_n^L) + (2a - b_1^C - \dots - b_n^C) = 0. \tag{1}$$

It follows that:

$$\sum_{i=1}^{i=n} b_i^L = a, \tag{2}$$

and

$$\sum_{j=1}^{j=m} b_j^C = 2a. \tag{3}$$

Henceforth if either n or m vanishes, then the integer a also vanishes. If neither n nor m vanishes, then consider the reals $(\xi_i^L)_{i\in\{1,\dots,n\}}$ and $(\xi_j^C)_{j\in\{1,\dots,m\}}$ defined by:

$$\xi_i^L = \left(b_i^L - \frac{a}{n}\right), \text{ for every } i = 1, \dots, n, \tag{4}$$

and

$$\xi_j^C = \left(b_j^C - \frac{2a}{m}\right), \text{ for every } i = 1, \dots, m. \tag{5}$$

It follows from the equations (2) and (3) that:

$$\sum_{i=1}^{i=n} \xi_i^L = 0, \tag{6}$$

and

$$\sum_{j=1}^{j=m} \xi_j^C = 0. \tag{7}$$

On the other hand, the nonnegativity of D^2 gives rise to the following inequality:

$$a^2\left(1-\frac{1}{n}-\frac{4}{m}\right) \geq \sum_{i=1}^{i=n}(\xi_i^L)^2+\sum_{j=1}^{j=m}(\xi_j^C)^2, \tag{8}$$

which in turn, by assumption, proves that a vanishes. Consequently, in all cases, D equals the zero divisor and we are done.

ACKNOWLEDGEMENTS. We would like to thank very much Professor Gaetana Restuccia for providing us a very nice working atmosphere. The first author acknowledges supports from G.N.S.A.G.A at the Department of Mathematics of the University of Messina (Messina, Italy) and from Concyteg (Consejo de Ciencias y Tecnología del Estado de Guanajuato), (Guanajuato, México) under the project whose number is: 07-02-K662-069A04. Also he acknowledges the very nice hospitalities of the Centro de Investigación en Matemáticas (CIMAT), (Guanajuato, México) and the Instituto de Fisíca y Matemáticas of the Universidad Michoacana de San Nicolás de Hidalgo (Morelia, México).

References

[1] W. Barth, C. Peters, A. Van de Ven, *Compact Complex Surfaces,* Berlin, Springer (1984).

[2] G. Failla, M. Lahyane and G. Molica Bisci, On the finite generation of the monoid of effective divisor classes on rational surfaces of type (n, m), *Atti dell' Accademia Peloritana dei Pericolanti Classe di Scienze Fisiche, Matematiche e Naturali* Vol. **LXXXIV**, C1A0601001 (2005) Adunanza del 28 novembre 2005, 1 - 9. 2006.

[3] G. Failla, M. Lahyane and G. Molica Bisci, The finite generation of the monoid of effective divisor classes on Platonic rational surfaces, *Singularity Theory, Proceedings of the 2005 Marseille Singularity School and*

Conference, 565 - 576, CIRM, Marseille, France. World Scientific Publishing, Hackensack, New Jersey (United States), 2007.

[4] G. Failla, M. Lahyane and G. Molica Bisci, Rational surfaces of Kodaira type IV, *Bollettino della Unione Matematica Italiana. Serie VIII. Sezione B. Articoli di Ricerca Matematica*. Volume **10** (2007), No. 3., pp. 741 - 750.

[5] B. Harbourne, Blowings-up of $\mathbb{P}^2$ and their blowings-down, *Duke Mathematical Journal* **52**:1 (1985), 129 - 148.

[6] B. Harbourne, The geometry of rational surfaces and Hilbert functions of points in the plane, *Proceedings of the 1984 Vancouver conference in algebraic geometry*, 95 - 111, CMS Conference Proceedings, **6**, American Mathematical Society, Providence, RI, 1986.

[7] B. Harbourne, Complete linear systems on rational surfaces, *Transactions of the American Mathematical Society*, Vol. **289**, No. 1. (May, 1985), pp. 231 - 226. on $\mathbb{P}^2$,

[8] B. Harbourne and R. Miranda, Exceptional curves on rational numerically elliptic surfaces, *Journal of Algebra*, **128** (1990), 405 - 433.

[9] B. Harbourne, Rational surfaces with $K^2 > 0$, *Proceedings of the American Mathematical Society*, Volume **124**, Number 3, March 1996.

[10] B. Harbourne, Anticanonical rational surfaces, *Transactions of the American Mathematical Society*, Volume **349** (1997), Number 3, 1191-1208.

[11] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, Springer Verlag (1977).

[12] A. Hirschowitz, La méthode d'Horace pour l'interpolation à plusieurs variables, *Manuscripta Mathematica*, **50** (1985), 337 - 388.

[13] A. Hirschowitz, Une conjecture pour la cohomologie des diviseurs sur les surfaces rationnelles génériques, *Journal für die reine und angewandte Mathematik*, **397** (1989), 208 - 213.

[14] M. Lahyane, Exceptional curves on rational surfaces having $K^2 \geq 0$, *C. R. Acad. Sci. Paris, Ser. I*, **338** (2004), 873 - 878.

[15] M. Lahyane, Rational surfaces having only a finite number of exceptional curves, *Mathematische Zeitschrift* Volume **247**, Number 1, 213 - 221 (May 2004).

[16] M. Lahyane, Exceptional curves on smooth rational surfaces with $-K$ not nef and of self-intersection zero, *Proceedings of the American Mathematical Society*, **133** (2005), 1593 - 1599.

[17] M. Lahyane and B. Harbourne, Irreducibility of -1-classes on anticanonical rational surfaces and finite generation of the effective monoid, *Pacific Journal of Mathematics*, Volume **218** Number 1 (2005), pp. 101 - 114.

[18] E. Looijenga, Rational surfaces with an anticanonical cycle, *Annals of Mathematics* (2), **114** (1981), no. 2, 267 - 322.

[19] R. Miranda and U. Persson, On extremal rational elliptic surfaces, *Mathematische Zeitschrift*, **193** (1986), 537 - 558.

[20] S. Mori, Threefolds whose canonical bundles are not numerically effective, *Annals of Mathematics* (2), **116** (1982), no. 1, 133 - 176.

[21] M. Nagata, On rational surfaces II, *Memoirs of the College of Science, University of Kyoto, Series A*, **33** (1960), no. 2, 271 - 293.

[22] J. Rosoff, Effective divisor classes and blowings-up of $\mathbb{P}^2$, *Pacific Journal of Mathematics*, Volume **89**, Number 2(1980), pp. 419 - 429.

Received: December 20, 2006

Revised Version Received: March 17, 2008