A Note on Three-Dimensional Lorentzian Manifolds

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Abstract

In this paper, we construct the necessary and sufficient condition for a three-dimensional Lorentz manifolds admitting a parallel one-dimensional degenerate plane field to be Locally homogeneous. Also Killing vector fields on Walker Lorentz manifolds are studied.

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1 Introduction

A pseudo- Riemannian manifold $(M, g)$ is said to be curvature homogeneous up to order $k \in \mathbb{N}$ [11] or, equivalently, to satisfy the condition $P(k)$ if, for every pair of points $p, q$ in $M$, there exists a linear isometry $\phi : T_p M \longrightarrow T_q M$ such that, for all $i = 0, \ldots, k$, we have

$$\varphi^*(\nabla^i R(q)) = \nabla^i R(p)$$

where, $\nabla^i R(p)$ is the i-th covariant derivative of the Riemannian curvature tensor at $p \in M$. For $k \equiv 0$, the manifold $(M, g)$ is said to be curvature homogeneous. It is easily seen that a locally homogeneous pseudo-Riemannian

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manifold satisfies the condition $P(k)$ for every $k \in \mathbb{N}$. In [11], Singer showed that, conversely, a Riemannian manifold $(M, g)$ is locally homogeneous if it satisfies the condition $P(k)$ for some $k > k_M$, where $k_M$ is the so-called Singer index of $(M, g)$. In [8] Sekigawa proved that every three-dimensional Riemannian manifold satisfying the condition $P(1)$ is locally homogeneous. In [9], it was shown that a four-dimensional Riemannian manifold which is curvature homogeneous up to order one is also locally homogeneous, improving the result in [10], where the condition $P(2)$ was considered. For Lorentzian manifolds, contrary to the Riemannian case, the condition $P(1)$ is not sufficient to characterize local homogeneity, not even in dimension three. Moreover, at every point $p \in M$ of a three-dimensional Lorentzian manifold $(M, g)$, that is, the self-adjoint linear operator in $T_pM$ associated to the Ricci curvature tensor, can be classified [4, 5] according to its eigenvalues and the associated eigenspaces (so-called Segre type). In particular, one can always construct a pseudo-orthonormal basis $E_1, E_2, E_3$ for the tangent space $T_pM$ (with $E_3$ time like and $E_1, E_2$ space like ) such that, with respect to this basis, the Ricci operator at $p$ takes one of the following forms:

\[
\text{Segre type}{\{11,1\}}: \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \quad \text{Segre type}{\{1\cdot z\}}: \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & -c & b \end{pmatrix}
\]

\[
\text{Segre type}{\{3\}}: \begin{pmatrix} b & a & -a \\ a & b & 0 \\ a & 0 & b \end{pmatrix}, \quad \text{Segre type}{\{21\}}: \begin{pmatrix} a & 0 & 0 \\ 0 & b & 1 \\ 0 & -1 & b \pm 2 \end{pmatrix}
\]

The Segre type $\{11,1\}$ (the comma is used to separate the space-like and time-like eigenvectors) denotes a diagonalizable Ricci operator. In the special case where the two space-like (resp. one space-like or time-like) eigenvectors have equal eigenvalue, we denote this by the degenerate Segre type $\{11,1\}$ (resp.$\{1(1,1)\}$), and the case of three equal eigenvalues is denoted by the Segre type $\{(11,1)\}$. A Ricci operator of Segre type $\{3\}$ has three equal eigenvalues associated to a one-dimensional eigenspace, and Segre type $\{1\cdot z\}$ denotes a Ricci operator with one real and two complex conjugate eigenvalues. Finally, Segre type $\{21\}$ Ricci operators have two eigenvalues (one of which has multiplicity two), each associated to a one-dimensional eigenspace, and the degenerate case where these two eigenvalues are equal, is denoted by $\{(21)\}$.

2 curvature homogeneous walker manifolds

**Definition 2.1** A 3-dimensional Lorentzian manifold admitting a parallel one-dimensional degenerate plane has local coordinates $(t, x, y)$ where the
Lorentzian metric tensor expresses as

\[
g = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \varepsilon & 0 \\ 1 & 0 & f(t, x, y) \end{pmatrix}
\]  

(2)

for some function \( f(t, x, y) \), where \( \varepsilon = \pm 1 \) and the parallel degenerate one-dimensional plane field becomes \( D = \langle \partial_t \rangle \).

It follows after a straightforward calculation that the Levi-Civita connection of any metric (2.1) is given by:

\[
\begin{align*}
\nabla_{\partial_y} \partial_t &= \frac{1}{2} f_t \partial_t \\
\nabla_{\partial_y} \partial_x &= \frac{1}{2} f_t \partial_x \\
\nabla_{\partial_y} \partial_y &= \frac{1}{2} (f f_t + f_y) \partial_t - \frac{1}{2 \varepsilon} f_x \partial_x - \frac{1}{2} f_t \partial_y,
\end{align*}
\]

(3)

where \( \partial_t, \partial_x, \partial_y \) are the coordinate vector fields \( \frac{\partial}{\partial t}, \frac{\partial}{\partial x} \) and \( \frac{\partial}{\partial y} \), respectively. Hence, if \( (M, g) \) admits a parallel null vector field, then the associated Levi-Civita connection satisfies

\[
\begin{align*}
\nabla_{\partial_y} \partial_x &= \frac{1}{2} f_t \partial_x \\
\nabla_{\partial_y} \partial_y &= \frac{1}{2} f_y \partial_t - \frac{1}{2 \varepsilon} f_x \partial_x.
\end{align*}
\]

(4)

Let \( R \) denote the curvature tensor taken with the sign convention \( R(X, Y) = \nabla_X Y - [\nabla_X, \nabla_Y] \). Then the nonzero components of the curvature tensor of any metric (2.1) are given by

\[
\begin{align*}
R(\partial_t, \partial_y) \partial_t &= -\frac{1}{2} f_{tt} \partial_t \\
R(\partial_t, \partial_y) \partial_x &= -\frac{1}{2} f_{tx} \partial_t \\
R(\partial_t, \partial_y) \partial_x &= -\frac{1}{2} f f_{tt} \partial_t + \frac{1}{2 \varepsilon} f_{tx} \partial_x + \frac{1}{2} f_{tt} \partial_y \\
R(\partial_x, \partial_y) \partial_t &= -\frac{1}{2} f_{tx} \partial_t \\
R(\partial_x, \partial_y) \partial_x &= -\frac{1}{2} f_{xx} \partial_t \\
R(\partial_x, \partial_y) \partial_x &= -\frac{1}{2} f f_{tx} \partial_t + \frac{1}{2 \varepsilon} f_{xx} \partial_x + \frac{1}{2} f_{tx} \partial_y,
\end{align*}
\]

(5)
further, note that the existence of parallel null vector field simplifies (2.4) as follows:

\[ R(\partial_x, \partial_y)\partial_x = -\frac{1}{2} f_{xx} \partial_t \]
\[ R(\partial_x, \partial_y)\partial_y = \frac{1}{2\varepsilon} f_{xx} \partial_x. \]

(6)

**Definition 2.2** Let Ric and Sc be the Ricci tensor and the scalar curvature of \((M, g)\), defined by \(Ric = \text{trace} \{ Z \rightarrow R(X, Z)Y \} \) and \(Sc = \text{trace} Ric\), respectively. Moreover, let \(\hat{Ric}\) be the Ricci operator defined by \(\langle \hat{Ric}(X), Y \rangle = Ric(X, Y)\).

The Ricci tensor of the metric (2.1) is

\[
Ric = \begin{pmatrix}
0 & 0 & \frac{1}{2} f_{tt} \\
0 & 0 & \frac{1}{2} f_{tx} \\
\frac{1}{2} f_{tt} & \frac{1}{2} f_{tx} & \frac{1}{2\varepsilon} (\varepsilon f_{tt} - f_{xx})
\end{pmatrix}
\]

when expressed in the local coordinate basis. Moreover, the Ricci operator \(\hat{Ric}\) takes the form

\[
\hat{Ric} = \begin{pmatrix}
\frac{1}{2} f_{tt} & \frac{1}{2} f_{tx} & -\frac{1}{2\varepsilon} f_{xx} \\
0 & 0 & \frac{1}{2\varepsilon} f_{tx} \\
0 & 0 & \frac{1}{2} f_{tt}
\end{pmatrix}
\]

hence, the Ricci operator has eigenvalues

\[ \lambda_1 = 0, \quad \lambda_2 = \lambda_3 = \frac{1}{2} f_{tt}. \]

The Ricci operator of a curvature homogeneous 3-dimensional Lorentzian manifold has the same Segre type at every point \(p \in M\), and that, at least locally, there exists a pseudo-orthonormal frame field \(\{E_1, E_2, E_3\}\) such that the Ricci operator is given by one of the expressions in (1.1) where \(a, b\) and \(c\) are constants.

In [1], Bueken and Djoric showed that all three-dimensional Lorentzian manifolds whose Ricci operator has Segre type \(\{3\}\) or \(\{1z\}\) and which are curvature homogeneous up to order one, are locally homogeneous. Following [2], in this paper we make necessary and sufficient condition for 3-dimensional Lorentz manifolds admitting a parallel one-dimensional degenerate plane field which is curvature homogeneous up to order one to be locally homogeneous i.e.,

**Theorem 2.3** Let \((M, g)\) be a three-dimensional Lorentzian manifold which is curvature homogeneous up to order one and admitting a parallel one-dimensional
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A degenerate plane field. If the Ricci operator of \( (M,g) \) has Segre type \( \{3\} \) or \( \{1zz\} \) at a point \( p \in M \), then \( (M,g) \) is a locally homogenous manifold if and only if \( g \) is locally given by (2.1) where the function \( f \) is one of the following two types:

Type I: \( f \) is linear function with respect to \( t \) and \( x \), i.e.,

\[
f(t, x, y) = xR(y) + tS(y) + \xi(y),
\]

for any functions \( R(y), S(y), \xi(y) \),

or

Type II: \( f \) is quadratic function respect to \( t \), i.e,

\[
f(t, x, y) = \kappa t^2 + xR(y) + tS(y) + \xi(y),
\]

for any functions \( R(y), S(y), P(y), Q(y) \) and any constant \( \kappa \).

Proof: We know that the function \( f \) defined in the metric (1.1) is real. Then each three eigenvalue of the \( \hat{Ric} \), are real. Hence, we can follow with respect to eigenvalues, in two case:

Type I: \( \lambda_1 = \lambda_2 = \lambda_3 = 0 \), then from the case Segre type \( \{3\} \) we have three equations,

1) \( f_{tx} = 0 \)
2) \( f_{xx} = 0 \)
3) \( f_{tt} = 0 \)

from the first equation, we have,

\[
f(t, x, y) = \tilde{f}(t, y) + \hat{f}(x, y),
\]

from the second and third equations, we have

\[
\tilde{f}(t, y) = tS(y) + \xi(y), \quad \hat{f}(x, y) = xR(y) + \eta(y).
\]

Hence, \( f(t, x, y) = xR(y) + tS(y) + \xi(y) \). This completes the first case of the theorem.

Type II: \( \lambda_1 = 0, \lambda_2 = \lambda_3 = \kappa \) for any constant \( \kappa \), then from the Segre type \( \{1zz\} \) we have three equations:

1) \( f_{tx} = 0 \)
2) \( f_{xx} = 0 \)
3) \( f_{tt} = 2\kappa \)

from the first equation, we get,

\[
f(t, x, y) = \tilde{f}(t, y) + \hat{f}(x, y),
\]
from the second equation, we have
\[ \hat{f}(x, y) = xR(y) + \eta(y), \]
and from the third equation we have \( \mathcal{T}_{tt} = 2\kappa \).
Then
\[ \mathcal{T}(t, y) = \kappa t^2 + tS(y) + \xi(y). \]
Therefor
\[ f(t, x, y) = \kappa t^2 + tS(y) + xR(y) + \xi(y). \]
This complete the second case of the theorem.

3 killing vector fields on 3-dimensional walker manifolds

**Definition 3.1** Let \( M \) be a Riemannian manifold and \( X \in \mathcal{X}(M) \). For any \( p \in M \) let \( U \subset M \) be a neighborhood of \( p \). Let \( \varphi : (-\varepsilon, \varepsilon) \times U \longrightarrow M \) be a differential mapping such that for any \( q \in U \) the curve \( t \longrightarrow \varphi(t, q) \) is a trajectory of \( X \) passing through \( q \) at \( t = 0 \). \( X \) is called a Killing vector field (or an infinitesimal isometry) if, for each \( t_0 \in (-\varepsilon, \varepsilon) \), the mapping \( \varphi(t_0, \cdot) : U \subset M \longrightarrow M \) is an isometry.

**Remark 3.2** \( X \) is Killing \( \Leftrightarrow \langle \nabla_Y X, Z \rangle + \langle \nabla_Z X, Y \rangle = 0 \) for all \( Y, Z \in \mathcal{X}(M) \)(the equation above is called the Killing equation) [3].

**Theorem 3.3** Let \( M \) be a three dimensional lorentz manifold admitting a parallel degenerate line field, Any vector field \( X \) on \( M \) is Killing if and only if the function \( f \) in the metric (2.1) takes the form of \( f(t, x, y) = \alpha(y) + c \), for any function \( \alpha(y) \) and \( c \) constant.

**Proof** : Let \( \{ \partial_t, \partial_x, \partial_y \} \) to be the base of three dimensional lorentz manifold \( M \). It follows after a straightforward calculation that the only nonzero component of metric product of metric (2.1) in Killing conditions are:

\[ \langle \nabla_{\partial_y} \partial_t, \partial_t \rangle = 0 \]
\[ \langle \nabla_{\partial_y} \partial_x, \partial_y \rangle = 0 \]

From (2.2), we conclude that \( f_1(t, x, y) = 0 \) and \( f_x(t, x, y) = 0 \) so \( f(t, x, y) = \alpha(y) + c \), for any \( \alpha(y) \) and \( c \) constant.
References


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