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On Holomorphic Solutions of Vector Differential Equations

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Abstract

By studying a first order Briot-Bouquet vector differential equation we obtain conditions on the existence and uniqueness of solutions to higher order non-homogeneous vector differential equations. These solutions are vector functions whose components are complex valued holomorphic functions in the unit disc.

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1. Introduction

The second order nonlinear differential equation

$$\frac{d^2x}{dt^2} = e^{\alpha\lambda t} x^{1+\alpha}, \tag{1.1}$$

where t, x are real variables and α, λ are real parameters, as it is well known [1], is deduced from a very important second order nonlinear differential equation which has many physical applications, since it contains the Emden equation in astrophysics and the Fermi-Thomas equation in atomic physics. In addition to this, (1.1) is closely related to a positive radial solution of an elliptic partial differential equation [8].

The asymptotic behavior of the solutions of the equation (1.1) was recently examined for $-1 \prec \alpha \prec 0$ [9] and for $\alpha \prec 0$ [10]. In doing so, the following result was proved and used.

Lemma. [9,10] Let $w = w(\xi)$ be a solution of the Briot-Bouquet differential equation

$$\xi \frac{dw}{d\xi} = f(\xi, w), \tag{1.2}$$

where $f(\xi, w)$ is a holomorphic function in the neighborhood of $(\xi, w) = (0,0)$, with $f(0,0) = 0$ and $f_w(0,0) \prec 0$. If 0 is an accumulation point of $w(\xi)$ as $\xi \to 0$ so that $\arg \xi$ is bounded, then $w(\xi)$ is the unique holomorphic solution.

In [5], by the use of a well known functional-analytic method, the study of the holomorphic solutions of the nonlinear scalar Briot-Bouquet differential equation

$$z \frac{df(z)}{dz} = G(f(z)), \tag{1.3}$$

where $f(z)$ is a complex function and $G(f(z))$ is holomorphic in $f(z)$, satisfying the conditions $G(0) = 0,\ G'(0) \neq 0$, where $G'(0)$ is the derivative of the function $G(w)$ at $w = 0$, led to the existence of families of complex-valued solutions of the second order nonlinear differential equation

$$\frac{d^2 g(z)}{dt^2} = G(g(t)). \tag{1.4}$$

These solutions are periodic for $G'(0) = -\tau^2 \prec 0$ and of exponential type for $G'(0) = \tau^2 \succ 0$. Moreover, these solutions imply the existence of infinite families of real-valued solutions, periodic and non-periodic, of a larger class of second order differential systems.

For the non-homogeneous equation $Ly = g$, where L is the operator defined by

$$Ly(z) = z^D y'(z) + A(z) y(z), \tag{1.5}$$

with $D = diag(d_1, d_2, ...d_n)$, d_i nonnegative integers and $A(z)$ an $n \times n$ matrix of function holomorphic at $z = 0$, it was proved in [3] that its solvability depends on the solutions of the conjugate homogeneous equation $L^* y = 0$, where L^* is the conjugate operator of L. In particular it was shown that $L^* y = 0$ has polynomial solutions if the constant matrix A has at least one non-positive integer eigenvalue.

The above problem was studied in the Banach space $A_{p,n}$ of all $n-$vector functions whose components are holomorphic in the open unit disc $\Delta=\{z:|z|\prec 1\}$ and p times continuously differentiable on the closed unit disc $\overline{\Delta}=\{z:|z|\le 1\}$. The main difficulty encountered there was the description of the cokernel of the operator L.

The differential operator $L:A_n\to A_0$, defined by

$$Ly(z)=z^n y^{(n)}(z)+z^{n-1}a_1(z)y^{(n-1)}(z)+\cdots+a_n(z)y(z), \qquad (1.6)$$

where $a_1(z),...,a_n(z)$ are holomorphic at $z=0$ and A_p is the Banach space of functions $u(z)$ holomorphic in Δ and p times continuously differentiable on $\overline{\Delta}$, whose norm is given by

$$\|u(z)\|_p=\left\{\max\left|u^{(i)}(z)\right|,\ \ 0\le i\le p,\ \ |z|=1\right\}$$

was studied in [2].

Using certain representation theorems from functional analysis, it was shown that the cokernel of the operator L is either zero dimensional or it is spanned by polynomials. Thus in the case when the non-homogeneous term is represented by

$$g(z)=\sum_{n=0}^{\infty}g_k z^k\ , \qquad (1.7)$$

the solvability of the differential equation $Ly=g$ in the Banach space A_p, is determined by a finite number of the coefficients in (1.7).

In [7] conditions were presented on the holomorphic coefficients $A_j(z)$ of the higher order homogeneous differential equation

$$f^{(n)}(z)+A_{n-1}(z)f^{(n-1)}(z)+\cdots+A_1(z)f'(z)+A_0(z)f(z)=0 \qquad (1.8)$$

So that all its solutions belong to the space $H_p(\Delta)$ of holomorphic functions in the unit disc Δ, which have p summable Taylor coefficients, i.e the space which consists of elements $f(z)=\sum_{n=1}^{\infty}f_n z^{n-1}$, holomorphic in Δ that satisfy the condition $\sum_{n=1}^{\infty}|f_n|^p\prec\infty$. Thus generalizing similar results, proved previously in the space $H_2(\Delta)$, concerning the second order homogeneous differential equation

$$f''(z)+A(z)f(z)=0. \qquad (1.9)$$

In this paper, by studying the homogeneous vector Briot-Bouquet differential equation

$$z\frac{df(z)}{dz} - A(z)\,f(z) = 0 \tag{1.10}$$

where $f(z)$ is a vector-valued complex function and $A(z)$ is an $m \times m$ matrix with elements $a_{ij}(z),\ \ i, j = 1, 2, ..., m$, holomorphic in $\overline{\Delta}$, we obtain conditions on the existence and uniqueness of holomorphic solutions to certain vector differential equations. Existence and uniqueness conditions of entire solutions are also predicted.

We follow the well-known functional analytic method, described in [4] and [5] that converts the question of the existence of holomorphic solutions of a differential equation into the study of an operator equation in an abstract Banach space, which is embedded in a separable Hilbert space. Moreover, the constructive nature of the method used, enables us to obtain information about the form of the existing solutions.

1. The abstract representation of a differential equation

Let H be an abstract, separable Hilbert space with an orthonormal basis $\{e_n\}_1^\infty$ and H^m its product, $H^m = \underbrace{H \times H \times \cdots \times H}_{m-times}$. The members f, g of H^m are denoted by $f = (f_1, f_2, ..., f_m)$ and $g = (g_1, g_2, ..., g_m)$, where each $f_i, g_i \in H,\ \ i = 1, 2, ..., m$ and their scalar product is defined by

$$\langle f, g \rangle = \sum_{i=1}^{m} (f_i, g_i),$$

where (f_i, g_i) denotes the scalar product in H. Every element $f_i = \sum_{n=1}^{\infty} (f_i, e_n) e_n$ in H satisfies the condition $\sum_{n=1}^{\infty} \left|(f_i, e_n)\right|^2 \prec \infty$. We shall use the same symbol for the norm in both spaces H and H^m, denoted by $\|\ \cdot\ \|$.

We define the operator $\overline{C}_0$ in H^m by the relation $\overline{C}_0 f = (C_0 f_1, C_0 f_2, ..., C_0 f_m)$, where C_0 is a diagonal operator acting on the elements $e_n,\ \ n = 1, 2, ...$ of H, as follows $C_0 e_n = n e_n,\ \ n = 1, 2, ...$ (see [4] for details).

The eigenvalues of $\overline{C}_0$ in H^m are the values $n = 1, 2, ...$, each of which has multiplicity m. Also, the eigenelements of $\overline{C}_0$, corresponding to an eigenvalue k are

$$\begin{aligned} e_{k1} &= (e_k, 0, 0, ..., 0) \\ e_{k2} &= (0, e_k, 0, ..., 0) \\ &\cdots\cdots\cdots \\ &\cdots\cdots\cdots \\ &\cdots\cdots\cdots \\ e_{km} &= (0, 0, 0, ..., e_k) \end{aligned} .$$

The set e_{ki}, $k = 1, 2, ...,$ $i = 1, 2, ..., m$ consisting of all the eigenelements of the operator $\overline{C}_0$ is an orhtonormal basis in the space H^m.
In fact if $\langle e_{ki}, f \rangle = 0$, for some $f \in H^m$, then $\langle e_{ki}, f \rangle = (e_k, f) = 0, \ \forall k$. This implies that $f_i = 0$ for $i = 1, 2, ..., m$ and thus $f = 0$.

Similarly, we define the operator $\overline{B}$ on H^m by $\overline{B}f = (Bf_1, Bf_2, ..., Bf_m)$, where B is the compact, self-adjoint, diagonal, inverse operator of C_0, i.e.

$$Be_n = \frac{1}{n} e_n, \ \ n = 1, 2, ... \text{ and } Bf_i = \sum_{n=1}^{\infty} \frac{1}{n} (f_i, e_n) e_n .$$

From the above it follows easily that $\overline{B}$ is also a compact self-adjoint operator on H^m and since C_0 has a self adjoint extension on the range of B [4], the same holds for $\overline{C}_0$ in H^m, i.e. the inverse of $\overline{C}_0$ is the compact operator $\overline{B}$.

Now let H_1 be the Banach space consisting of those elements $f_i \in H$, $i = 1, 2, \dots$, for which $\sum_{n=1}^{\infty} |(f_i, e_n)| \prec \infty$. The norm in H_1 is defined by

$$\| f_i \|_1 = \sum_{n=1}^{\infty} |(f_i, e_n)| .$$

Let H_1^m denote the product space $H_1^m = \underbrace{H_1 \times H_1 \times \cdots \times H_1}_{m-times}$, then the norm of an element $f = (f_1, f_2, ..., f_m) \in H_1^m$, denoted by the same symbol as in H_1 is given by

$$\| f \|_1 = \sum_{i=1}^{m} \| f_i \|_1 .$$

The Banach space H_1^m is embedded in H^m in the sense that if $f \in H_1^m$, then $f \in H^m$ and $\| f \| \leq \| f \|_1$.

The Hilbert space $H_2(\varDelta)$ of holomorphic functions in the open unit disc $\varDelta = \{z : |z| \prec 1\}$ is a realization of the abstract Hilbert space H and the product space $H_2(\varDelta)^m = \underbrace{H_2(\varDelta) \times \cdots \times H_2(\varDelta)}_{m-times}$ is a realization of the abstract Hilbert space H^m.

Similarly, the Banach space $H_1(\varDelta)$ of holomorphic functions having absolutely convergent power series expansions in the closed unit disc $\overline{\varDelta} = \{z : |z| \le 1\}$, is a realization of the abstract Banach space H_1 and also, $H_1(\varDelta)^m = \underbrace{H_1(\varDelta) \times \cdots \times H_1(\varDelta)}_{m-times}$ is a realization of the space H_1^m.

It was shown in [5] that if $a(z) = \sum_{n=1}^{\infty} a_n z^{n-1}$ is a holomorphic function in some neighborhood of the closed unit disc, then the linear scalar Briot-Bouquet differential equation

$$z \frac{df(z)}{dz} = a(z) f(z) \tag{2.1}$$

in the space $H_2(\varDelta)$ is equivalent to an abstract operator equation in H of the form

$$(C_0 - I) f = a^*(V) f , \tag{2.2}$$

where I is the identity operator, C_0 is the diagonal operator defined above and $a^*(V)$ is a bounded operator on H defined by the relation

$$a^*(V) = \sum_{n=1}^{\infty} \overline{a}_n V^{n-1}, \ \ V^0 = I . \tag{2.3}$$

In (2.3) V is the unilateral shift operator on H, such that $Ve_n = e_{n+1}, \ \ n = 1, 2, ...$ and $\overline{a}_n$ is the complex conjugate of the coefficients of the function $a(z)$. Every function $f(z) \in H_2(\varDelta)$ is represented by

$$f(z) = (f_z, f),$$

where $f \in H$ and $f_z = \sum_{n=1}^{\infty} z^{n-1} e_n$ are the eigenelements of the operator V^*, the adjoint of V, which is defined by $V^* : V^* e_n = e_{n-1}, \ \ n \ne 1, \ V^* e_1 = 0$.

If $a(z) \in H_1(\varDelta)$, then $a^*(V)$ is a bounded operator on H_1 and in addition,

$$\left\|a^*(V)\right\|_1 = \left\|a(z)\right\|_{H_1(\Delta)} = \sum_{n=1}^{\infty}|a_n| .$$

Moreover the operators V, V^* and C_0 fulfill the following properties:

(i) The space H_1 is invariant under V, V^* and $\|V\|_1 = \|V^*\|_1 = 1$.

(ii) If V^* is restricted to any subspace $H - \{e_1, e_2, ..., e_{n+1}\}$, then $V^* e_{n+2} = 0$.

(iii) Since $V^*V = I$ and $(C_0 - I)e_1 = 0$, we have that $VC_0V^* = C_0 - I$ and $(C_0 - I)^k = VC_0{}^kV^*$. (See ref. [4]).

Therefore, the differential equation (2.1) in the space $H_1(\Delta)$ is equivalent to the abstract equation (2.2) in H_1, or equivalently the differential equation (2.1) is the realization of the operator equation (2.2).

By applying the operator B on the left in (2.2) we get

$$(I - B)\, f = Ba^*(V)\, f . \tag{2.4}$$

In a similar manner if $A(z) = (a_{ij}(z))$, $i, j = 1, 2, ...m$ is a matrix consisting of elements belonging to $H_1(\Delta)$, then the homogeneous vector Briot-Bouquet differential equation

$$z\frac{df(z)}{dz} = A(z)\, f(z) , \tag{2.5}$$

where $f(z) = (f_1(z), f_2(z), ..., f_m(z))$, is equivalent to the abstract equation

$$(\bar{C}_0 - I)\, f = \bar{A}(V)\, f$$

or

$$(I - \bar{B})\, f = \bar{B}\bar{A}(V)\, f , \tag{2.6}$$

where $\bar{C}_0$ and $\bar{B}$ are the operators defined above and $\bar{A}(V)$ is given by the relation

$\bar{A}(V)\, f = g$, where $g = (g_1, g_2, ..., g_m)$ with $g_i = \sum_{j=1}^{m} a_{ij}^*(V)\, f_j$, $i = 1, 2, ..., m$.

3. Main results

We begin this section by proving some interesting and useful results about the operators $\bar{C}_0$ and $\bar{A}(V)$ in the abstract spaces H^m and H_1^m .

Let $A(z)$ be given by $A(z) = A(0) + zA_1(z)$, with $A(0)$ being the constant matrix, $A(0) = \{a_{ij}(0)\}$, $i, j = 1, 2, ...m$, then its abstract equivalent is

$$\bar{A}(V) = \bar{A}(0) + V\bar{A}_1(V), \tag{3.1}$$

where $\bar{A}(0)$ and V are defined on H^m or on H_1^m by the relations

$$Vf = (Vf_1, Vf_2, ..., Vf_m)$$

and

$$\bar{A}(0)\, f = (g_i), \quad g_i = \sum_{j=1}^{m} \bar{a}_{ij}(0)\, f_j, \quad i = 1, 2, ..., m.$$

Lemma 3.1. Let $k \geq 1$ be an integer and consider the operator

$$\left(\bar{C}_0 - I\right)^k - \bar{A}(0). \tag{3.2}$$

A necessary and sufficient condition for its kernel to be nontrivial in H^m or in H_1^m is that at least one of $0, 1, 2^k, 3^k, ...,$ to be an eigenvalue of the constant matrix $A(0)$.

Proof. Assume that λ^k, $\lambda = 0, 1, 2, ...$ is an eigenvalue of the constant matrix $A(0)$ and $\zeta = (\zeta_1, \zeta_2, ..., \zeta_m)$ be a corresponding to it eigenvector.

Then the elements $\zeta_q = \left(\bar{\zeta}_1 q, \bar{\zeta}_2 q, ..., \bar{\zeta}_m q\right)$, $q \in H(H_1)$ are eigenelements of the operator $\bar{A}(0)$ in $H^m\left(H_1^m\right)$, which correspond to the eigenvalue λ^k. In particular, $f_0 = \left(\bar{\zeta}_1 e_{\lambda+1}, \bar{\zeta}_2 e_{\lambda+1}, ..., \bar{\zeta}_m e_{\lambda+1},\right)$ is an eigenelement of the operator $\bar{A}(0)$ in H_1^m corresponding to the eigevalue λ^k.

Furthermore, f_0 is also an eigenelement of the operator $\left(\bar{C}_0 - I\right)^k$ in H_1^m corresponding to λ^k, which means that

$$\left[\left(\bar{C}_0 - I\right)^k - \bar{A}(0)\right] f_0 = 0$$

and therefore the kernel of the operator (3.2) is not trivial.

Conversely, assume that the kernel of the operator (3.2) is not trivial. Then there exists an $f=(f_1, f_2,..., f_m)\neq 0$ in H_1^m, such that

$$\left[\left(\overline{C}_0-I\right)^k-\overline{A}(0)\right]f=0. \tag{3.4}$$

or

$$\begin{array}{l}(C_0-I)^k f_1=\overline{a}_{11}(0)f_1+\overline{a}_{12}(0)f_2+\cdots+\overline{a}_{1m}(0)f_m\\(C_0-I)^k f_2=\overline{a}_{21}(0)f_1+\overline{a}_{22}(0)f_2+\cdots+\overline{a}_{2m}(0)f_m\\\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\(C_0-I)^k f_m=\overline{a}_{m1}(0)f_1+\overline{a}_{m2}(0)f_2+\cdots+\overline{a}_{mm}(0)f_m\end{array}.$$

Taking scalar product of the above by e_n, we have that $(C_0-I)e_n=(n-1)e_n$ and thus for every $n=1,2,...$ we get

$$\begin{array}{l}(n-1)^k(e_n,f_1)=\overline{a}_{11}(0)(e_n,f_1)+\overline{a}_{12}(0)(e_n,f_2)+\cdots+\overline{a}_{1m}(0)(e_n,f_m)\\(n-1)^k(e_n,f_2)=\overline{a}_{21}(0)(e_n,f_1)+\overline{a}_{22}(0)(e_n,f_2)+\cdots+\overline{a}_{2m}(0)(e_n,f_m)\\\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\(n-1)^k(e_n,f_m)=\overline{a}_{m1}(0)(e_n,f_1)+\overline{a}_{m2}(0)(e_n,f_2)+\cdots+\overline{a}_{mm}(0)(e_n,f_m)\end{array} \tag{3.5}$$

This implies that at least one of $0,1,2^k,3^k,...$ is an eigenvalue of the constant matrix $A(0)$, for otherwise, $(e_n, f_i)=0,\ \forall n=1,2,...$, i.e. $f_i=0,\ i=1,2,...,m$, which is a contradiction.

Theorem 3.2. Let $A(0)$ be a constant matrix and $f=(f_1, f_2,..., f_m)\neq 0$ be a solution of the operator equation

$$\left(\overline{C}_0-I\right)^k f=\overline{A}(0)f. \tag{3.6}$$

Assume also that for some $n=1,2,...$, at least one of $(e_n,f_1),(e_n,f_2),...,(e_n,f_m)$ is not equal to zero, then

$$(n-1)^k\leq\|A(0)\|. \tag{3.7}$$

Proof. For a nonzero element $u_n=((e_n,f_1),(e_n,f_2),...,(e_n,f_m))\in H_1^m$ from (3.5) we have that

$$A(0)u_n=(n-1)^k u_n \tag{3.8}$$

or $\left\|A(0)u_n\right\|_1 = \left|(n-1)^k\right|\left\|u_n\right\|_1$ and since $n \geq 1$, $\left\|A(0)u_n\right\|_1 = (n-1)^k \left\|u_n\right\|_1$.

Hence (3.7) follows from the fact that (see [6])

$$\left\|A(0)\right\| = \sup_{u_n \neq 0} \frac{\left\|A(0)u_n\right\|_1}{\left\|u_n\right\|_1}.$$

For $k=1$, (3.7) implies that $n-1 \leq \left\|A(0)\right\|$, which means that $(e_n, f_i) = 0$, $i = 1,2,...,m$ for every $n \succ \left\|A(0)\right\| + 1$. Therefore we obtain the following result concerning the vector Briot-Bouquet differential equation.

Theorem 3.3. Assume that there exists a function $f(z) = (f_1(z), f_2(z),..., f_m(z))$ in $H_2(\Delta)^m$, satisfying the vector Briot-Bouquet differential equation

$$z\frac{df(z)}{dz} = Af(z), \tag{3.9}$$

where A is any constant matrix, then the components of $f(z)$ are polynomials of degree less or equal to $1 + \left\|A\right\|$.

Moreover if A is a nonsingular matrix such that $1 \succ \left\|A\right\|$, then the vector differential equation (3.9) has no solutions in $H_2(\Delta)^m$.

Now consider the subspace $M(\lambda)$ of H^m, which is spanned by the $m(\lambda+1)$ elements e_{ni}, $n = 1,2,...,(\lambda+1)$, $i = 1,2,...,m$ and denote by $M^c(\lambda)$ its orthogonal complement. Furthermore, let $M_1(\lambda)$ be the intersection

$$M_1(\lambda) = M^c(\lambda) \cap H_1^m.$$

Clearly, this is a Banach space, which is invariant under the operators $\overline{B}$ and $\overline{A}(V)$.

Lemma 3.4. Assume that λ^k is the greatest, positive eigenvalue of the matrix $A(0)$, then the bounded operator

$$\left(I - \overline{B}\right)^k - \overline{A}(V)\overline{B}^k, \tag{3.10}$$

when restricted to the subspace $M_1(\lambda)$ of H_1^m, possesses a bounded inverse.

Proof. By lemma 3 of theorem 3.2 in [5], it can be seen that $\overline{B}^k$ is a compact operator on $M_1(\lambda)$ and since $\overline{A}(V)$ is bounded, the operator (3.10) is of the form $I-K$, where K is a compact operator on $M_1(\lambda)$.
Therefore, by the Fredholm alternative, (3.10) is invertible if the kernel of $I-K$ is trivial.
On the other hand if $g \neq 0$ is an element of the kernel of the operator (3.10) in $M_1(\lambda)$, then by setting $g = \overline{C}_0^{\,k} f$, i.e. $f = \overline{B}^k g$ in the equation

$$\left(\left(I-\overline{B}\right)^k - \overline{A}(V)\,\overline{B}^k\right)g = 0$$

we get

$$\left(\overline{C}_0 - I\right)^k f = \overline{A}(V)\, f\,. \tag{3.11}$$

Since $M_1(\lambda)$ is invariant under $\overline{B}$ and $g \in M_1(\lambda)$ it is implied that f is also an element of the subspace $M_1(\lambda)$, thus

$$(f_i, e_1) = (f_i, e_2) = \cdots = (f_i, e_{\lambda+1}) = 0, \quad i = 1,2,...,m$$

and

$$\left(a_{ij}^*(V)\, f_j, e_{\lambda+2}\right) = \overline{a}_{ij}(0)\left(f_j, e_{\lambda+2}\right), \quad i, j = 1,2,...,m\,. \tag{3.12}$$

Taking scalar product multiplication of the components of (3.11) by $e_{\lambda+2}$ we obtain

$$\begin{aligned}
(\lambda+1)^k (e_{\lambda+2}, f_1) &= \overline{a}_{11}(0)(e_{\lambda+2}, f_1) + \overline{a}_{12}(0)(e_{\lambda+2}, f_2) + \cdots + \overline{a}_{1m}(0)(e_{\lambda+2}, f_m)\\
(\lambda+1)^k (e_{\lambda+2}, f_2) &= \overline{a}_{21}(0)(e_{\lambda+2}, f_1) + \overline{a}_{22}(0)(e_{\lambda+2}, f_2) + \cdots + \overline{a}_{2m}(0)(e_{\lambda+2}, f_m)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
(\lambda+1)^k (e_{\lambda+2}, f_m) &= \overline{a}_{m1}(0)(e_{\lambda+2}, f_1) + \overline{a}_{m2}(0)(e_{\lambda+2}, f_2) + \cdots + \overline{a}_{mm}(0)(e_{\lambda+2}, f_m)
\end{aligned} \quad . \tag{3.13}$$

Now, since λ^k is the greatest positive integer eigenvalue of $A(0)$ and $V^* e_{n+2} = 0$ when restricted to $H - \{e_1, e_2, ..., e_{\lambda+1}\}$, it is implied by (3.13) that $(e_{\lambda+2}, f_i) = 0$, $i = 1,2,...,m$.
Continuing in this same way we obtain $(e_{\lambda+3}, f_i) = 0$, $i = 1,2,...,m$ and so on. Thus $f = 0$.

We note here that in case the constant matrix $A(0) = 0$, the above lemma is true for the operator

$$\left(I-\overline{B}\right) - V\overline{A}_1(V)\,\overline{B}\,,$$

for every positive integer λ^k.

Following the proof of lemma 3.4 above it can be shown that the operator

$$\left(\bar{C}_0 - I\right)^k - \bar{A}(V) \tag{3.14}$$

has a bounded inverse in H^m, as well as in H_1^m, on the condition that the non-zero constant matrix $A(0)$ possesses eigenvalues different from $0, 1, 2^k, 3^k, \ldots$.

More precisely, since the operator (3.14) is of the form $\bar{C}_0^{\,k}\left(I - K\right)$, where K is a compact operator, it is sufficient to show, as in lemma 3.1, that its kernel is trivial.

To that end let

$$\left(\bar{C}_0 - I\right)^k f = \bar{A}(V) f, \tag{3.15}$$

from which, taking scalar product by e_1, we have

$$\begin{array}{l} 0 = \bar{a}_{11}(0)(e_1, f_1) + \bar{a}_{12}(0)(e_1, f_2) + \cdots + \bar{a}_{1m}(0)(e_1, f_m) \\ 0 = \bar{a}_{21}(0)(e_1, f_1) + \bar{a}_{22}(0)(e_1, f_2) + \cdots + \bar{a}_{2m}(0)(e_1, f_m) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 0 = \bar{a}_{m1}(0)(e_1, f_1) + \bar{a}_{m2}(0)(e_1, f_2) + \cdots + \bar{a}_{mm}(0)(e_1, f_m) \end{array} . \tag{3.16}$$

Since 0 is not an eigenvalue of $A(0)$, it follows that $(e_1, f_i) = 0, \quad i = 1, 2, \ldots, m$ and hence we get, by scalar multiplication of (3.15) by e_2

$$\begin{array}{l} (e_2, f_1) = \bar{a}_{11}(0)(e_2, f_1) + \bar{a}_{12}(0)(e_2, f_2) + \cdots + \bar{a}_{1m}(0)(e_2, f_m) \\ (e_2, f_2) = \bar{a}_{21}(0)(e_2, f_1) + \bar{a}_{22}(0)(e_2, f_2) + \cdots + \bar{a}_{2m}(0)(e_2, f_m) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (e_2, f_m) = \bar{a}_{m1}(0)(e_2, f_1) + \bar{a}_{m2}(0)(e_2, f_2) + \cdots + \bar{a}_{mm}(0)(e_2, f_m) \end{array} .$$

Since 1 is not an eigenvalue of $A(0)$, it follows that $(e_2, f_i) = 0, \quad i = 1, 2, \ldots, m$.

Repeating this process we find $(e_3, f_i) = (e_4, f_i) = \cdots = 0, \quad i = 1, 2, \ldots, m$.

Therefore $f = 0$ and the operator (3.14) has a bounded inverse. Hence we have proved the following.

Theorem 3.5. The non-homogeneous abstract equation

$$\left[\left(\bar{C}_0 - I\right)^k - \bar{A}(V)\right] f = g \tag{3.17}$$

possesses a unique solution in $H^m\left(H_1^m\right)$ for every $g\in H^m\left(H_1^m\right)$. This solution has the form

$$f=\left(I-K\right)^{-1}\bar{B}^k g\ ,$$

where K is a compact operator.

Following theorem 3.3 of ref. [5], it can be shown that the range of the operator $\bar{B}^k$ in $H^m\left(H_1^m\right)$ is invariant under the operator $\left(I-K\right)^{-1}$, which means that if $g\in H^m\left(H_1^m\right)$, then the solution of the differential equation that realizes the abstract equation (3.17) in $H\left(\Delta\right)^m$, together with its derivatives up to order $k-1\left(k\right)$ belong to $H_1\left(\Delta\right)^m$.

The following result provides even more information about the form of the solution of the equation (3.17).

Theorem 3.6. Let the constant matrix $A\left(0\right)$ possess eigenvalues different from $0,1,2^k,3^k,\ldots$. Then for every g in H^m there exists a constant vector $\mu=\left(\mu_1,\mu_2,\ldots,\mu_m\right)$ and an element $q=\left(q_1,q_2,\ldots,q_m\right)\in H^m$ such that the unique solution of the abstract equation (3.17) is given by

$$f=\mu_0+\bar{B}^k q,\quad \left(q_i,e_i\right)=0,\quad i=1,2,\ldots,m\ , \tag{3.18}$$

where $\mu_0=\left(\bar{\mu}_1 e_1,\bar{\mu}_2 e_1,\ldots,\bar{\mu}_m e_1\right)\in H_1^m$.

Proof. Inserting (3.18) into (3.17) we get

$$\left(I-\bar{B}\right)^k q-\bar{A}\left(V\right)\mu_0-\bar{A}\left(V\right)\bar{B}^k q=g$$

or equivalently, by the relation (3.1) we have

$$\left(I-\bar{B}\right)^k q-\left[\bar{A}\left(0\right)+V\bar{A}_1\left(V\right)\right]\mu_0-\bar{A}\left(V\right)\bar{B}^k q=g\ . \tag{3.19}$$

Denote by μ_1 the element $\mu_1=\left(\left(g_1,e_1\right)e_1,\left(g_2,e_1\right)e_1,\ldots,\left(g_m,e_1\right)e_1\right)$, where g_i, $i=1,2,\ldots,m$ are the components of the function g. The fact that the matrix $A\left(0\right)$ in non-singular, implies that a constant vector $\mu=\left(\mu_1,\mu_2,\ldots,\mu_m\right)$ can be found for which $\bar{A}\left(0\right)\mu_0=-\mu_1$. Hence, from (3.19) we get

$$\left[\left(I-\bar{B}\right)^k-\bar{A}\left(V\right)\bar{B}^k\right]q=g-\mu_1+V\bar{A}_1\left(V\right)\mu_0\ . \tag{3.20}$$

Let w_i, $i=1,2,\ldots,m$ be the components of the element

$$w = g - \mu_1 + V\overline{A}_1(V)\mu_0 .$$

They obviously satisfy the condition $(w_i, e_1) = 0, \ \ i = 1, 2, ..., m$, which means that w belongs to the subspace $H^m - \{e_{1i}, \ \ i = 1, 2, ..., m\}$, which is invariant under the operators $(I - \overline{B})^k$ and $\overline{A}(V)\overline{B}^k$. Also by lemma 3.4, the operator

$$(I - \overline{B})^k - \overline{A}(V)\overline{B}^k ,$$

when restricted to that subspace, has a bounded inverse, i.e. equation (3.17) has a unique solution of the form (3.18).

Next we consider the case in which the elements of the matrix $A(z)$ are holomorphic functions in the closed unit disc and thus the operator $a^*_{ij}(V)$ is bounded on H, whereas the operator $\overline{A}(V)$ is bounded on H^m.

We proceed by proving the existence of a holomorphic solution for first and second order non-homogeneous vector differential equations, but these results can be readily extended to equations of higher order due to theorems 3.5 and 3.6.

Theorem 3.7. Consider the non-homogeneous vector differential equation

$$z\frac{df(z)}{dz} - A(z) f(z) = g(z), \tag{3.21}$$

where $g(z) = (g_1(z), g_2(z), ..., g_m(z)) \in H_2(\Delta)^m$ and the constant matrix $A(0)$, has eigenvalues different from $0, 1, 2, ...$. Then there exists a vector-valued solution of (3.21), $f(z) = (f_1(z), f_2(z), ..., f_m(z))$ having the form

$$f_i(z) = \mu_i + \sum_{n=1}^{\infty} b_{in} z^n, \ \ i = 1, 2, ..., m, \tag{3.22}$$

where the vector $\mu = (\mu_1, \mu_2, ..., \mu_m)$ is the unique solution of the system $A(0)\mu = (g_1(0), g_2(0), ..., g_m(0))$. Moreover the series (3.22) is absolutely convergent in the closed unit disc and if $g(z) \in H_1(\Delta)^m$, then the first order derivatives $\dfrac{df_i(z)}{dz}, \ \ i = 1, 2, ..., m$ are also absolutely convergent.

Proof. The differential equation (3.21) is equivalent to the operator equation

$$\left[\left(\bar{C}_0 - I\right) - \bar{A}(V)\right] f = g\ , \tag{3.23}$$

where $g_i(z) = (f_z, g_i)$, $i = 1, 2, \ldots m$.

Theorems 3.5 and 3.6 imply that the abstract equation (3.23) possesses a unique solution in the space H^m of the form $f = \mu_0 + \bar{B}q$, which is the realization of an element of the form (3.22) in the space $H_2(\Delta)^m$.

In the case where q belongs to H_1^m, we have that $\bar{B}q \in H_1^m$ and so the series (3.22) converges absolutely in $\bar{\Delta} = \{z : |z| \le 1\}$. Moreover, since the space $H_1(\Delta)^m$ remains invariant under the operator $\bar{B}$, $f = \mu_0 + \bar{B}q$ is an element of the definition domain of $\bar{C}_0$ in H_1^m. This means that the first order derivatives of (3.22) converge absolutely in $\bar{\Delta}$.

Theorem 3.8. Consider the second order vector differential equation

$$z^2 \frac{d^2 f(z)}{dz^2} + z\frac{df(z)}{dz} = A(z)\, f(z) + g(z), \tag{3.24}$$

where $g(z) = \left(g_1(z), g_2(z), \ldots, g_m(z)\right)$ is an element of $H_2(\Delta)^m \left(H_1(\Delta)^m\right)$ and the constant matrix $A(0)$ has eigenvalues different from $0, 1, 2^2, 3^2, \ldots$.
Then there exists a vector-valued solution of (3.24) of the form (3.22), which together with its derivatives up to order one (two) converge absolutely in the closed unit disc.

Proof. The product $z^2 \frac{d^2 f(z)}{dz^2}$, when $f(z)$ is a function in $H_2(\Delta)\left(H_1(\Delta)\right)$, is equivalent to $V^2\left(C_0 V^*\right)^2 f$ in $H_2(H_1)$, [4].
Also, as we have seen $z\frac{df(z)}{dz}$ is equivalent to $VC_0V^* f$ or $\left(C_0 - I\right) f$.
It follows that the left hand side of the given vector differential equation (3.24) in $H_2(\Delta)^m \left(H_1(\Delta)^m\right)$, is equivalent in $H^m\left(H_1^m\right)$ to

$$V^2\left(\bar{C}_0 V^*\right)^2 f + VC_0V^* f$$

or

$$V^2\overline{C}_0V^*\overline{C}_0V^*f+V\overline{C}_0V^*f=V\left(\overline{C}_0-I\right)\overline{C}_0V^*f+V\overline{C}_0V^*f$$
$$=V\overline{C}_0{}^2V^*f$$
$$=\left(\overline{C}_0-I\right)^2f$$

Hence, the equivalent operator equation of (3.24) is given by

$$\left(\overline{C}_0-I\right)^2f=\overline{A}\left(V\right)f+g\,. \tag{3.25}$$

By theorems 3.5 and 3.6 there exists a unique solution of (3.25) of the form $f=\mu_0+\overline{B}^2q$, which belongs to the definition domain of $\overline{C}_0$ in H_1^m, for $q\in H_1^m$ and to the definition domain of $\overline{C}_0{}^2$ in H_1^m, for $q\in H_1^m$. Hence the proof of theorem (3.8) follows.

Theorems 3.5 and 3.6 also predict the existence of entire solutions for the vector non-homogeneous differential equations (3.21) and (3.24) as the following corollaries show.

Corollary 3.9. Let the element $a_{ij}\left(z\right),\ i,j=1,2,...,m$ of the matrix $A\left(z\right)$ and the components $g_i\left(z\right),\ i=1,2,...,m$ of $g\left(z\right)$ be entire functions. If the conditions of theorem (3.7) are true, then the differential equation (3.21) possesses a unique entire solution $f\left(z\right)$ whose components are given by (3.22).

Corollary 3.10. Let the element $a_{ij}\left(z\right),\ i,j=1,2,...,m$ of the matrix $A\left(z\right)$ and the components $g_i\left(z\right),\ i=1,2,...,m$ of $g\left(z\right)$ be entire functions. If the conditions of theorem (3.8) are true, then the differential equation (3.24) possesses a unique entire solution $f\left(z\right)$ with components of the form (3.22).

The proofs of both the above corollaries follow easily if we use the transformation $z=r\psi$ and thus obtain unique solutions $f\left(r\psi\right)$ of (3.21) and (3.24) that are holomorphic in $\Delta=\left\{\psi:\left|\psi\right|\prec 1\right\}$, for every $r\succ 0$.

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