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S-C-Permutably Embedded Subgroups of Finite Groups[1]

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Abstract

We call a subgroup H of a group G s-c-permutably embedded in G if for each prime $p \in \pi(H)$, every Sylow p-subgroup of H is a Sylow p-subgroup of some s-conditionally permutable subgroup of G. In this paper, we obtain some results on s-c-permutably embedded subgroups and by using these results, we determine the structures of some groups.

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1 Introduction

All subgroups considered in this paper are finite.

Recall that a subgroup A of a group G is permutable with a subgroup B if $AB = BA$. If A is permutable with all subgroups of G, then A is called a permutable subgroup [3] (or quasinormal subgroup) [13] of G. The permutable subgroups have many interesting properties. For example, Ore [13] proved that every permutable subgroup of a group is subnormal. Itó and Szêp [12] showed that H/H_G is nilpotent for every permutable subgroup H of a group G. Kegel and Deskins [2] showed that the subgroups H of a group G which are permutable with all Sylow subgroups of G inherit a series of key properties of permutable subgroups. Recently, Guo, Shum and Skiba [8] introduce the concept of conditionally permutable subgroup. They say that a subgroup H of a group G is conditionally permutable in G if for any subgroup T of G, there exists some $x \in G$ such that $HT^x = T^xH$. Using the new idea, people have

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obtained a series of elegant results on the structure of groups (cf [6-10]). A subgroup H of G is said to be s-conditionally permutable in G (cf. [10, 16]) if for every Sylow subgroup T of G, there exists $x \in G$ such that $HT^x = T^xH$. By Sylow theorem, we know that a subgroup H of G is s-conditionally permutable if and only if for any $p \in \pi(G)$, there exists a Sylow p-subgroup P such that $PH = HP$. As a continuation, we now introduce the following concept:

Definition 1.1 *Let H be a subgroup of a group G. H is said to be s-c-permutably embedded in G if for every Sylow subgroup of H is a Sylow subgroup of some s-conditionally permutable subgroup of G.*

Clearly, every s-conditionally permutable subgroup is a s-c-permutably embedded subgroup of G. However, the following examples shows that an s-c-permutably embedded subgroup is not necessarily s-conditionally permutable in G.

Example 1. Let $N \lhd G$. The every Sylow subgroup T of N is s-c-permutably embedded in G, but clearly T is not necessarily be an s-conditionally permutable subgroup of G if G is non-soluble.

Example 2. Let $G = S_5$ and P be a Sylow 3-subgroup of G. Then P is not an s-conditionally permutable subgroup. In fact, we know that S_5 has no a subgroup of order 15. Hence P_3 can not permute with any Sylow 5-subgroup of G. However, G is itself an s-conditionally permuted subgroup of G. Hence P_3 is an s-c-permutably embedded subgroup in G.

All unexplained notations and terminology are standard. The reader is referred to Huppert [11] or Guo [4].

2 Preliminaries

We first give some basic results on s-conditionally subgroups and s-c-permutably embedded subgroups.

Lemma 2.1 [16, Lemma 2.1] *Let G be a group, $K \lhd G$ and $H \leq G$. Then:*

(1) If H is s-conditionally permutable in G, then HK/K is s-conditionally permutable in G/K.

(2) If $K \leq H$ and H/K is s-conditionally permutable in G/K, then H is s-conditionally permutable in G.

(3) Suppose that HK/K is s-conditionally permutable in G/K and $(|H|, |K|) = 1$. If G is soluble or K is nilpotent, then H is s-conditionally permutable in G.

(4) If H is s-conditionally permutable in G, then $H \cap K$ is s-conditionally permutable in K.

Lemma 2.2 *Suppose that G is a group, $K \lhd G$ and $H \leq G$. Then:*

(1) If H is s-c-permutably embedded in G, then HK/K is s-c-permutably embedded in G/K.

(2) If $K \leq H$ and H/K is s-c-permutably embedded in G/K, then H is s-c-permutably embedded in G.

(3) If HK/K is s-c-permutably embedded in G/K and $(|H|, |K|) = 1$, then H is s-c-permutably embedded in G.

(4) If H is s-c-permutably embedded in G, then $H \cap K$ is s-c-permutably embedded in K.

Proof. (1) It is obvious.

(2) Let $p \in \pi(H)$. By the hypothesis, there exists an s-conditionally permutable subgroup N/K of G/K such that every Sylow p-subgroup P/K of H/K is a Sylow p-subgroup of N/K. Hence $p \nmid |N/K : P/K|$. If $p \nmid |H/K|$, then $P = K$. In this case, every Sylow p-subgroup of H is also a Sylow p-subgroup of K. Since $K \trianglelefteq G$, K is clearly s-conditionally permutable in G. Therefore H is s-c-permutably embedded in G. Now, suppose that $p \mid |H/K|$. By Lemma 2.1(2), N is s-conditionally permuted in G. Let P_1 be a Sylow p-subgroup of H. We need only to prove that P_1 is also a Sylow p-subgroup of N. Obviously, $P/K = P_1K/K$ and $|N : P_1| = |N : P| \mid P_1K : P_1 \mid$ is a p'-number. This means that P_1 is a Sylow p-subgroup of N. Thus H is s-c-permutably embedded in G.

(3) By (2), we can see that HK is s-c-permutably embedded in G. Let p be an arbitrary prime dividing the order of H. Then $p \mid |HK|$. By the hypothesis, there exists an s-conditionally permutable subgroup N of G such that every Sylow p-subgroup of HK is also a Sylow p-subgroup of N. Since $(|H|, |K|) = 1$, we have that every Sylow p-subgroup of H is also a Sylow p-subgroup of N. Hence, H is s-c-permutably embedded in G.

(4) Let $p \in \pi(H)$. By the hypothesis, there exists a s-conditionally permutable subgroup N of G such that every Sylow p-subgroup P of H is also a Sylow p-subgroup of N. Since $K \trianglelefteq G$, obviously $N \cap K$ is also s-conditionally permutable in K. We now prove that every Sylow p-subgroup of $H \cap K$ is also a Sylow p-subgroup of $N \cap K$. In fact, since $K \lhd G$, $P \cap K$ is a Sylow p-subgroup of $H \cap K$. By Sylow theorem, we may assume without loss of generality that $P \cap K$ is an arbitrary Sylow p-subgroup of $H \cap K$. Since $|(N \cap K) : (P \cap K)| = |(N \cap K) : (P \cap N \cap K)| = |P(N \cap K) : P|$, $p \nmid |(N \cap K) : (P \cap K)|$. This shows that $P \cap K$ is a Sylow p-subgroup of $N \cap K$. Hence H is s-c-permutably embedded in K.

Lemma 2.3 *Let G be a group and P a subgroup of G contained in $O_p(G)$. If P is s-c-permutably embedded in G, then P is s-conditionally permutable in G.*

Proof. Obviously, P is a subnormal subgroup of G. Since P is s-c-permutably embedded in G, there exists an s-conditionally permutable subgroup A of G such that P is a Sylow p-subgroup of A. Hence, for any $q \in \pi(G)$, there exists a Sylow q-subgroup G_q of G such that $AG_q = G_qA$. If $p = q$, then $P \leq G_p$ and so $PG_p = G_pP$. If $p \neq q$, then P is a subnormal Hall subgroup of $AG_q = G_qA$ and consequently P is normal in AG_q. Hence $PG_q = G_qP$. This shows that P is s-conditionally permuted in G.

Lemma 2.4 *Let P be a minimal normal p-subgroup of G. If every subgroup of P with order p is s-c-permutably embedded in G, then P is a group of order p.*

Proof. Suppose that E is a Sylow p-subgroup of G. Then $P \cap Z(E) \neq 1$. Let L be a subgroup of $P \cap Z(E)$ of order p. By the hypothesis, L is s-c-permutably embedded in G. Hence there exists a s-conditionally permutable subgroup N of G such that L is a Sylow p-subgroup of N. This means that for every $q \in \pi(G)$ and $p \neq q$, there exists a Sylow q-subgroup Q of G such that $NQ = QN$. Since $L = P \cap NQ \lhd NQ$, $NQ \subseteq N_G(L)$. On the other hand, $E \leq N_G(L)$. Thus $L \lhd G$. But since P is a minimal normal p-subgroup of G, we have $P = L$. Thus P is a group of order p.

For the sake of convenience, we now list some known results for the proofs in this paper.

Lemma 2.5 [8, Lemma 3.1]. *Let N and L be normal subgroups of a group G. Let P/L be a Sylow p-subgroup of NL/L and M/L a maximal subgroup of P/L. If P_p is a Sylow p-subgroup of $P \cap N$, then P_p is a Sylow p-subgroup in N such that $D = M \cap N \cap P_p$ is a maximal subgroup in P_p and $M = LD$.*

3 Main Results

Theorem 3.1 *Let p be a prime and G a p-soluble group. If every cyclic p-subgroup of G is s-c-permutably embedded in G, then G is p-supersoluble.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order. Then:

(1) If N is a proper normal subgroup of G, then G/N is p-supersoluble.

In fact, for the cyclic p-subgroup K/N of G/N, we have $K/N = \langle x\rangle N/N$, where $x \in K$. By Sylow theorem, there exists a Sylow p-subgroup G_p such that $KN/N \leq G_pN/N$ and so $K \leq G_pN$. Therefore, we may assume that $x = gn$, where $g \in G_p$, $n \in N$. Then $\langle x\rangle N = \langle g\rangle N$. By the hypothesis, $\langle g\rangle$ is s-c-permutably embedded in G. It follows from Lemma 2.2 that $K/N = \langle x\rangle N/N = \langle g\rangle N/N$ is s-c-permutably embedded in G/N. Hence G/N

satisfies the condition of the theorem. The choice of G implies that G/N is p-supersoluble.

(2) $\Phi(G) = 1$ and G has a unique minimal normal subgroup N such that $N = C_G(N) = O_p(G)$ and $G = [N]M$, where M is a maximal subgroup of G with $O_p(M) = 1$.

Since the class of all p-supersoluble groups is a saturated formation, by (1), obviously, $\Phi(G) = 1$ and G has a minimal normal subgroup N. Hence there exists a maximal subgroup M of G such that $G = NM$. Since G is p-soluble, N is a clearly p-subgroup of G (Otherwise, N is a p' group and consequently G is p-supersoluble.) Thus N is an elementary abelian p-subgroup of G. It follows that $G = [N]M$. Let $C = C_G(N)$. Obviously, $C \cap M = 1$. By the Dedekind identity, $C = C \cap NM = N(C \cap M) = N$. This shows that $N = O_p(G) = C_G(N)$ and $M \cong G/N$ is a supersoluble group with $O_p(M) = 1$ (cf. [4, Lemma 1.7.11]).

(3) Final contradiction follows.

By Lemma 2.4, $|N| = p$. Then by (1), G is p-supersoluble. This is a contradiction. Thus, the proof of the theorem is completed.

Corollary 3.1.1 *Let G be a p-soluble group and p a prime dividing the order of G. $(|G|, p-1) = 1$ and P is a Sylow p-subgroup of G. If every maximal subgroup of P is s-c-permutably embedded in G, then G is p-nilpotent.*

Theorem 3.2 *Let G be a soluble group. If every maximal subgroup of every non-cyclic Sylow subgroup of G having no supersoluble supplement in G is s-c-permutably embedded in G, then G is supersoluble.*

Proof. Suppose that the result is false and let G be a counterexample of minimal order. Then:

(1) G is not a simple group.

If G is a simple group, then G is a cyclic group of prime order and so G is supersoluble, a contradiction.

(2) For every minimal normal subgroup N of G, G/N is supersoluble.

Let Q/N be a non-cyclic Sylow p-subgroup of G/N and K/N a maximal subgroup of Q/N. Then there exists a Sylow p-subgroup P of G such that $Q = PN$ and $K = N(P \cap K)$. Clearly, $P \cap K$ is a maximal subgroup of P and P is non-cyclic. If $P \cap K$ possesses a supersoluble supplement T in G, then $TN/N \cong T/T \cap N$ is a supersoluble supplement to K/N in G/N. If $P \cap K$ is s-c-permutably embedded in G, then by Lemma 2.2, $K/N = N(P \cap K)/N$ is s-c-permutably embedded in G/N. These shows that G/N satisfies the hypothesis of the theorem. Thus, by the choice of G, we have that G/N is supersoluble.

(3) $\Phi(G) = 1$ and G has a unique minimal normal subgroup H such that $H = C_G(H) = O_p(G) = F(G)$ for some prime p, and $G = [H]M$, where M is

a maximal subgroup of G with $O_p(M) = 1$. (See the proof of (2) in Theorem 3.1.)

(4) Any Sylow subgroup of G is not normal subgroup in G.

Suppose that for some $q \in \pi(G)$, G has a normal Sylow q-subgroup G_p. Then by (3)$q = p$ and $H = G_p$. Sine $G = [H]M$, $|G : M| = |H|$. Assume that H_1 is a maximal subgroup of H. By the hypothesis, either H_1 possesses a supersoluble supplement T in G or H_1 is s-c-permutably embedded in G. In the first case, the choice of G implies that $T \neq G$ and so $G = [H_1]T$, which contradicts the minimality of H. In the second case, there exists an s-conditionally permutable subgroup A of G such that H is a Sylow p-subgroup of A. Let q be an arbitrary prime divisor of $|G|$ with $q \neq p$. Since A is s-conditionally permutable, there exists a Sylow q-subgroup Q of G such that $AQ = QA$. Then $H_1 = H \cap AQ \trianglelefteq G$. It follows that $Q \leq N_G(H_1)$ for any $q \neq p$ and $q \in \pi(G)$. On the other hand, since $H_1 \lhd H = G_p$, $G_p \leq N_G(H_1)$. Hence $H_1 \lhd G$. But since H is a minimal normal subgroup, we have $|H| = p$. This induces that G is supersoluble, a contradiction.

(5) The number p is not the largest prime divisor of $|G|$.

Indeed, if p is the largest divisor of $|G|$, then (2) and (5) implies that $O_p(G/N) \neq 1$ which contradicts to (4).

(6) Final contradiction.

By (3), we have that $G = [H]M$. Pick some Sylow p-subgroup M_p of M and let P be a Sylow p-subgroup of G including M_p. Let P_1 be maximal subgroup of P such that $M_p \leq P_1$. Then $H_1 = H \cap P_1$ is a maximal subgroup of H. By (2), it is clear that $|H| > p$. Hence P is not cyclic. By the hypothesis, P_1 is s-c-permutably embedded in G or P_1 possesses a supersoluble supplement T in G. In the first case, there exists an s-conditionally permutable subgroup A of G such that P_1 is a Sylow p-subgroup of A. This means that for an arbitrary prime divisor q of $|G|$ with $p \neq q$, there exists a Sylow q-subgroup Q of G such that $AQ = QA$. Since $H_1 < \cdot H$ and $H_1 = H \cap P_1 \leq H \cap A \leq H \cap AQ \leq H$, $H_1 = A \cap HQ$ or $H = H \cap AQ$. If $H = H \cap AQ$, then $H \leq AQ$ and thereby $P = P_1H \leq P_1AQ = AQ$. This implies that $P = P_1$, which is impossible. Hence we may assume that $H_1 = H \cap AQ$. Because $H \lhd G$, $H_1 \lhd AQ$. It follows that $Q \leq N_G(H_1)$. On the other hand, since $H_1 \lhd H$ and $H_1 \lhd P_1$, $H_1 \lhd P_1H = P$. This means that $H_1 \lhd G$ and so $|H| = p$, a contradiction. Now assume the second case applies. Let q be the largest prime divisor of $|T|$ and T_q a Sylow q-subgroup of T. Since T is supersoluble, $T_q \lhd T$ where T_q is a Sylow q-subgroup of T. Obviously, T_q is also a Sylow q-subgroup of G. Since by (3), M is supersoluble, $T_q \lhd M^x$ for some $x \in G$. It follows that $M^x \subseteq N_G(T_q)$. But since $M < \cdot G$, by (3) and (4), $M^x = N_G(T_q)$ and consequently $T \subseteq M^x$. Thus $G = P_1T = P_1M$, which implies that $P = P_1$. This contradiction completes the proof.

Corollary 3.2.1 *Let G be a soluble group. If every maximal subgroup of each*

Sylow p-subgroup of G is s-c-permutably embedded in G. Then G is supersoluble.

Corollary 3.2.2 [10] *Let G be a soluble group. If every maximal subgroup of each Sylow subgroup of G is s-conditionally permuted in G, then G is supersoluble.*

Theorem 3.3 *Let $\mathfrak{F}$ be a saturated formation containing the class $\mathfrak{U}$ of all supersoluble group. A group $G \in \mathfrak{F}$ if and only if G has a soluble normal subgroup H such that $G/H \in \mathfrak{F}$ and every maximal subgroup of every Sylow subgroup of H is s-c-permutably embedded in G.*

Proof. The necessary part is obvious and so we only need to prove the sufficient part. Suppose that the sufficient part is false and let G be a counterexample of minimal order, we process with our proof as follows:

(1) If R is a minimal normal subgroup of G, then $G/R \in \mathfrak{F}$.

In fact, if $R = H$, then, of course, $G/R \in \mathfrak{F}$. Now we assume that $R \neq H$. Then RH/R is a soluble normal subgroup of G/R such that the factor group $(G/R)/(RH/R) \cong G/RH \cong (G/H)/(RH/R) \in \mathfrak{F}$. Let P/R be a Sylow p-subgroup in RH/R and let M/R be a maximal subgroup in P/R. If P_p is a Sylow p-subgroup in $P \cap H$ then by Lemma 2.5, P_p is a Sylow p-subgroup in H and $D = M \cap H \cap P_p$ is a maximal subgroup in P_p and $M = RD$. Thus, by the hypothesis, D is s-c-permutably embedded in G. It follows from Lemma 2.2 that $M/R = LR/R$ is s-c-permutably embedded in G/R. This shows that the conditions of the theorem are inherited on G/R. Hence, the choice of G, we have that $G/R \in \mathfrak{F}$.

(2) G has the unique minimal normal subgroup R and $R = C_G(R) = O_p(G) = F(G) \nsubseteq$ for some $p \in \pi(G)$ and $\Phi(G) = 1$.

Since $\mathfrak{F}$ is a saturate formation, by (1) we see that (2) holds clearly.

(3) $|R| = p$.

Assume $|R| = p^\alpha$ for some natural number $\alpha > 1$. Let P be a Sylow p-subgroup of G. Since $R \nsubseteq \Phi(G)$, $R \nsubseteq \Phi(P)$. Hence, there exists a maximal subgroup P_1 of P such that $R \nsubseteq P_1$. Since $R \subseteq H$, $P_1 \cap H$ is a maximal subgroup of some Sylow p-subgroups of H. By the hypothesis, there exists an s-conditionally permutable subgroup A of G such that $P_1 \cap H$ is a Sylow p-subgroup of A. Then, for arbitrary $q \in \pi(G)$ with $p \neq q$, there exists a Sylow q-subgroup Q of G such that $AQ = QA$. Hence $R \cap P_1 = R \cap (P_1 \cap H) \subseteq R \cap AQ \trianglelefteq AQ$ and thereby $Q \subseteq N_G(R \cap P_1)$ for any $q \neq p$. On the other hand, $R \cap P_1 \trianglelefteq P$. This shows that $R \cap P_1 \trianglelefteq G$. Consequently $R \cap P_1 = 1$, that is, $|R| = p$.

(4) Final contradiction.

Since $\mathfrak{F}$ is a saturated formation containing $\mathfrak{U}$, $\mathfrak{F}$ has a formation function f such that $\mathfrak{A}(p-1) \subseteq f(p)$ for all prime p, where $\mathfrak{A}(p-1)$ is the formation of all abelian group with exponents dividing $p-1$ (see [4, p. 98]). By (2) and (3), $G/R = G/C_G(R) \in \mathfrak{A}(p-1) \subseteq f(p)$. Then since $G/R \in \mathfrak{F}$, we obtain that $G \in \mathfrak{F}$. This contradiction completes the proof.

Theorem 3.4 *Let $\mathfrak{F}$ be a saturated formation containing the class $\mathfrak{U}$ of all supersoluble groups. A group $G \in \mathfrak{F}$ if and only if G has a soluble normal subgroup H such that $G/H \in \mathfrak{F}$ and every maximal subgroup of each Sylow subgroup of $F(H)$ is s-c-permutably embedded in G.*

Proof. The necessary part is clear, we only need to prove the sufficiency part.

Suppose that the assertion is false and let G be a counterexample of minimal order. Let P be a Sylow p-subgroup of $F(H)$ for an arbitrary prime divisor p of $|G|$. Then P char $F(H) \lhd G$ and so $P \lhd G$. We proceed with our proof as follows:

(1) $P \cap \Phi(G) = 1$.

Assume that $R = P \cap \Phi(G) \neq 1$. Then $(G/R)/(H/R) \in \mathfrak{F}$. Let $F(H/R) = T/R$. Obviously $F(H) \subseteq T$. On the other hand, since $R \subseteq \Phi(G)$, T is nilpotent. Thus $T \subseteq F(H)$ and so $F(H)/R = F(H/R)$. Let P_1/R be a maximal subgroup of P/R. Then P_1 is a maximal subgroup of P. By the hypothesis, P_1 is s-c-permutably embedded in G. It follows from Lemma 2.2 that P_1/R is s-c-permutably embedded in G/R. Now let Q/R be a maximal subgroup of a Sylow q-subgroup of $F(H)/R$, where $q \neq p$. Then $Q = Q_1R$, where Q_1 is a Sylow q-subgroup of $F(G)$. By the hypothesis, Q_1 is s-c-permutably embedded in G and so $Q/R = Q_1R/R$ is s-c-permutably embedded in G/R by Lemma 2.2. This shows that G/R satisfies the hypothesis. Thus, by the choice of G, we have $G/R \in \mathfrak{F}$. Since $G/\Phi(G) \cong (G/R)/(\Phi(G)/R)$ and $\mathfrak{F}$ is a saturated formation, we have that $G \in \mathfrak{F}$, a contradiction.

(2) $P = R_1 \times R_2 \times \cdots \times R_m$, where R_i $(i = 1, 2, \cdots, m)$ is a minimal normal subgroup of G with order p.

Since $P \lhd G$ and $P \cap \Phi(G) = 1$, it is easy to see that $P = R_1 \times R_2 \times \cdots \times R_m$, where R_i $(i = 1, 2, \cdots, m)$ is a minimal normal subgroup of G (cf. [4, Theorem 1.8.17]). We now prove that $|R_i| = p$. Since $R \nsubseteq \Phi(G)$, there exists a maximal subgroup M of G such that $G = MR_i$ and clearly $M \cap R_i = 1$. Let M_p be a Sylow p-subgroup of M. Then $G_p = M_pR_i = M_pP$ is a Sylow p-subgroup of G. Let H_1 be a maximal subgroup of G_p containing M_p. Then $P_1 = H_1 \cap P$ is a maximal subgroup of P. By the hypothesis, P_1 is s-c-permutably embedded in G. Hence, there exists a subgroup s-conditionally permutable subgroup N of G such that P_1 is a Sylow p-subgroup of N. Hence, for any $q \in \pi(G)$ with $p \neq q$, there exists a Sylow q-subgroup Q of G such that $NQ = QN$. Similar to the proof of Theorem 3.3, we obtain that $P_1 \lhd G$ and so $P_1 \cap R_i \lhd G$. But since $R_i \nleq P_1$ and R_i is a minimal normal subgroup of G, $P_1 \cap R_i = 1$. Hence, $|R_i| = |R_i : P_1 \cap R_i| = |R_i : H_1 \cap P \cap R_i| = |R_i : H_1 \cap R_i| = |R_iH_1 : H_1| =$

$|G_p : H_1| = |PH_1 : H_1| = |P : P_1 \cap H_1| = |P : P_1| = p$. This shows that R_i is a cyclic group of order p.

(3) Final contradiction follows.

By (2), $F(H) = N_1 \times N_2 \cdots \times N_m$, where N_i $(i = 1, 2, \cdots m)$ is a minimal normal subgroup of G of prime order. Since $G/C_G(N_i)$ is isomorphic to a subgroup of $Aut(N_i)$, $G/C_G(N_i)$ is cyclic. It follows that $G/\cap_{i=1}^{n} C_G(N_i) = G/C_G(F(H)) \in \mathfrak{U} \subseteq \mathfrak{F}$, and consequently $G/C_H(F(H)) = G/(H \cap C_G(F(H))) \in \mathfrak{F}$. Since $F(H)$ is abelian, $F(H) = C_H(F(H))$. Therefore $G/F(H) \in \mathfrak{F}$. Then, by Theorem 3.3, we obtain that $G \in \mathfrak{F}$. This contradiction complete the proof.

Corollary 3.4.1 *Let G be a group with a soluble normal subgroup E such that G/E is supersoluble. If every maximal subgroup of every Sylow subgroup of $F(E)$ is s-conditionally permutable in G, then G is supersoluble.*

Remark 3.4.1 *Theorem 3.4 and corollary 3.4.1 can not necessarily hold for non-soluble groups. For example, let $G = SL(2, 5)$. Then $F(G)$ is a cyclic group of order 2. Thus each maximal subgroup of Sylow Subgroup of $F(G)$ is s-c-permutably embedded in G. However G is not a supersoluble group.*

Remark 3.4.2 *Theorem 3.4 and corollary 3.4.1 are not necessarily hold if the saturated formation $\mathfrak{F}$ does not contain the class $\mathfrak{U}$ of all supersoluble subgroups. For example, let $\mathfrak{F}$ be a nilpotent formation. Then the symmetric group of degree 3 is a counterexample.*

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