S-C-Permutably Embedded Subgroups of Finite Groups

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Abstract

We call a subgroup $H$ of a group $G$ $s$-$c$-permutably embedded in $G$ if for each prime $p \in \pi(H)$, every Sylow $p$-subgroup of $H$ is a Sylow $p$-subgroup of some $s$-conditionally permutable subgroup of $G$. In this paper, we obtain some results on $s$-$c$-permutably embedded subgroups and by using these results, we determine the structures of some groups.

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1 Introduction

All subgroups considered in this paper are finite.

Recall that a subgroup $A$ of a group $G$ is permutable with a subgroup $B$ if $AB = BA$. If $A$ is permutable with all subgroups of $G$, then $A$ is called a permutable subgroup [3] (or quasinormal subgroup) [13] of $G$. The permutable subgroups have many interesting properties. For example, Ore [13] proved that every permutable subgroup of a group is subnormal. Itó and Szép [12] showed that $H/H_G$ is nilpotent for every permutable subgroup $H$ of a group $G$. Kegel and Deskins [2] showed that the subgroups $H$ of a group $G$ which are permutable with all Sylow subgroups of $G$ inherit a series of key properties of permutable subgroups. Recently, Guo, Shum and Skiba [8] introduce the concept of conditionally permutable subgroup. They say that a subgroup $H$ of a group $G$ is conditionally permutable in $G$ if for any subgroup $T$ of $G$, there exists some $x \in G$ such that $HT^x = T^xH$. Using the new idea, people have

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obtained a series of elegant results on the structure of groups (cf [6-10]). A subgroup $H$ of $G$ is said to be $s$-conditionally permutable in $G$ (cf. [10, 16]) if for every Sylow subgroup $T$ of $G$, there exists $x \in G$ such that $HT^x = T^xH$. By Sylow theorem, we know that a subgroup $H$ of $G$ is $s$-conditionally permutable if and only if for any $p \in \pi(G)$, there exists a Sylow $p$-subgroup $P$ such that $PH = HP$. As a continuation, we now introduce the following concept:

**Definition 1.1** Let $H$ be a subgroup of a group $G$. $H$ is said to be $s$-c-permutably embedded in $G$ if for every Sylow subgroup of $H$ is a Sylow subgroup of some $s$-conditionally permutable subgroup of $G$.

Clearly, every $s$-conditionally permutable subgroup is a $s$-c-permutably embedded subgroup of $G$. However, the following examples shows that an $s$-c-permutably embedded subgroup is not necessarily $s$-conditionally permutable in $G$.

**Example 1.** Let $N \triangleleft G$. The every Sylow subgroup $T$ of $N$ is $s$-c-permutably embedded in $G$, but clearly $T$ is not necessarily be an $s$-conditionally permutable subgroup of $G$ if $G$ is non-soluble.

**Example 2.** Let $G = S_5$ and $P$ be a Sylow 3-subgroup of $G$. Then $P$ is not an $s$-conditionally permutable subgroup. In fact, we know that $S_5$ has no a subgroup of order 15. Hence $P_3$ can not permute with any Sylow 5-subgroup of $G$. However, $G$ is itself an $s$-conditionally permutated subgroup of $G$. Hence $P_3$ is an $s$-c-permutably embedded subgroup in $G$.

All unexplained notations and terminology are standard. The reader is referred to Huppert [11] or Guo [4].

## 2 Preliminaries

We first give some basic results on $s$-conditionally subgroups and $s$-c-permutably embedded subgroups.

**Lemma 2.1** [16, Lemma 2.1] Let $G$ be a group, $K \triangleleft G$ and $H \leq G$. Then:

1. If $H$ is $s$-conditionally permutable in $G$, then $HK/K$ is $s$-conditionally permutable in $G/K$.
2. If $K \leq H$ and $H/K$ is $s$-conditionally permutable in $G/K$, then $H$ is $s$-conditionally permutable in $G$.
3. Suppose that $HK/K$ is $s$-conditionally permutable in $G/K$ and $(|H|, |K|) = 1$. If $G$ is soluble or $K$ is nilpotent, then $H$ is $s$-conditionally permutable in $G$.
4. If $H$ is $s$-conditionally permutable in $G$, then $H \cap K$ is $s$-conditionally permutable in $K$. 
Lemma 2.2 Suppose that $G$ is a group, $K \triangleleft G$ and $H \triangleleft G$. Then:

1. If $H$ is $s$-$c$-permutably embedded in $G$, then $HK/K$ is $s$-$c$-permutably embedded in $G/K$.
2. If $K \leq H$ and $H/K$ is $s$-$c$-permutably embedded in $G/K$, then $H$ is $s$-$c$-permutably embedded in $G$.
3. If $HK/K$ is $s$-$c$-permutably embedded in $G/K$ and $(|H|, |K|) = 1$, then $H$ is $s$-$c$-permutably embedded in $G$.
4. If $H$ is $s$-$c$-permutably embedded in $G$, then $H \cap K$ is $s$-$c$-permutably embedded in $K$.

Proof. (1) It is obvious.

(2) Let $p \in \pi(H)$. By the hypothesis, there exists an $s$-conditionally permutable subgroup $N/K$ of $G/K$ such that every Sylow $p$-subgroup $P/K$ of $H/K$ is a Sylow $p$-subgroup of $N/K$. Hence $p \nmid |N/K : P/K|$. If $p \nmid |H/K|$, then $P = K$. In this case, every Sylow $p$-subgroup of $H$ is also a Sylow $p$-subgroup of $K$. Since $K \triangleleft G$, $K$ is clearly $s$-conditionally permutable in $G$. Therefore $H$ is $s$-$c$-permutably embedded in $G$. Now, suppose that $p \mid |H/K|$. By Lemma 2.1(2), $N$ is $s$-conditionally permuted in $G$. Let $P_1$ be a Sylow $p$-subgroup of $H$. We need only to prove that $P_1$ is also a Sylow $p$-subgroup of $N$. Obviously, $P/K = P_1K/K$ and $|N : P_1| = |N : P| |P_1K : P_1|$ is a $p'$-number. This means that $P_1$ is a Sylow $p$-subgroup of $N$. Thus $H$ is $s$-$c$-permutably embedded in $G$.

(3) By (2), we can see that $HK$ is $s$-$c$-permutably embedded in $G$. Let $p$ be an arbitrary prime dividing the order of $H$. Then $p \mid |HK|$. By the hypothesis, there exists an $s$-conditionally permutable subgroup $N$ of $G$ such that every Sylow $p$-subgroup of $HK$ is also a Sylow $p$-subgroup of $N$. Since $(|H|, |K|) = 1$, we have that every Sylow $p$-subgroup of $H$ is also a Sylow $p$-subgroup of $N$. Hence, $H$ is $s$-$c$-permutably embedded in $G$.

(4) Let $p \in \pi(H)$. By the hypothesis, there exists a $s$-conditionally permutable subgroup $N$ of $G$ such that every Sylow $p$-subgroup $P$ of $H$ is also a Sylow $p$-subgroup of $N$. Since $K \triangleleft G$, obviously $N \cap K$ is also $s$-conditionally permutable in $K$. We now prove that every Sylow $p$-subgroup of $H \cap K$ is also a Sylow $p$-subgroup of $N \cap K$. In fact, since $K \triangleleft G$, $P \cap K$ is a Sylow $p$-subgroup of $H \cap K$. By Sylow theorem, we may assume without loss of generality that $P \cap K$ is an arbitrary Sylow $p$-subgroup of $H \cap K$. Since $|(N \cap K) : (P \cap K)| = |(N \cap K) : (P \cap N \cap K)| \mid |P(N \cap K) : P|$, $p \nmid |(N \cap K) : (P \cap K)|$. This shows that $P \cap K$ is a Sylow $p$-subgroup of $N \cap K$. Hence $H$ is $s$-$c$-permutably embedded in $K$.

Lemma 2.3 Let $G$ be a group and $P$ a subgroup of $G$ contained in $O_p(G)$. If $P$ is $s$-$c$-permutably embedded in $G$, then $P$ is $s$-conditionally permutable in $G$.
Proof. Obviously, $P$ is a subnormal subgroup of $G$. Since $P$ is $s$-$c$-permutably embedded in $G$, there exists an $s$-conditionally permutably embeddable subgroup $A$ of $G$ such that $P$ is a Sylow $p$-subgroup of $A$. Hence, for any $q \in \pi(G)$, there exists a Sylow $q$-subgroup $G_q$ of $G$ such that $AG_q = G_qA$. If $p = q$, then $P \leq G_p$ and so $PG_p = G_pP$. If $p \neq q$, then $P$ is a subnormal Hall subgroup of $AG_q = G_qA$ and consequently $P$ is normal in $AG_q$. Hence $PG_q = G_qP$. This shows that $P$ is $s$-conditionally permuted in $G$.

Lemma 2.4 Let $P$ be a minimal normal $p$-subgroup of $G$. If every subgroup of $P$ with order $p$ is $s$-$c$-permutably embedded in $G$, then $P$ is a group of order $p$.

Proof. Suppose that $E$ is a Sylow $p$-subgroup of $G$. Then $P \cap Z(E) \neq 1$. Let $L$ be a subgroup of $P \cap Z(E)$ of order $p$. By the hypothesis, $L$ is $s$-$c$-permutably embedded in $G$. Hence there exists a $s$-conditionally permutable subgroup $N$ of $G$ such that $L$ is a Sylow $p$-subgroup of $N$. This means that for every $q \in \pi(G)$ and $p \neq q$, there exists a Sylow $q$-subgroup $Q$ of $G$ such that $NQ = QN$. Since $L = P \cap NQ \triangleleft NQ$, $NQ \subseteq N_G(L)$. On the other hand, $E \leq N_G(L)$. Thus $L \triangleleft G$. But since $P$ is a minimal normal $p$-subgroup of $G$, we have $P = L$. Thus $P$ is a group of order $p$.

For the sake of convenience, we now list some known results for the proofs in this paper.

Lemma 2.5 [8, Lemma 3.1]. Let $N$ and $L$ be normal subgroups of a group $G$. Let $P/L$ be a Sylow $p$-subgroup of $NL/L$ and $M/L$ a maximal subgroup of $P/L$. If $P_p$ is a Sylow $p$-subgroup of $P \cap N$, then $P_p$ is a Sylow $p$-subgroup in $N$ such that $D = M \cap N \cap P_p$ is a maximal subgroup in $P_p$ and $M = LD$.

3 Main Results

Theorem 3.1 Let $p$ be a prime and $G$ a $p$-soluble group. If every cyclic $p$-subgroup of $G$ is $s$-$c$-permutably embedded in $G$, then $G$ is $p$-supersoluble.

Proof. Suppose that the theorem is false and let $G$ be a counterexample of minimal order. Then:

(1) If $N$ is a proper normal subgroup of $G$, then $G/N$ is $p$-supersoluble.

In fact, for the cyclic $p$-subgroup $K/N$ of $G/N$, we have $K/N = \langle x \rangle N/N$, where $x \in K$. By Sylow theorem, there exists a Sylow $p$-subgroup $G_p$ such that $KN/N \leq G_pN/N$ and so $K \leq G_pN$. Therefore, we may assume that $x = gn$, where $g \in G_p$, $n \in N$. Then $\langle x \rangle N = \langle g \rangle N$. By the hypothesis, $\langle g \rangle$ is $s$-$c$-permutably embedded in $G$. It follows from Lemma 2.2 that $K/N = \langle x \rangle N/N = \langle g \rangle N/N$ is $s$-$c$-permutably embedded in $G/N$. Hence $G/N$
S-

C-permutably embedded subgroups of finite groups

satisfies the condition of the theorem. The choice of \( G \) implies that \( G/N \) is \( p \)-supersoluble.

(2) \( \Phi(G) = 1 \) and \( G \) has a unique minimal normal subgroup \( N \) such that \( N = C_G(N) = O_p(G) \) and \( G = [N]M \), where \( M \) is a maximal subgroup of \( G \) with \( O_p(M) = 1 \).

Since the class of all \( p \)-supersoluble groups is a saturated formation, by (1), obviously, \( \Phi(G) = 1 \) and \( G \) has a minimal normal subgroup \( N \). Hence there exists a maximal subgroup \( M \) of \( G \) such that \( G = NM \). Since \( G \) is \( p \)-soluble, \( N \) is a clearly \( p \)-subgroup of \( G \) (Otherwise, \( N \) is a \( p' \) group and consequently \( G \) is \( p \)-supersoluble.) Thus \( N \) is an elementary abelian \( p \)-subgroup of \( G \). It follows that \( G = [N]M \). Let \( C = C_G(N) \). Obviously, \( C \cap M = 1 \). By the Dedekind identity, \( C = C \cap NM = N(C \cap M) = N \). This shows that \( N = O_p(G) = C_G(N) \) and \( M \cong G/N \) is a supersoluble group with \( O_p(M) = 1 \) (cf. [4, Lemma 1.7.11]).

(3) Final contradiction follows.

By Lemma 2.4, \( |N| = p \). Then by (1), \( G \) is \( p \)-supersoluble. This is a contradiction. Thus, the proof of the theorem is completed.

Corollary 3.1.1 Let \( G \) be a \( p \)-soluble group and \( p \) a prime dividing the order of \( G \). \((|G|, p - 1) = 1 \) and \( P \) is a Sylow \( p \)-subgroup of \( G \). If every maximal subgroup of \( P \) is \( s \)-\( c \)-permutably embedded in \( G \), then \( G \) is \( p \)-nilpotent.

Theorem 3.2 Let \( G \) be a soluble group. If every maximal subgroup of every non-cyclic Sylow subgroup of \( G \) having no supersoluble supplement in \( G \) is \( s \)-\( c \)-permutably embedded in \( G \), then \( G \) is supersoluble.

Proof. Suppose that the result is false and let \( G \) be a counterexample of minimal order. Then:

(1) \( G \) is not a simple group.

If \( G \) is a simple group, then \( G \) is a cyclic group of prime order and so \( G \) is supersoluble, a contradiction.

(2) For every minimal normal subgroup \( N \) of \( G \), \( G/N \) is supersoluble.

Let \( Q/N \) be a non-cyclic Sylow \( p \)-subgroup of \( G/N \) and \( K/N \) a maximal subgroup of \( Q/N \). Then there exists a Sylow \( p \)-subgroup \( P \) of \( G \) such that \( Q = PN \) and \( K = N(P \cap K) \). Clearly, \( P \cap K \) is a maximal subgroup of \( P \) and \( P \) is non-cyclic. If \( P \cap K \) possesses a supersoluble supplement \( T \) in \( G \), then \( TN/N \cong T/T \cap N \) is a supersoluble supplement to \( K/N \) in \( G/N \). If \( P \cap K \) is \( s \)-\( c \)-permutably embedded in \( G \), then by Lemma 2.2, \( K/N = N(P \cap K)/N \) is \( s \)-\( c \)-permutably embedded in \( G/N \). These shows that \( G/N \) satisfies the hypothesis of the theorem. Thus, by the choice of \( G \), we have that \( G/N \) is supersoluble.

(3) \( \Phi(G) = 1 \) and \( G \) has a unique minimal normal subgroup \( H \) such that \( H = C_G(H) = O_p(G) = F(G) \) for some prime \( p \), and \( G = [H]M \), where \( M \) is
a maximal subgroup of $G$ with $O_p(M) = 1$. (See the proof of (2) in Theorem 3.1.)

(4) Any Sylow subgroup of $G$ is not normal subgroup in $G$.

Suppose that for some $q \in \pi(G)$, $G$ has a normal Sylow $q$-subgroup $G_q$. Then by (3) $q = p$ and $H = G_q$. Since $G = [H]M$, $|G : M| = |H|$. Assume that $H_1$ is a maximal subgroup of $H$. By the hypothesis, either $H_1$ possesses a supersoluble supplement $T$ in $G$ or $H_1$ is $s$-$c$-permutably embedded in $G$. In the first case, the choice of $T$ implies that $T \not\subseteq G$ and so $G = [H_1]T$, which contradicts the minimality of $H$. In the second case, there exists an $s$-conditionally permutable subgroup $A$ of $G$ such that $H$ is a Sylow $p$-subgroup of $A$. Let $q$ be an arbitrary prime divisor of $|G|$ with $q \neq p$. Since $A$ is $s$-conditionally permutable, there exists a Sylow $q$-subgroup $Q$ of $G$ such that $AQ = QA$. Then $H_1 = H \cap AQ \leq G$. It follows that $Q \leq N_G(H_1)$ for any $q \neq p$ and $q \in \pi(G)$. On the other hand, since $H_1 \triangleleft H = G_q$, $G_p \leq N_G(H_1)$. Hence $H_1 \triangleleft G$. But since $H$ is a minimal normal subgroup, we have $|H| = p$. This induces that $G$ is supersoluble, a contradiction.

(5) The number $p$ is not the largest prime divisor of $|G|$.

Indeed, if $p$ is the largest divisor of $|G|$, then (2) and (5) implies that $O_p(G/N) \neq 1$ which contradicts to (4).

(6) Final contradiction.

By (3), we have that $G = [H]M$. Pick some Sylow $p$-subgroup $M_p$ of $M$ and let $P$ be a Sylow $p$-subgroup of $G$ including $M_p$. Let $P_1$ be maximal subgroup of $P$ such that $M_p \leq P_1$. Then $H_1 = H \cap P_1$ is a maximal subgroup of $H$. By (2), it is clear that $|H| > p$. Hence $P$ is not cyclic. By the hypothesis, $P_1$ is $s$-$c$-permutably embedded in $G$ or $P_1$ possesses a supersoluble supplement $T$ in $G$. In the first case, there exists an $s$-conditionally permutable subgroup $A$ of $G$ such that $P_1$ is a Sylow $p$-subgroup of $A$. This means that for an arbitrary prime divisor $q$ of $|G|$ with $p \neq q$, there exists a Sylow $q$-subgroup $Q$ of $G$ such that $AQ = QA$. Since $H_1 \triangleleft H$ and $H_1 = H \cap P_1 \leq H \cap A \leq H \cap AQ \leq H$, $H_1 = A \cap HQ$ or $H = H \cap AQ$. If $H = H \cap AQ$, then $H \leq AQ$ and thereby $P = P_1H \leq P_1AQ = AQ$. This implies that $P = P_1$, which is impossible. Hence we may assume that $H_1 = H \cap AQ$. Because $H \triangleleft G$, $H_1 \triangleleft AQ$. It follows that $Q \leq N_G(H_1)$. On the other hand, since $H_1 \triangleleft H$ and $H_1 \triangleleft P_1$, $H_1 \triangleleft P_1H = P$. This means that $H_1 \triangleleft G$ and so $|H| = p$, a contradiction. Now assume the second case applies. Let $q$ be the largest prime divisor of $|T|$ and $T_q$ a Sylow $q$-subgroup of $T$. Since $T$ is supersoluble, $T_q \triangleleft T$ where $T_q$ is a Sylow $q$-subgroup of $T$. Obviously, $T_q$ is also a Sylow $q$-subgroup of $G$. Since by (3), $M$ is supersoluble, $T_q \triangleleft M_x$ for some $x \in G$. It follows that $M_x \subseteq N_G(T_q)$. But since $M \triangleleft G$, by (3) and (4), $M_x = N_G(T_q)$ and consequently $T \subseteq M_x$. Thus $G = P_1T = P_1M$, which implies that $P = P_1$. This contradiction completes the proof.

**Corollary 3.2.1** Let $G$ be a soluble group. If every maximal subgroup of each
Sylow $p$-subgroup of $G$ is $s$-$c$-permutably embedded in $G$. Then $G$ is supersoluble.

**Corollary 3.2.2** [10] Let $G$ be a soluble group. If every maximal subgroup of each Sylow subgroup of $G$ is $s$-conditionally permuted in $G$, then $G$ is supersoluble.

**Theorem 3.3** Let $\mathfrak{F}$ be a saturated formation containing the class $\mathfrak{U}$ of all supersoluble group. A group $G \in \mathfrak{F}$ if and only if $G$ has a soluble normal subgroup $H$ such that $G/H \in \mathfrak{F}$ and every maximal subgroup of every Sylow subgroup of $H$ is $s$-$c$-permutably embedded in $G$.

**Proof.** The necessary part is obvious and so we only need to prove the sufficient part. Suppose that the sufficient part is false and let $G$ be a counterexample of minimal order, we process with our proof as follows:

1. If $R$ is a minimal normal subgroup of $G$, then $G/R \in \mathfrak{F}$.

   In fact, if $R = H$, then, of course, $G/R \in \mathfrak{F}$. Now we assume that $R \neq H$. Then $RH/R$ is a normal subgroup of $G/R$ such that the factor group $(G/R)/(RH/R) \cong G/RH \cong (G/H)/(RH/R) \in \mathfrak{F}$. Let $P/R$ be a Sylow $p$-subgroup of $RH/R$ and let $M/R$ be a maximal subgroup in $P/R$. If $P_p$ is a Sylow $p$-subgroup in $P \cap H$ then by Lemma 2.5, $P_p$ is a Sylow $p$-subgroup in $H$ and $D = M \cap H \cap P_p$ is a maximal subgroup in $P_p$ and $M = RD$. Thus, by the hypothesis, $D$ is $s$-$c$-permutably embedded in $G$. It follows from Lemma 2.2 that $M/R = LR/R$ is $s$-$c$-permutably embedded in $G/R$. This shows that the conditions of the theorem are inherited on $G/R$. Hence, the choice of $G$, we have that $G/R \in \mathfrak{F}$.

2. $G$ has the unique minimal normal subgroup $R$ and $R = C_G(R) = O_p(G) = F(G) \nsubseteq p$ for some $p \in \pi(G)$ and $\Phi(G) = 1$.

   Since $\mathfrak{F}$ is a saturate formation, by (1) we see that (2) holds clearly.


   Assume $|R| = p^\alpha$ for some natural number $\alpha > 1$. Let $P$ be a Sylow $p$-subgroup of $G$. Since $R \nsubseteq \Phi(G)$, $R \nsubseteq \Phi(P)$. Hence, there exists a maximal subgroup $P_1$ of $P$ such that $R \nsubseteq P_1$. Since $R \subseteq H$, $P_1 \cap H$ is a maximal subgroup of some Sylow $p$-subgroups of $H$. By the hypothesis, there exists an $s$-conditionally permutable subgroup $A$ of $G$ such that $P_1 \cap H$ is a Sylow $p$-subgroup of $A$. Then, for arbitrary $q \in \pi(G)$ with $p \neq q$, there exists a Sylow $q$-subgroup $Q$ of $G$ such that $AQ = QA$. Hence $R \cap P_1 = R \cap (P_1 \cap H) \subseteq R \cap AQ \subseteq AQ$ and thereby $Q \subseteq N_G(R \cap P_1)$ for any $q \neq p$. On the other hand, $R \cap P_1 \subseteq P$. This shows that $R \cap P_1 \subseteq G$. Consequently $R \cap P_1 = 1$, that is, $|R| = p$.

4. Final contradiction.

   Since $\mathfrak{F}$ is a saturated formation containing $\mathfrak{U}$, $\mathfrak{F}$ has a formation function $f$ such that $\mathfrak{A}(p - 1) \subseteq f(p)$ for all prime $p$, where $\mathfrak{A}(p - 1)$ is the formation
Proof. The necessary part is clear, we only need to prove the sufficiency part.

Suppose that the assertion is false and let \( G \) be a counterexample of minimal order. Let \( P \) be a Sylow \( p \)-subgroup of \( F(H) \) for an arbitrary prime divisor \( p \) of \( |G| \). Then \( P \) char \( F(H) \triangleleft G \) and so \( P \triangleleft G \). We proceed with our proof as follows:

1. \( P \cap \Phi(G) = 1 \).
   Assume that \( R = P \cap \Phi(G) \neq 1 \). Then \((G/R)/(H/R) \in \mathfrak{F} \). Let \( F(H/R) = T/R \). Obviously \( F(H) \subseteq T \). On the other hand, since \( R \subseteq \Phi(G) \), \( T \) is nilpotent. Thus \( T \subseteq F(H) \) and so \( F(H)/R = F(H/R) \). Let \( P_1/R \) be a maximal subgroup of \( P/R \). Then \( P_1 \) is a maximal subgroup of \( P \). By the hypothesis, \( P_1 \) is \( s \)-\( c \)-permutably embedded in \( G \). It follows from Lemma 2.2 that \( P_1/R \) is \( s \)-\( c \)-permutably embedded in \( G/R \). Now let \( Q/R \) be a maximal subgroup of a Sylow \( q \)-subgroup of \( F(H)/R \), where \( q \neq p \). Then \( Q = Q_1R \), where \( Q_1 \) is a Sylow \( q \)-subgroup of \( F(G) \). By the hypothesis, \( Q_1 \) is \( s \)-\( c \)-permutably embedded in \( G \) and so \( Q/R = Q_1R/R \) is \( s \)-\( c \)-permutably embedded in \( G/R \) by Lemma 2.2. This shows that \( G/R \) satisfies the hypothesis. Thus, by the choice of \( G \), we have \( G/R \in \mathfrak{F} \). Since \( G/\Phi(G) \cong (G/R)/(\Phi(G)/R) \) and \( \mathfrak{F} \) is a saturated formation, we have that \( G \in \mathfrak{F} \), a contradiction.

2. \( P = R_1 \times R_2 \times \cdots \times R_m \), where \( R_i \) (\( i = 1, 2, \ldots, m \)) is a minimal normal subgroup of \( G \) with order \( p \).

Since \( P \triangleleft G \) and \( P \cap \Phi(G) = 1 \), it is easy to see that \( P = R_1 \times R_2 \times \cdots \times R_m \), where \( R_i \) (\( i = 1, 2, \ldots, m \)) is a minimal normal subgroup of \( G \) (cf. [4, Theorem 1.8.17]). We now prove that \(|R_i| = p \). Since \( R \not\subseteq \Phi(G) \), there exists a maximal subgroup \( M \) of \( G \) such that \( G = MR_i \) and clearly \( M \cap R_i = 1 \). Let \( M_p \) be a Sylow \( p \)-subgroup of \( M \). Then \( G_p = M_pR_i = M_pP \) is a Sylow \( p \)-subgroup of \( G \). Let \( H_1 \) be a maximal subgroup of \( G_p \) containing \( M_p \). Then \( P_1 = H_1 \cap P \) is a maximal subgroup of \( P \). By the hypothesis, \( P_1 \) is \( s \)-\( c \)-permutably embedded in \( G \). Hence, \( P_1 \) is a Sylow \( p \)-subgroup of \( N \) of \( G \) such that \( P_1 \triangleleft G \). Hence, for any \( q \in \pi(G) \) with \( p \neq q \), there exists a Sylow \( q \)-subgroup \( Q \) of \( G \) such that \( NQ = QN \). Similar to the proof of Theorem 3.3, we obtain that \( P_1 \triangleleft G \) and so \( P_1 \cap R_i \triangleleft G \). But since \( R_i \not\subseteq P_1 \) and \( R_i \) is a minimal normal subgroup of \( G \), \( P_1 \cap R_i = 1 \). Hence, \(|R_i| = |R_i : P_1 \cap R_i| = |R_i : H_1 \cap P \cap R_i| = |R_i : H_1 \cap R_i| = |R_iH_1 : H_1| = |R_i : H_1| = 1\).
\[ |G_p : H_1| = |P H_1 : H_1| = |P : P_1 \cap H_1| = |P : P_1| = p. \] This shows that \( R_i \) is a cyclic group of order \( p \).

(3) Final contradiction follows.

By (2), \( F(H) = N_1 \times N_2 \times \cdots \times N_m \), where \( N_i \) \((i = 1, 2, \cdots, m)\) is a minimal normal subgroup of \( G \) of prime order. Since \( G/C_G(N_i) \) is isomorphic to a subgroup of \( Aut(N_i) \), \( G/C_G(N_i) \) is cyclic. It follows that \( G/\bigcap_{i=1}^{\infty} C_G(N_i) = G/C_G(F(H)) \in \mathfrak{U} \subseteq \mathfrak{F} \), and consequently \( G/C_H(F(H)) = G/(H \cap C_G(F(H))) \in \mathfrak{F} \). Since \( F(H) \) is abelian, \( F(H) = C_H(F(H)) \). Therefore \( G/F(H) \in \mathfrak{F} \). Then, by Theorem 3.3, we obtain that \( G \in \mathfrak{F} \). This contradiction complete the proof.

**Corollary 3.4.1** Let \( G \) be a group with a soluble normal subgroup \( E \) such that \( G/E \) is supersoluble. If every maximal subgroup of every Sylow subgroup of \( F(E) \) is \( s \)-conditionally permutable in \( G \), then \( G \) is supersoluble.

**Remark 3.4.1** Theorem 3.4 and corollary 3.4.1 can not necessarily hold for non-soluble groups. For example, let \( G = SL(2,5) \). Then \( F(G) \) is a cyclic group of order 2. Thus each maximal subgroup of Sylow Subgroup of \( F(G) \) is \( s \)-c-permutably embedded in \( G \). However \( G \) is not a supersoluble group.

**Remark 3.4.2** Theorem 3.4 and corollary 3.4.1 are not necessarily hold if the saturated formation \( \mathfrak{F} \) does not contain the class \( \mathfrak{U} \) of all supersoluble subgroups. For example, let \( \mathfrak{F} \) be a nilpotent formation. Then the symmetric group of degree 3 is a counterexample.

**References**


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