Derivations and Isomorphisms of a Non-Associative Algebra with Finitely Many Right Annihilators

Seul Hee Choi

Dept. of Mathematics, Jeonju University
Chon-ju 560-759, Korea
chois@jj.ac.kr

Abstract

We prove that the non-associative algebra $W(n, 0, 0)$ and its symmetrized algebra are simple (see [1], [2], [3]). We find all the derivations of the algebra $W(n, 0, 0)$. Thus we can prove that for $n_1 \neq n_2$, the algebras $W(n_1, 0, 0)$ and $W(n_2, 0, 0)$ are not isomorphic. Because of all the derivations of a non-associative algebra, we can find the associator of the algebra. Depending on the dimension of the right annihilator of an algebra, we define the class $\mathcal{R}_{\text{ann}}$ of the algebras in this work.

Mathematics Subject Classification : Primary 17B40, 17B56

Keywords: simple, non-associative algebra, right identity, (right) annihilator, derivation, associator

1 Preliminaries

Let $\mathbb{N}$ be the set of all non-negative integers and $\mathbb{Z}$ be the set of all integers. Let $\mathbb{F}$ be a field of characteristic zero. Let $\mathbb{F}^*$ be the multiplicative group of non-zero elements of $\mathbb{F}$. Let us define the $\mathbb{F}$-algebra $\mathbb{F}[e^{\pm x_1}, \cdots , e^{\pm x_n}, x_1^{\pm 1}, \cdots , x_m^{\pm 1}, x_{m+1}, \cdots , x_{m+s}]$ spanned by

$$\{ e^{a_1 x_1} \cdots e^{a_n x_n} x_1^{i_1} \cdots x_m^{i_m} x_{m+1}^{i_{m+1}} \cdots x_{m+s}^{i_{m+s}} \mid a_1, \cdots , a_n, i_1, \cdots , i_m \in \mathbb{Z}, i_{m+1}, \cdots , i_{m+s} \in \mathbb{N} \}.$$  \hspace{1cm} (1)

with the obvious addition and its multiplication (see [12]). For a positive integer $r$, the non-associative algebra $W(n, m, m+s)[r]$ is the vector space spanned by

$$\{ f \partial_u^r | f \in \mathbb{F}[e^{\pm x_1}, \cdots , e^{\pm x_n}, x_1^{\pm 1}, \cdots , x_m^{\pm 1}, x_{m+1}, \cdots , x_{m+s}] , 1 \leq u \leq n \}.$$  \hspace{1cm} (2)
with the obvious addition and the multiplication $\ast$ where $\partial_u$ is the usual partial derivative on the $\mathbb{F}$-algebra $\mathbb{F}[^{\pm x_1}, \ldots, ^{\pm x_n}, _{x_1}, \ldots, _{x_m}, _{x_{m+1}}, \ldots, _{x_{m+s}}]$ with respect to $x_u$, $1 \leq u \leq \text{max}\{n, m + s\}$ in the papers (see [1], [3], [9], [10], [14]). The non-associative algebra $\mathbb{W}(n, m, m + s)_{[r]}$ is a subalgebra of the algebra in the papers (see [6-11]). Note that $\partial_u^0 = 1$, $1 \leq u \leq n$. Thus if $r = 0$, then the algebra $\mathbb{W}(n, m, m + s)_{[r]}$ is the $\mathbb{F}$-algebra $\mathbb{F}[^{\pm x_1}, \ldots, ^{\pm x_n}, _{x_1}, \ldots, _{x_m}, _{x_{m+1}}, \ldots, _{x_{m+s}}]$. For an element $l$ of an algebra $\mathbb{A}$ and an ideal $I$ of $\mathbb{A}$, $l$ is full, if the ideal contains the element $l$, then $l$ is $\mathbb{A}$. The matrix ring $M_{m+s}(\mathbb{F})$ is imbedded in the algebra $\mathbb{W}(n, m, m + s)_{[r]}$. The matrix ring $M_n(\mathbb{F})$ is not imbedded in $\mathbb{W}(n, 0, 0)_{[r]}$. The non-associative algebra $\mathbb{W}(n, 0, 0)_{[r]}$ has neither a right nor a left multiplicative identity element. Note that the definition of a non-associative algebra in this paper is different from the definitions of the algebras in the papers (see [3], [9], [10]), because of some results in the paper. Similarly to the non-associative algebra $\mathbb{W}(n, m, m + s)_{[r]}$, we can define the non-associative algebra $\mathbb{W}(n, 0, s)_{[r]}$ spanned by $\langle e^{a_1 x_1}, \ldots, e^{a_n x_n}, _{x_1}, \ldots, _{x_{n+1}}, \ldots, _{x_{n+s}}, \partial_u | a_1, \ldots, a_n \rangle$, $\mathbb{W}(n, m, m + s)_{[r]}$ has idempotents and $\mathbb{W}(n, m, m + s)_{[r]}$ is simple. Using the commutator $[,]$ of $\mathbb{W}(n, m, m + s)_{[r]}$, we can define the antisymmetrized algebra $\mathbb{W}(n, m, m + s)_{[r],[r]}$. The Lie algebra $\mathbb{W}(n, m, m + s)_{[1],[1]}$ is the simple Lie algebra in the papers (see [9] and [13]). The algebra $\mathbb{W}(n, 0, 0)_{[r]}$ is a subalgebra of the algebra $\mathbb{W}(n, m, m + s)_{[r]}$ spanned by $\langle \prod_{r = 1}^{n} e^{a_r x_r}, \partial_u | a_1, \ldots, a_n \rangle$. The algebra $\mathbb{W}(n, 0, 0)_{[r]}$ is $\mathbb{Z}^n$-graded as follows:

$$\mathbb{W}(n, 0, 0)_{[r]} = \bigoplus_{(a_1, \ldots, a_n)} N(a_1, \ldots, a_n)$$

where the $(a_1, \ldots, a_n)$-homogeneous component $N(a_1, \ldots, a_n)$ is the vector subspace of $\mathbb{W}(n, 0, 0)_{[r]}$ spanned by $\langle \prod_{r = 1}^{n} e^{a_r x_r}, \partial_u | 1 \leq u \leq n \rangle$. Let $\mathcal{R}_{\text{ann}}(n)$ be the set of all algebra $\mathbb{A}$ such that the dimension of all the right annihilators of $\mathbb{A}$ is $n$. The class $\mathcal{R}_{\text{ann}}$ is defined as follows:

$$\mathcal{R}_{\text{ann}} = \{\mathcal{R}_{\text{ann}}(n) | n \in \mathbb{N}\}.$$ 

2 Simplicities

**Theorem 2.1** The algebra $\mathbb{W}(n, 0, 0)_{[r]}$ is simple.

**Proof.** It is easy to prove that for any ideal $I$ of $\mathbb{W}(n, 0, 0)_{[r]}$, if $I$ contains an element of the homogeneous component $N(0, \ldots, 0)$, then $I$ is the algebra...
Thus it is enough to show that for every non-zero ideal $I$ of $W(n,0,0)_{[r]}$, $I$ contains an element in the homogeneous component $N_{(0,...,0)}$. Let $I_1$ be a non-zero ideal of the homogeneous component $N_{(0,...,0)}$. Let $l$ be a non-zero element in $I_1$. Since every term of $l$ in a homogeneous component, let us prove the theorem by induction on the number of homogeneous components which contain a term of $l$. If $l$ is in the homogeneous component $N_{(0,...,0)}$, then there is nothing to prove. Let us assume that $l$ is in the homogeneous component $N_{(a_1,...,a_n)}$. We can also assume that $a_1$ is a non-zero integer. By taking an appropriate element $l_2$ in the homogeneous component $N_{(-a_1,...,-a_n)}$, we have that $l_2 \ast l$ is a non-zero element in $N_{(0,...,0)}$. Thus $I_1$ is the algebra $W(n,0,0)_{[r]}$. By induction, we can assume that if $l$ has terms in $k$ homogeneous components, then $I_1$ is the algebra $W(n,0,0)_{[r]}$. Let us assume that $l$ has terms in $k+1$ homogeneous components. We can find an appropriate elements $l_1$ and $\partial^r$, so that $\partial^r \ast (l_1 \ast l)$ is non-zero and it has at most $k$ homogeneous components. By induction, $I_1$ is the algebra $W(n,0,0)_{[r]}$. Therefore we have proven the theorem. \qed

**Theorem 2.2** The antisymmetrized algebra $W(n,0,0)_{[r]}$ of the algebra $W(n,0,0)_{[r]}$ is simple. The algebra $W(n,0,0)_{[r]}$ is self-centralizing (see [8]).

**Proof.** The proof of the theorem is almost the same as the proof of Theorem 2.1 and the remaining result of the proof is easy, so it is omitted. \qed

**Remarks.** The algebra $W(n,0,0)$ is in $\mathfrak{R}_{ann}(n)$. For any $A \in \mathfrak{R}_{ann}(n)$ and $B \in \mathfrak{R}_{ann}(m)$, if $n \neq m$, then the algebras $A$ and $B$ are not isomorphic. Thus their antisymmetrized algebras $A_{[r]}$ and $B_{[r]}$ are not isomorphic. \qed

### 3 Derivations of a non-associative algebra

For a non-associative algebra, we need the following obvious results.

**Proposition 3.1** Let $A$ be a non-associative algebra and $B$ be the set of all associators of the algebra $A$. For any $l \in B$, $l$ induces the inner derivation $ad_l$ of $A$ (see [14]).

**Note 1.** For any basis element $\prod_{u=1}^{3} e^{k_u x_u} \partial_v$ of the algebra $W(3,0,0)_{[1]}$ and $v \in \{1,2,3\}$, if we define $\mathbb{F}$-linear maps $D_w$, $1 \leq w \leq 3$, of the non-associative algebra $W(3,0,0)_{[1]}$ as follows:

$$D_w(\prod_{u=1}^{3} e^{k_u x_u} \partial_v) = d_w k_w \prod_{u=1}^{3} e^{k_u x_u} \partial_v,$$
then $D_w, 1 \leq w \leq 3$, can be linearly extended to derivations of the algebra $\overline{W(3,0,0)}_1$ with appropriate coefficients. □ □

**Lemma 3.1** For any derivation $D$ of the algebra $\overline{W(3,0,0)}_1$, we have that $D(\partial_u) = 0, 1 \leq u \leq 3$.

**Proof.** Let $D$ be the derivation of the algebra $\overline{W(3,0,0)}_1$ in the lemma. Since $\partial_1$ annihilates itself, we have that $D(\partial_1) \ast \partial_1 + \partial_1 \ast D(\partial_1)$ is zero. This implies that

$$D(\partial_1) = s_1 e^{a_1 x_1} e^{a_2 x_2} e^{a_3 x_3} \partial_1 + s_2 e^{a_4 x_1} e^{a_5 x_2} e^{a_6 x_3} \partial_2 + s_3 e^{a_7 x_1} e^{a_8 x_2} e^{a_9 x_3} \partial_3$$

for $s_1, s_2, s_3 \in \mathbb{F}$. Since $\partial_2$ is in the left annihilator of $\partial_1$, $s_1 p_2 e^{a_2 x_2} e^{a_3 x_3} \partial_1 + s_2 q_2 e^{a_4 x_1} e^{a_5 x_2} e^{a_6 x_3} \partial_2 + s_3 r_2 e^{a_7 x_1} e^{a_8 x_2} e^{a_9 x_3} \partial_3 = 0$

This implies that $p_2 = q_2 = r_2 = 0$. Since $\partial_3$ is in the left annihilator of $\partial_1$, similarly we are able to prove that $p_3 = q_3 = r_3 = 0$. So we have that $D(\partial_1) = s_1 \partial_1 + s_2 \partial_2 + s_3 \partial_3$. Since $\partial_1$ is a left identity of $e^{x_1} \partial_1$, we have that

$$\partial_1 \ast D(e^{x_1} \partial_1) = -s_1 e^{x_1} \partial_1 + D(e^{x_1} \partial_1).$$

This implies that $D(e^{x_1} \partial_1)$ can be written as follows:

$$D(e^{x_1} \partial_1) = c_1 e^{a_1 x_1} e^{a_2 x_2} e^{a_3 x_3} \partial_1 + c_2 e^{a_4 x_1} e^{a_5 x_2} e^{a_6 x_3} \partial_2 + c_3 e^{a_7 x_1} e^{a_8 x_2} e^{a_9 x_3} \partial_3$$

where $c_1, \ldots, c_9 \in \mathbb{F}$. By (4) we are able to prove that $a_1 = a_4 = a_7 = 1$ and $s_1$ is zero. Since $\partial_1$ is in the left annihilators of $e^{x_2} \partial_1$, and $e^{x_3} \partial_1$, we are also able to prove that $s_2$ and $s_3$ are zeroes, i.e., $D(\partial_1)$ is zero. Similarly, we can also prove that $D(\partial_2)$ and $D(\partial_3)$ are zeroes. Therefore we have proven the lemma. □

**Lemma 3.2** For any derivation $D$ of the algebra $\overline{W(3,0,0)}_1$, $D(e^{k_i x_i} \partial_j)$ holds, where $c_i \in \mathbb{F}, 1 \leq i, j \leq 3$.

**Proof.** Let $D$ be the derivation of the non-associative algebra $\overline{W(3,0,0)}_1$ in the lemma. Since $\partial_1$ is a left identity of $e^{x_1} \partial_1$ and by Lemma 3.1, we have that $\partial_1 \ast D(e^{x_1} \partial_1) = D(e^{x_1} \partial_1)$. Let us put $D(e^{x_1} \partial_1) = c_1 \prod_{u=1}^3 e^{a_{u+x_1}} \partial_1 + c_2 \prod_{u=1}^3 e^{a_{u+x_2}} \partial_2 + c_3 \prod_{u=1}^3 e^{a_{u+x_3}} \partial_3$ where $c_1, c_2, c_3 \in \mathbb{F}$. Since

$$c_1 a_1 \prod_{u=1}^3 e^{a_{u+x_1}} \partial_1 + c_2 a_4 \prod_{u=1}^3 e^{a_{u+x_2}} \partial_2 + c_3 a_7 \prod_{u=1}^3 e^{a_{u+x_3}} \partial_3 =$$

$$c_1 \prod_{u=1}^3 e^{a_{u+x_1}} \partial_1 + c_2 \prod_{u=1}^3 e^{a_{u+x_2}} \partial_2 + c_3 \prod_{u=1}^3 e^{a_{u+x_3}} \partial_3$$

(6)
we have that $a_1, a_4,$ and $a_7$ are ones. Since $\partial_2$ and $\partial_3$ are in the left annihilator of $e_{x_1}\partial_1$, we can easily prove that $a_2 = a_3 = a_5 = a_6 = a_8 = a_9 = 0$ and $D(e_{x_1}\partial_1) = \sum_{u=1}^{3} c_u e_{x_u} \partial_u$. Similarly, we can also prove the followings:

\[
D(e_{x_2}\partial_1) = \sum_{u=1}^{3} c_{3+u} e_{x_u} \partial_u \quad \text{and} \quad D(e_{x_3}\partial_1) = \sum_{u=1}^{3} c_{6+u} e_{x_u} \partial_u
\]

with appropriate coefficients. Since $e_{x_1}\partial_1$ is in the left annihilator of $e_{x_2}\partial_1$, we also have that $c_2$ is zero. Since $e_{x_1}\partial_1$ is in the left annihilator of $e_{x_3}\partial_1$, we are able to prove that $c_3$ is zero. This implies that $D(e_{x_1}\partial_1) = e_{x_1}\partial_1$. Similarly, we can prove that $D(e_{x_2}\partial_1) = c_4 e_{x_1}\partial_1$ and $D(e_{x_3}\partial_1) = c_7 e_{x_1}\partial_1$. By denoting $c_{1,1} = c_1, c_{2,1} = c_4,$ and $c_{3,1} = c_7$, we have that $D(e_{x_1}\partial_1) = c_{u,1} e_{x_u} \partial_1$ where $1 \leq u \leq 3$. Similarly, we can prove that $D(e_{x_u}\partial_j) = c_{u,v} e_{x_u} \partial_v$ where $1 \leq u,v \leq 3$. By $D(e_{x_1}\partial_1 * e_{x_1}\partial_1) = D(e_{x_1}\partial_1)$, we also have that $c_{1,1} e_{x_1}\partial_1 * e_{x_1}\partial_1 * c_{1,1} e_{x_1}\partial_1 = D(e_{x_1}\partial_1)$. This implies that $D(e_{x_1}\partial_1) = 2c_{1,1} e_{x_1}\partial_1$.

By induction on $k_1 \in \mathbb{N}$ of $e_{k_1 x_1}\partial_1$, we can prove that

\[
D(e_{k_1 x_1}\partial_1) = c_{1,1} k_1 e_{k_1 x_1}\partial_1.
\]

Similarly to (7), by induction on $k_1 \in \mathbb{Z}$ of $e_{k_1 x_1}\partial_1$, we can also prove that

\[
D(e_{k_1 x_1}\partial_1) = c_{1,1} k_1 e_{k_1 x_1}\partial_1.
\]

Since $D(e_{x_1}\partial_1 * e_{x_1}\partial_2) = D(e_{x_2}\partial_2)$, we have that $(c_{1,1} + c_{1,2}) e_{x_2}\partial_2 = 2c_{1,2} e_{x_2}\partial_2$. So we have that $c_{1,1} = c_{1,2}$. Since $D(e_{x_1}\partial_1 * e_{x_2}\partial_2) = D(e_{x_3}\partial_2)$, we have that $(c_{1,1} + c_{1,3}) e_{x_3}\partial_3 = 2c_{1,3} e_{x_3}\partial_3$. So we also have that $c_{1,1} = c_{1,3}$. This implies that $c_{1,1} = c_{1,2} = c_{1,3}$. Similarly, we can prove that $c_{2,1} = c_{2,2} = c_{2,3}$ and $c_{3,1} = c_{3,2} = c_{3,3}$. Let us denote $d_1 = c_{1,1} = c_{1,2} = c_{1,3}, d_2 = c_{2,1} = c_{2,2} = c_{2,3}$, and $d_3 = c_{3,1} = c_{3,2} = c_{3,3}$ respectively. The formula (8) becomes

\[
D(e_{k_1 x_1}\partial_1) = d_1 k_1 e_{k_1 x_1}\partial_1.
\]

Similarly, we can prove that

\[
D(e_{k_u x_u}\partial_v) = d_u k_u e_{k_u x_u}\partial_v
\]

for $c_{u,v} \in \mathbb{F}$ and $1 \leq u,v \leq 3$. Therefore we have proven the lemma. \(\square\)
Lemma 3.3 For any derivation D of the algebra \( WN_{3,0,0} \) and for a basis element \( \prod_{u=1}^{3} e^{k_u x_u} \partial_v \), \( 1 \leq v \leq 3 \), in the algebra \( W(3,0,0)_{[1]} \), we have the following:

\[
D(\prod_{u=1}^{3} e^{k_u x_u} \partial_v) = \sum_{w=1}^{3} d_w k_w (\prod_{u=1}^{3} e^{k_u x_u}) \partial_v.
\]

Proof. Let D be the derivation of the algebra \( W(3,0,0)_{[1]} \) in the lemma. By Lemma 3.2, we have that

\[
D(e^{k_2 x_2} \partial_3 \ast e^{k_3 x_3} \partial_1) = k_3 D(e^{k_2 x_2} e^{k_3 x_3} \partial_1)
\]

Since the left side of (11) is \( d_2 k_2 k_3 e^{k_2 x_2} e^{k_3 x_3} \partial_1 + d_3 k_3 e^{k_2 x_2} e^{k_3 x_3} \partial_1 \), we have that

\[
D(e^{k_2 x_2} e^{k_3 x_3} \partial_1) = d_2 k_2 e^{k_2 x_2} e^{k_3 x_3} \partial_1 + d_3 k_3 e^{k_2 x_2} e^{k_3 x_3} \partial_1.
\]

Since \( D(e^{k_1 x_1} \partial_2 \ast e^{k_2 x_2} e^{k_3 x_3} \partial_1) = k_2 D(\prod_{u=1}^{3} e^{k_u x_u} \partial_1) \) holds, we can prove that

\[
D(\prod_{u=1}^{3} e^{k_u x_u} \partial_1) = \sum_{v=1}^{3} d_v k_v (\prod_{u=1}^{3} e^{k_u x_u}) \partial_1
\]

Similarly we can also prove the following cases \( D(\prod_{u=1}^{3} e^{k_u x_u} \partial_2) \) and \( D(\prod_{u=1}^{3} e^{k_u x_u} \partial_3) \) as \( D(\prod_{u=1}^{3} e^{k_u x_u} \partial_1) \). Thus we have proven the lemma. \( \square \)

Theorem 3.1 The additive group \( Der(WN_{3,0,0}) \) of all the derivations of the algebra \( WN_{3,0,0} \) is spanned by \( D_w \), \( 1 \leq w \leq 3 \), which are defined in Note 1.

Proof. The proof of the lemma is straightforward by Lemmas 3.1-3.3, and Note 1. So it is omitted. \( \square \)

Proposition 3.2 If two algebras \( A_1 \) and \( A_2 \) are isomorphic, then the dimension \( Dim(Der(A_1)) \) of the algebra \( A_1 \) is equal to the dimension \( Dim(Der(A_2)) \) of the algebra \( A_2 \).

Proof. The proof of proposition is straightforward, so it is omitted. \( \square \)

Since there are non-isomorphic algebras \( A_1 \) and \( A_2 \) such that \( Dim(Der(A_1)) = Dim(Der(A_2)) \), the converse of Proposition 3.1 is not true (see [3] and [4]).

Note 2. For any basis element \( \prod_{u=1}^{n} e^{x_u} \partial_v \) of the algebra \( W(n,0,0)_{[1]} \) and for \( d_u \in \mathbb{F}, 1 \leq u \leq n \), if we define \( \mathbb{F} \)-linear map \( D_w, 1 \leq w \leq 3 \), from the algebra \( W(n,0,0)_{[1]} \) to itself as follows:

\[
D_w(\prod_{u=1}^{n} e^{k_u x_u} \partial_v) = d_w k_w (\prod_{u=1}^{n} e^{k_u x_u}) \partial_v
\]

then \( D_w \) can be linearly extended to a derivation of \( W(n,0,0)_{[1]} \). \( \square \)
Lemma 3.4 For any derivation $D$ in $\text{Der}_{\text{non}}(\overline{W(n,0,0)}_{[1]}^{1})$ of the algebra $\overline{W(n,0,0)}_{[1]}^{1}$, $D(\partial_v), 1 \leq v \leq n$, are zeroes.

Proof. Let $D$ be any derivation of $D \in \text{Der}_{\text{non}}(\overline{W(n,0,0)}_{[1]}^{1})$. Since $\partial_i$ annihilates $\partial_j$, $1 \leq i, j \leq n$, we have that $D(\partial_1) = \sum_{u=1}^{n} r_u \partial_u$ with $r_u \in \mathbb{F}$, $1 \leq u \leq n$. Since $e^{x_1} \partial_1$ is in the right annihilator of $\partial_1$, we have that $\partial_1 \ast D(e^{x_2} \partial_1) = -r_2 e^{x_2} \partial_1$. This implies that $r_2$ is zero. Symmetrically, we can prove that $r_3 = \cdots = r_n = 0$, i.e., $D(\partial_1) = r_1 \partial_1$. Since $\partial_1$ is a left identity of $e^{x_1} \partial_1$, we have that

$$r_1 e^{x_1} \partial_1 + \partial_1 \ast D(e^{x_1} \partial_1) = D(e^{x_1} \partial_1)$$

(12)

Let us put $D(e^{x_1} \partial_1)$ as follows:

$$D(e^{x_1} \partial_1) = \sum_{u=1}^{n} a_u (\prod_{v=1}^{n} e^{x_v}) \partial_u$$

(13)

where $a_u \in \mathbb{F}$, $1 \leq u \leq n$. By (12) and (13), we can prove that $a_1 = 1$ and $r_1 = 0$, i.e., $D(\partial_1) = 0$. Similarly, we can also prove that $D(\partial_w) = 0$, $2 \leq w \leq n$. So we have proven the lemma. \qed

Lemma 3.5 For any $D \in \text{Der}_{\text{non}}(\overline{W(n,0,0)}_{[1]}^{1})$ and a basis element $\prod_{u=1}^{n} e^{k_u x_u} \partial_v$ of the algebra $\overline{W(n,0,0)}_{[1]}^{1}$, we have that

$$D(\prod_{u=1}^{n} e^{k_u x_u} \partial_v) = \sum_{u=1}^{n} k_u d_u (\prod_{u=1}^{n} e^{k_u x_u}) \partial_v$$

(14)

where $u, v \in \{1, 2, \ldots, n\}$ and $d_u \in \mathbb{F}$.

Proof. Let $D$ be the derivation of $\overline{W(n,0,0)}_{[1]}^{1}$ in the lemma. By Lemma 3.3, we are able to prove (14) for $n \leq 3$ easily. These results can be naturally extended to the algebra $\overline{W(n,0,0)}_{[1]}^{1}$ for $n \geq 4$. The remaining proof of the proof of the lemma is straightforward, so it is omitted. \qed

Theorem 3.2 $\text{Der}_{\text{non}}(\overline{W(n,0,0)}_{[1]}^{1})$ of the algebra $\overline{W(n,0,0)}_{[1]}^{1}$ is generated by $D_w, 1 \leq w \leq n$, which are defined in Note 2.

Proof. Let $D$ be any derivation of $\overline{W(n,0,0)}_{[1]}^{1}$. By Lemma 3.6, for any basis element $\prod_{u=1}^{n} e^{k_u x_u} \partial_v$ of the algebra $\overline{W(n,0,0)}_{[1]}^{1}$, (14) holds with appropriate coefficients. This implies that $D$ is the linear sum of the derivations $D_1, \ldots, D_n$ which are defined in Note 2. Therefore we have proven the theorem. \qed
Corollary 3.1 The dimension $\text{Dim}(\text{Der}(\overline{W(n,0,0)}_1))$ of the algebra $\overline{W(n,0,0)}_1$ is $n$.

Proof. Since the proof of the corollary is straightforward by Theorem 3.2, it is omitted. □

Corollary 3.2 For a derivation $D_u$, $1 \leq u \leq n$, of the algebra $\overline{W(n,0,0)}_1$, $D_u$ is the inner derivation $\text{ad}_{\partial_u}$ induced by $\partial_u$. Furthermore the associator of the algebra $\overline{W(n,0,0)}_1$ is zero (see [14]).

Proof. The proof of the corollary is straightforward by Proposition 3.1, Note 2, and Theorem 3.2. So it is omitted. □

Corollary 3.3 If $n_1 \neq n_2$, then the algebras $\overline{W(n_1,0,0)}_1$ and $\overline{W(n_2,0,0)}_1$ are not isomorphic.

Proof. Since $n_1 \neq n_2$, the results of the corollary is straightforward by Lemma 3.4 and Theorem 3.2. □

Proposition 3.3 The algebras $\overline{W(n,0,0)}_1$ and $\overline{W(0,n,0)}_1$ (resp. $\overline{W(0,0,n)}_1$) are not isomorphic.

Proof. Since the dimension $\text{Dim}(\text{Der}(\overline{W(n,0,0)}_1))$ of the algebra $\overline{W(n,0,0)}_1$ is $n$ and the dimension $\text{Dim}(\text{Der}(\overline{W(0,n,0)}_1))$ (resp. $\text{Dim}(\text{Der}(\overline{W(0,0,n)}_1))$) of the algebra $\overline{W(0,n,0)}_1$ (resp. $\overline{W(0,0,n)}_1$) is $n^2 + n$, by Lemma 3.4, the algebras $\overline{W(n,0,0)}_1$ and $\overline{W(0,n,0)}_1$ (resp. $\overline{W(0,0,n)}_1$) are not isomorphic. □

Proposition 3.4 For an algebra $A$, if $A$ has a right (resp. left) identity, then the algebras $A$ and $\overline{W(n,0,0)}_1$ are not isomorphic.

Proof. It is easy to prove that the algebra $\overline{W(n,0,0)}_1$ has no right (resp. left) identity, Since $A$ has a right (resp. left) identity, the algebras $A$ and $\overline{W(n,0,0)}_1$ are not isomorphic. Thus we have proven the proposition. □

Corollary 3.4 If the matrix ring $M_n(\mathbb{F})$ is a subalgebra of an algebra $A$ and $n$ is greater than 1, then the algebras $A$ and $\overline{W(n,0,0)}_1$ are not isomorphic.

Proof. It is easy to prove that the matrix ring $M_n(\mathbb{F})$ is not a subalgebra of $\overline{W(n,0,0)}_1$. By Proposition 3.3, the the proof of the corollary is easy to prove, so it is omitted. □
References


**Received: March 11, 2008**