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Derivations and Isomorphisms of a Non-Associative Algebra with Finitely Many Right Annihilators

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Abstract

We prove that the non-associative algebra $\overline{W(n,0,0)}_{[r]}$ and its symmetrized algebra are simple (see [1], [2], [3]). We find all the derivations of the algebra $\overline{W(n,0,0)}_1$. Thus we can prove that for $n_1 \neq n_2$, the algebras $\overline{W(n_1,0,0)}_1$ and $\overline{W(n_2,0,0)}_1$ are not isomorphic. Because of all the derivations of a non-associative algebra, we can find the associator of the algebra. Depending on the dimension of the right annihilator of an algebra, we define the class $\mathfrak{R}_{ann}$ of the algebras in this work.

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1 Preliminaries

Let $\mathbb{N}$ be the set of all non-negative integers and $\mathbb{Z}$ be the set of all integers. Let $\mathbb{F}$ be a field of characteristic zero. Let $\mathbb{F}^{\bullet}$ be the multiplicative group of non-zero elements of $\mathbb{F}$. Let us define the $\mathbb{F}$-algebra $\mathbb{F}[e^{\pm x_1}, \cdots, e^{\pm x_n}, x_1^{\pm 1}, \cdots, x_m^{\pm 1}, x_{m+1}, \cdots, x_{m+s}]$ spanned by

$$\{e^{a_1x_1} \cdots e^{a_nx_n} x_1^{i_1} \cdots x_m^{i_m} x_{m+1}^{i_{m+1}} \cdots x_{m+s}^{i_{m+s}} | \\ a_1, \cdots, a_n, i_1, \cdots, i_m \in \mathbb{Z}, i_{m+1}, \cdots, i_{m+s} \in \mathbb{N}\}. \quad (1)$$

with the obvious addition and its multiplication (see [12]). For a positive integer r, the non-associative algebra $\overline{W(n,m,m+s)}_{[r]}$ is the vector space spanned by

$$\{f\partial_u^r | f \in \mathbb{F}[e^{\pm x_1}, \cdots, e^{\pm x_n}, x_1^{\pm 1}, \cdots, x_m^{\pm 1}, x_{m+1}, \cdots, x_{m+s}], 1 \leq u \leq n\} \quad (2)$$

with the obvious addition and the multiplication $*$ where ∂_u is the usual partial derivative on the $\mathbb{F}$-algebra $\mathbb{F}[e^{\pm x_1}, \cdots, e^{\pm x_n}, x_1^{\pm 1}, \cdots, x_m^{\pm 1}, x_{m+1}, \cdots, x_{m+s}]$ with respect to x_u, $1 \leq u \leq max\{n, m+s\}$ in the papers (see [1], [3], [9], [10], [14]). The non-associative algebra $\overline{W(n,m,m+s)}_{[r]}$ is a subalgebra of the algebra in the papers (see [6-11]). Note that $\partial_u^0 = 1$, $1 \leq u \leq n$. Thus if $r = 0$, then the algebra $\overline{W(n,m,m+s)}_{[r]}$ is the $\mathbb{F}$-algebra $\mathbb{F}[e^{\pm x_1}, \cdots, e^{\pm x_n}, x_1^{\pm 1}, \cdots, x_m^{\pm 1}, x_{m+1}, \cdots, x_{m+s}]$. For an element l of an algebra A and an ideal I of A, l is full, if the ideal contains the element l, then l is A. The matrix ring $M_{m+s}(\mathbb{F})$ is imbedded in the algebra $\overline{W(n,m,m+s)}_{[r]}$. The matrix ring $M_n(\mathbb{F})$ is not imbedded in $\overline{W(n,0,0)}_{[r]}$. The non-associative algebra $\overline{W(n,0,0)}_{[r]}$ has neither a right nor a left multiplicative identity element. Note that the definition of a non-associative algebra in this paper is different from the definitions of the algebras in the papers (see [3], [9], [10]), because of some results in the paper. Similarly to the non-associative algebra $\overline{W(n,m,m+s)}_{[r]}$, we can define the non-associative algebra $\overline{W(n,0,s)}_{[r]}$ spanned by $\{e^{a_1 x_1} \cdots e^{a_n x_n} x_{n+1}^{i_{n+1}} \cdots x_{n+s}^{i_{n+s}} \partial_u | a_1, \cdots, a_n \in \mathbb{Z}, i_{n+1}, \cdots, i_{n+s} \in \mathbb{N}, 1 \leq u \leq n+m\}$. The algebra $\overline{W(n,m,m+s)}_{[r]}$ has idempotents and $\overline{W(n,m,m+s)}_{[r]}$ is simple. Using the commutator $[,]$ of $\overline{W(n,m,m+s)}_{[r]}$, we can define the antisymmetrized algebra $\overline{W(n,m,m+s)}_{[r]_{[,]}}$. The Lie algebra $\overline{W(n,m,m+s)}_{[1]_{[,]}}$ is the simple Lie algebra in the papers (see [9] and [13]). The algebra $\overline{W(n,0,0)}_{[r]}$ is a subalgebra of the algebra $\overline{W(n,m,m+s)}_{[r]}$ spanned by $\{\prod_{v=1}^n e^{a_v x_v} \partial_u^r | a_1, \cdots, a_n \in \mathbb{Z}\}$. The algebra $\overline{W(n,0,0)}_{[r]}$ is $\mathbb{Z}^n$-graded as follows:

$$\overline{W(n,0,0)}_{[r]} = \bigoplus_{(a_1,\cdots,a_n)} N_{(a_1,\cdots,a_n)}$$

where the $(a_1, \cdots, a_n)$-homogeneous component $N_{(a_1,\cdots,a_n)}$ is the vector subspace of $\overline{W(n,0,0)}_{[r]}$ spanned by $\{\prod_{v=1}^n e^{a_v x_v} \partial_u^r | 1 \leq u \leq n\}$. Let $\mathfrak{R}_{ann}(n)$ be the set of all algebra A such that the dimension of all the right annihilators of A is n. The class $\mathfrak{R}_{ann}$ is defined as follows:

$$\mathfrak{R}_{ann} = \{\mathfrak{R}_{ann}(n) | n \in \mathbb{N}\}.$$

2 Simplicities

Theorem 2.1 *The algebra* $\overline{W(n,0,0)}_{[r]}$ *is simple.*

Proof. It is easy to prove that for any ideal I of $\overline{W(n,0,0)}_{[r]}$, if I contains an element of the homogeneous component $N_{(0,\cdots,0)}$, then I is the algebra

$\overline{W(n,0,0)}_{[r]}$. Thus it is enough to show that for every non-zero ideal I of $\overline{W(n,0,0)}_{[r]}$, I contains an element in the homogeneous component $N_{(0,\cdots,0)}$. Let I_1 be a non-zero ideal of the homogeneous component $N_{(0,\cdots,0)}$. Let l be a non-zero element in I_1. Since every term of l in a homogeneous component, let us prove the theorem by induction on the number of homogeneous components which contain a term of l. If l is in the homogeneous component $N_{(0,\cdots,0)}$, then there is nothing to prove. Let us assume that l is in the homogeneous component $N_{(a_1,\cdots,a_n)}$. We can also assume that a_1 is a non-zero integer. By taking an appropriate element l_2 in the homogeneous component $N_{(-a_1,\cdots,-a_n)}$, we have that $l_2 * l$ is a non-zero element in $N_{(0,\cdots,0)}$. Thus I_1 is the algebra $\overline{W(n,0,0)}_{[r]}$. By induction, we can assume that if l has terms in k homogeneous components, then I_1 is the algebra $\overline{W(n,0,0)}_{[r]}$. Let us assume that l has terms in $k+1$ homogeneous components. We can find an appropriate elements l_1 and ∂^r, so that $\partial_v^r * (l_1 * l)$ is non-zero and it has at most k homogeneous components. By induction, I_1 is the algebra $\overline{W(n,0,0)}_{[r]}$. Therefore we have proven the theorem. □

Theorem 2.2 *The antisymmetrized algebra* $\overline{W(n,0,0)}_{[r]_{[,]}}$ *of the algebra* $\overline{W(n,0,0)}_{[r]}$ *is simple. The algebra* $\overline{W(n,0,0)}_{[r]_{[,]}}$ *is self-centralizing (see [8]).*

Proof. The proof of the theorem is almost the same as the proof of Theorem 2.1 and the remaining result of the proof is easy, so it is omitted. □

Remarks. The algebra $\overline{W(n,0,0)}$ is in $\mathfrak{R}_{ann}(n)$. For any $A \in \mathfrak{R}_{ann}(n)$ and $B \in \mathfrak{R}_{ann}(m)$, if $n \neq m$, then the algebras A and B are not isomorphic. Thus their antisymmetrized algebras $A_{[,]}$ and $B_{[,]}$ are not isomorphic. □

3 Derivations of a non-associative algebra

For a non-associative algebra, we need the following obvious results.

Proposition 3.1 *Let A be a non-associative algebra and B be the set of all associators of the algebra A. For any $l \in B$, l induces the inner derivation ad_l of A (see [14]).*

Note 1. For any basis element $\prod_u^3 e^{k_u x_u}\partial_v$ of the algebra $\overline{W(3,0,0)}_{[1]}$ and $v \in \{1,2,3\}$, if we define $\mathbb{F}$-linear maps D_w, $1 \leq w \leq 3$, of the non-associative algebra $\overline{W(3,0,0)}_{[1]}$ as follows:

$$D_w(\prod_{u=1}^{3} e^{k_u x_u}\partial_v) = d_w k_w \prod_{u=1}^{3} e^{k_u x_u}\partial_v,$$

then D_w, $1 \le w \le 3$, can be linearly extended to derivations of the algebra $\overline{W(3,0,0)_{[1]}}$ with appropriate coefficients. □ □

Lemma 3.1 *For any derivation D of the algebra $\overline{W(3,0,0)_{[1]}}$, we have that $D(\partial_u) = 0$, $1 \le u \le 3$.*

Proof. Let D be the derivation of the algebra $\overline{W(3,0,0)_{[1]}}$ in the lemma. Since ∂_1 annihilates itself, we have that $D(\partial_1) * \partial_1 + \partial_1 * D(\partial_1)$ is zero. This implies that

$$D(\partial_1) = s_1 e^{p_2 x_2} e^{p_3 x_3} \partial_1 + s_2 e^{q_2 x_2} e^{q_3 x_3} \partial_2 + s_3 e^{r_2 x_2} e^{r_3 x_3} \partial_3 \tag{3}$$

for $s_1, s_2, s_3 \in \mathbb{F}$. Since ∂_2 is in the left annihilator of ∂_1,

$$s_1 p_2 e^{p_2 x_2} e^{p_3 x_3} \partial_1 + s_2 q_2 e^{q_2 x_2} e^{q_3 x_3} \partial_2 + s_3 r_2 e^{r_2 x_2} e^{r_3 x_3} \partial_3 = 0$$

This implies that $p_2 = q_2 = r_2 = 0$. Since ∂_3 is in the left annihilator of ∂_1, similarly we are able to prove that $p_3 = q_3 = r_3 = 0$. So we have that $D(\partial_1) = s_1\partial_1 + s_2\partial_2 + s_3\partial_3$. Since ∂_1 is a left identity of $e^{x_1}\partial_1$, we have that

$$\partial_1 * D(e^{x_1}\partial_1) = -s_1 e^{x_1}\partial_1 + D(e^{x_1}\partial_1). \tag{4}$$

This implies that $D(e^{x_1}\partial_1)$ can be written as follows:

$$D(e^{x_1}\partial_1) = c_1 e^{a_1 x_1} e^{a_2 x_2} e^{a_3 x_3} \partial_1 + c_2 e^{a_4 x_1} e^{a_5 x_2} e^{a_6 x_3} \partial_2 + c_3 e^{a_7 x_1} e^{a_8 x_2} e^{a_9 x_3} \partial_3 \tag{5}$$

where $c_1, \cdots, c_9 \in \mathbb{F}$. By (4) we are able to prove that $a_1 = a_4 = a_7 = 1$ and s_1 is zero. Since ∂_1 is in the left annihilators of $e^{x_2}\partial_1$, and $e^{x_3}\partial_1$, we are also able to prove that s_2 and s_3 are zeroes, i.e., $D(\partial_1)$ is zero. Similarly, we can also prove that $D(\partial_2)$ and $D(\partial_3)$ are zeroes. Therefore we have proven the lemma. □

Lemma 3.2 *For any derivation D of the algebra $\overline{W(3,0,0)_1}$, $D(e^{k_i x_i}\partial_j) = c_i k_i e^{k_i x_i}\partial_j$ holds, where $c_i \in \mathbb{F}$, $1 \le i, j \le 3$.*

Proof. Let D be the derivation of the non-associative algebra $\overline{W(3,0,0)_{[1]}}$ in the lemma. Since ∂_1 is a left identity of $e^{x_1}\partial_1$ and by Lemma 3.1, we have that $\partial_1 * D(e^{x_1}\partial_1) = D(e^{x_1}\partial_1)$. Let us put $D(e^{x_1}\partial_1) = c_1 \prod_{u=1}^{3} e^{a_u x_u}\partial_1 + c_2 \prod_{u=1}^{3} e^{a_{3+u} x_u}\partial_2 + c_3 \prod_{u=1}^{3} e^{a_{6+u} x_u}\partial_3$ where $c_1, c_2, c_3 \in \mathbb{F}$. Since

$$c_1 a_1 \prod_{u=1}^{3} e^{a_u x_u}\partial_1 + c_2 a_4 \prod_{u=1}^{3} e^{a_{3+u} x_u}\partial_2 + c_3 a_7 \prod_{u=1}^{3} e^{a_{6+u} x_u}\partial_3 =$$
$$c_1 \prod_{u=1}^{3} e^{a_u x_u}\partial_1 + c_2 \prod_{u=1}^{3} e^{a_{3+u} x_u}\partial_2 + c_3 \prod_{u=1}^{3} e^{a_{6+u} x_u}\partial_3 \tag{6}$$

we have that a_1, a_4, and a_7 are ones. Since ∂_2 and ∂_3 are in the left annihilator of $e^{x_1}\partial_1$, we can easily prove that $a_2 = a_3 = a_5 = a_6 = a_8 = a_9 = 0$ and $D(e^{x_1}\partial_1) = \sum_{u=1}^{3} c_u e^{x_u}\partial_u$. Similarly, we can also prove the followings:

$$D(e^{x_2}\partial_1) = \sum_{u=1}^{3} c_{3+u}e^{x_u}\partial_u \text{ and } D(e^{x_3}\partial_1) = \sum_{u=1}^{3} c_{6+u}e^{x_u}\partial_u$$

with appropriate coefficients. Since $e^{x_1}\partial_1$ is in the left annihilator of $e^{x_2}\partial_1$, we also have that c_2 is zero. Since $e^{x_1}\partial_1$ is in the left annihilator of $e^{x_3}\partial_1$, we are able to prove that c_3 is zero. This implies that $D(e^{x_1}\partial_1) = c_1 e^{x_1}\partial_1$. Similarly, we can prove that $D(e^{x_2}\partial_1) = c_4 e^{x_1}\partial_1$ and $D(e^{x_3}\partial_1) = c_7 e^{x_3}\partial_1$. By denoting $c_{1,1} = c_1$, $c_{2,1} = c_4$, and $c_{3,1} = c_7$, we have that $D(e^{x_u}\partial_1) = c_{u,1}e^{x_u}\partial_1$ where $1 \le u \le 3$. Similarly, we can prove that $D(e^{x_u}\partial_j) = c_{u,v}e^{x_u}\partial_v$ where $1 \le u, v \le 3$. By $D(e^{x_1}\partial_1 * e^{x_1}\partial_1) = D(e^{2x_1}\partial_1)$, we also have that $c_{1,1}e^{x_1}\partial_1 * e^{x_1}\partial_1 + e^{x_1}\partial_1 * c_{1,1}e^{x_1}\partial_1 = D(e^{2x_1}\partial_1)$. This implies that $D(e^{2x_1}\partial_1) = 2c_{1,1}e^{2x_1}\partial_1$. By induction on $k_1 \in \mathbb{N}$ of $e^{k_1x_1}\partial_1$, we can prove that

$$D(e^{k_1x_1}\partial_1) = c_{1,1}k_1e^{k_1x_1}\partial_1. \tag{7}$$

By $D(e^{-x_1}\partial_1 * e^{x_1}\partial_1) = D(\partial_1)$, we have that $D(e^{-x}\partial_1) * e^{x_1}\partial_1 + e^{-x}\partial_1 * D(e^{x_1}\partial_1) = 0$. This implies that $D(e^{-x_1}\partial_1) * e^{x_1}\partial_1 = -c_{1,1}\partial_1$ and $D(e^{-x_1}\partial_1) = -c_{1,1}e^{-x_1}\partial_1 + \sum_{l_1,l_2,l_3} \alpha_{l_1,l_2,l_3,2}e^{l_1x_1}\ e^{l_2x_2}e^{l_3x_3}\partial_2 + \sum_{l_4,l_5,l_6} \beta_{l_4,l_5,l_6,3}e^{l_4x_1}e^{l_5x_2}e^{l_6x_3}\partial_3$ holds. Since ∂_2 and ∂_3 are in the left annihilator of $e^{-x_1}\partial_1$, we prove that $D(e^{-x_1}\partial_1) = -c_{1,1}e^{-x_1}\partial_1$. Similarly to (7), by induction on $k_1 \in \mathbb{Z}$ of $e^{k_1x_1}\partial_1$, we can also prove that

$$D(e^{k_1x_1}\partial_1) = c_{1,1}k_1e^{k_1x_1}\partial_1. \tag{8}$$

Since $D(e^{x_1}\partial_1 * e^{x_1}\partial_2) = D(e^{2x_1}\partial_2)$, we have that $(c_{1,1}+c_{1,2})e^{2x_1}\partial_2 = 2c_{1,2}e^{2x_1}\partial_2$. So we have that $c_{1,1} = c_{1,2}$. Since $D(e^{x_1}\partial_1 * e^{x_1}\partial_3) = D(e^{2x_1}\partial_3)$, we have that $(c_{1,1} + c_{1,3})e^{2x_1}\partial_3 = 2c_{1,3}e^{2x_1}\partial_3$. So we also have that $c_{1,1} = c_{1,3}$. This implies that $c_{1,1} = c_{1,2} = c_{1,3}$. Similarly, we can prove that $c_{2,1} = c_{2,2} = c_{2,3}$ and $c_{3,1} = c_{3,2} = c_{3,3}$. Let us denote $d_1 = c_{1,1} = c_{1,2} = c_{1,3}$, $d_2 = c_{2,1} = c_{2,2} = c_{2,3}$, and $d_3 = c_{3,1} = c_{3,2} = c_{3,3}$ respectively. The formula (8) becomes

$$D(e^{k_1x_1}\partial_1) = d_1k_1e^{k_1x_1}\partial_1. \tag{9}$$

Similarly, we can prove that

$$D(e^{k_ux_u}\partial_v) = d_uk_ue^{k_ux_u}\partial_v \tag{10}$$

for $c_{u,v} \in \mathbb{F}$ and $1 \le u, v \le 3$. Therefore we have proven the lemma. $\square$

Lemma 3.3 *For any derivation D of the algebra $\overline{WN_{3,0,0_1}}$ and for a basis element $\prod_{u=1}^{3} e^{k_u x_u}\partial_v$, $1 \le v \le 3$, in the algebra $\overline{W(3,0,0)}_{[1]}$, we have the following:*

$$D(\prod_{u=1}^{3} e^{k_u x_u}\partial_v) = \sum_{w=1}^{3} d_w k_w (\prod_{u=1}^{3} e^{k_u x_u})\partial_v.$$

Proof. Let D be the derivation of the algebra $\overline{W(3,0,0)}_{[1]}$ in the lemma. By Lemma 3.2, we have that

$$D(e^{k_2x_2}\partial_3 * e^{k_3x_3}\partial_1) = k_3 D(e^{k_2x_2}e^{k_3x_3}\partial_1) \tag{11}$$

Since the left side of (11) is $d_2k_2k_3e^{k_2x_2}e^{k_3x_3}\partial_1 + d_3k_3^2e^{k_2x_2}e^{k_3x_3}\partial_1$, we have that

$$D(e^{k_2x_2}e^{k_3x_3}\partial_1) = d_2k_2e^{k_2x_2}e^{k_3x_3}\partial_1 + d_3k_3e^{k_2x_2}e^{k_3x_3}\partial_1.$$

Since $D(e^{k_1x_1}\partial_2 * e^{k_2x_2}e^{k_3x_3}\partial_1) = k_2 D(\prod_{u=1}^{3} e^{k_u x_u}\partial_1)$ holds, we can prove that

$$D(\prod_{u=1}^{3} e^{k_u x_u}\partial_1) = \sum_{v=1}^{3} d_v k_v (\prod_{u=1}^{3} e^{k_u x_u})\partial_1$$

Similarly we can also prove the following cases $D(\prod_{u=1}^{3} e^{k_u x_u}\partial_2)$ and $D(\prod_{u=1}^{3} e^{k_u x_u}\partial_3)$ as $D(\prod_{u=1}^{3} e^{k_u x_u}\partial_1)$. Thus we have proven the lemma. □

Theorem 3.1 *The additive group $Der(\overline{WN_{3,0,0}}_{[1]})$ of all the derivations of the algebra $\overline{WN_{3,0,0}}_{[1]}$ is spanned by D_w, $1 \le w \le 3$, which are defined in Note 1.*

Proof. The proof of the lemma is straightforward by Lemmas 3.1-3.3, and Note 1. So it is omitted. □

Proposition 3.2 *If two algebras A_1 and A_2 are isomorphic, then the dimension $Dim(Der(A_1))$ of the algebra A_1 is equal to the dimension $Dim(Der(A_2))$ of the algebra A_2.*

Proof. The proof of proposition is straightforward, so it is omitted. □

Since there are non-isomorphic algebras A_1 and A_2 such that $Dim(Der(A_1)) = Dim(Der(A_2))$, the converse of Proposition 3.1 is not true (see [3] and [4]).

Note 2. For any basis element $\prod_{u=1}^{n} e^{x_u}\partial_v$ of the algebra $\overline{W(n,0,0)}_{[1]}$ and for $d_u \in \mathbb{F}$, $1 \le u \le n$, if we define $\mathbb{F}$-linear map D_w, $1 \le w \le 3$, from the algebra $\overline{W(n,0,0)}_{[1]}$ to itself as follows:

$$D_w(\prod_{u=1}^{n} e^{k_u x_u}\partial_v) = d_w k_w (\prod_{u=1} e^{k_u x_u})\partial_v$$

then D_w can be linearly extended to a derivation of $\overline{W(n,0,0)}_{[1]}$. □

Lemma 3.4 *For any derivation D in $Der_{non}(\overline{W(n,0,0)}_{[1]})$ of the algebra $\overline{W(n,0,0)}_{[1]}$, $D(\partial_v)$, $1 \leq v \leq n$, are zeroes.*

Proof. Let D be any derivation of $D \in Der_{non}(\overline{W(n,0,0)}_{[1]})$. Since ∂_i annihilates ∂_j, $1 \leq i,j \leq n$, we have that $D(\partial_1) = \sum_{u=1}^{n} r_u \partial_u$ with $r_u \in \mathbb{F}$, $1 \leq u \leq n$. Since $e^{x_2}\partial_1$ is in the right annihilator of ∂_1, we have that $\partial_1 * D(e^{x_2}\partial_1) = -r_2 e^{x_2}\partial_1$. This implies that r_2 is zero. Symmetrically, we can prove that $r_3 = \cdots = r_n = 0$, i.e., $D(\partial_1) = r_1\partial_1$. Since ∂_1 is a left identity of $e^{x_1}\partial_1$, we have that

$$r_1 e^{x_1}\partial_1 + \partial_1 * D(e^{x_1}\partial_1) = D(e^{x_1}\partial_1) \tag{12}$$

Let us put $D(e^{x_1}\partial_1)$ as follows:

$$D(e^{x_1}\partial_1) = \sum_{u=1}^{n} a_u (\prod_{v=1}^{n} e^{x_v})\partial_u \tag{13}$$

where $a_u \in \mathbb{F}$, $1 \leq u \leq n$. By (12) and (13), we can prove that $a_1 = 1$ and $r_1 = 0$, i.e., $D(\partial_1) = 0$. Similarly, we can also prove that $D(\partial_w) = 0$, $2 \leq w \leq n$. So we have proven the lemma. □

Lemma 3.5 *For any $D \in Der_{non}(\overline{W(n,0,0)}_{[1]})$ and a basis element $\prod_{u=1}^{n} e^{k_u x_u}\partial_v$ of the algebra $\overline{W(n,0,0)}_{[1]}$, we have that*

$$D(\prod_{u=1}^{n} e^{k_u x_u}\partial_v) = \sum_{u=1}^{n} k_u d_u (\prod_{u=1}^{n} e^{k_u x_u})\partial_v \tag{14}$$

where $u, v \in \{1, 2, \cdots, n\}$ and $d_u \in \mathbb{F}$.

Proof. Let D be the derivation of $\overline{W(n,0,0)}_{[1]}$ in the lemma. By Lemma 3.3, we are able to prove (14) for $n \leq 3$ easily. These results can be naturally extended to the algebra $\overline{W(n,0,0)}_{[1]}$ for $n \geq 4$. The remaining proof of the proof of the lemma is straightforward, so it is omitted. □

Theorem 3.2 *$Der_{non}(\overline{W(n,0,0)}_{[1]})$ of the algebra $\overline{W(n,0,0)}_{[1]}$ is generated by D_w, $1 \leq w \leq n$, which are defined in Note 2.*

Proof. Let D be any derivation of $\overline{W(n,0,0)}_{[1]}$. By Lemma 3.6, for any basis element $\prod_{u=1}^{n} e^{k_u x_u}\partial_v$ of the algebra $\overline{W(n,0,0)}_{[1]}$, (14) holds with appropriate coefficients. This implies that D is the linear sum of the derivations $D_1, \cdots, D_n$ which are defined in Note 2. Therefore we have proven the theorem. □

Corollary 3.1 *The dimension* $Dim(Der(\overline{W(n,0,0)}_{[1]}))$ *of the algebra* $\overline{W(n,0,0)}_{[1]}$ *is* n.

Proof. Since the proof of the corollary is straightforward by Theorem 3.2, it is omitted. □

Corollary 3.2 *For a derivation* D_u, $1 \leq u \leq n$, *of the algebra* $\overline{W(n,0,0)}_{[1]}$, D_u *is the inner derivation* ad_{∂_u} *induced by* ∂_u. *Furthermore the associator of the algebra* $\overline{W(n,0,0)}_{[1]}$ *is zero (see [14]).*

Proof. The proof of the corollary is straightforward by Proposition 3.1, Note 2, and Theorem 3.2. So it is omitted. □

Corollary 3.3 *If* $n_1 \neq n_2$, *then the algebras* $\overline{W(n_1,0,0)}_{[1]}$ *and* $\overline{W(n_2,0,0)}_{[1]}$ *are not isomorphic.*

Proof. Since $n_1 \neq n_2$, the results of the corollary is straightforward by Lemma 3.4 and Theorem 3.2. □

Proposition 3.3 *The algebras* $\overline{W(n,0,0)}_{[1]}$ *and* $\overline{W(0,n,0)}_{[1]}$ *(resp.* $\overline{W(0,0,n)}_{[1]}$*) are not isomorphic.*

Proof. Since the dimension $Dim(Der(\overline{W(n,0,0)}_{[1]}))$ of the algebra $\overline{W(n,0,0)}_{[1]}$ is n and the dimension $Dim(Der(\overline{W(0,n,0)}_{[1]}))$ (resp. $Dim(Der(\overline{W(0,0,n)}_{[1]}))$) of the algebra $\overline{W(0,n,0)}_{[1]}$ (resp. $\overline{W(0,0,n)}_{[1]}$) is n^2+n, by Lemma 3.4, the algebras $\overline{W(n,0,0)}_{[1]}$ and $\overline{W(0,n,0)}_{[1]}$ (resp. $\overline{W(0,0,n)}_{[1]}$) are not isomorphic. □

Proposition 3.4 *For an algebra* A, *if* A *has a right (resp. left) identity, then the algebras* A *and* $\overline{W(n,0,0)}_{[r]}$ *are not isomorphic.*

Proof. It is easy to prove that the algebra $\overline{W(n,0,0)}_{[r]}$ has no right (resp. left) identity, Since A has a right (resp. left) identity, the algebras A and $\overline{W(n,0,0)}_{[r]}$ are not isomorphic. Thus we have proven the proposition. □

Corollary 3.4 *If the matrix ring* $M_n(\mathbb{F})$ *is a subalgebra of an algebra* A *and* n *is greater than* 1, *then the algebras* A *and* $\overline{W(n,0,0)}_{[r]}$ *are not isomorphic.*

Proof. It is easy to prove that the matrix ring $M_n(\mathbb{F})$ is not a subalgebra of $\overline{W(n,0,0)}_{[r]}$. By Proposition 3.3, the the proof of the corollary is easy to prove, so it is omitted. □

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