Isomorphism Theorems for QA-Mappings

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Abstract
In this paper we introduce the concept of QA-mappings and Q-quasi-antiorders in anti-ordered sets theory. Two isomorphism theorems for QA-mappings and Q-quasi-antiorders are presented.

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1 Introduction
Let $(X,\neq)$ be a set in the sense of books [1] - [3] and [10], where "$\neq$" is a binary relation on $X$ which satisfies the following properties:

$\neg(x \neq x)$, $x \neq y \implies y \neq x$, $x \neq z \implies x \neq y \lor y \neq z$,
$x \neq y \land y = z \implies x \neq z$,

called apartness (A. Heyting). The apartness is tight (D. Scott) if $\neg(x \neq y) \implies x = y$ holds. Let $Y$ be a subset of $X$ and $x \in X$. The subset $Y$ of $X$ is strongly extensional in $X$ if and only if $y \in Y \implies y \neq x \lor x \in Y$ ([3],[5]). If $x \in X$, it defined ([2]) $x \triangleleft Y$ by $(\forall y \in Y)(y \neq x)$.

Let $f : (X,=,\neq) \rightarrow (Y,=,\neq)$ be a function. We say that it is:
(a) $f$ is strongly extensional if and only if $(\forall a,b \in X)(f(a) \neq f(b) \implies a \neq b)$;
(b) $f$ is an embedding if and only if $(\forall a,b \in X)(a \neq b \implies f(a) \neq f(b))$.
Let $\alpha \subseteq X \times Y$ and $\beta \subseteq Y \times Z$ be relations. The filled product ([4]) of relations $\alpha$ and $\beta$ is the relation

$\beta \ast \alpha = \{(a,c) \in X \times Z : (\forall b \in Y)((a,b) \in \alpha \lor (b,c) \in \beta)\}$.

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A relation \( q \subseteq X \times X \) is a coequality relation on \( X \) if and only if holds:
\[
q \subseteq \neq, \; q \subseteq q^{-1}, \; q \subseteq q \ast q.
\]
If \( q \) is a coequality relation on set \((X, \neq, \neq)\), we can construct factor-set \((X/q, \neq_1, \neq_1)\) with
\[
aq = bq \iff (a, b) \neq q, \; aq \neq_1 bq \iff (a, b) \in q.
\]
A relation \( \alpha \) on \( X \) is an antiorder ([6],[7]) on \( X \) if and only if
\[
\alpha \subseteq \neq, \; \alpha \subseteq \alpha \ast \alpha, \; \neq \subseteq \alpha \cup \alpha^{-1}, (\alpha \cap \alpha^{-1} = \emptyset).
\]
Let \( f : (X, \neq, \alpha) \longrightarrow (Y, \neq, \beta) \) be a strongly extensional function of ordered sets under antiorders. \( f \) is called isotone if
\[
(\forall x, y \in S)((x, y) \in \alpha \implies (f(x), f(y)) \in \beta);
\]
\( f \) is called reverse isotone if and only if
\[
(\forall x, y \in S)((f(x), f(y)) \in \beta \implies (x, y) \in \alpha).
\]
The strongly extensional mapping \( f \) is called an isomorphism if it is injective and embedding, onto, isotone and reverse isotone. \( X \) and \( Y \) called isomorphic, in symbol \( X \cong Y \), if exists an isomorphism between them. As in [6], a relation \( \tau \subseteq X \times X \) is a quasi-antiorder on \( X \) if and only if
\[
\tau \subseteq \alpha (\subseteq \neq), \; \tau \subseteq \tau \ast \tau, (\tau \cap \tau^{-1} = \emptyset).
\]

## 2 Preliminaries

The first proposition gives some information about quasi-antiorder:

**Lemma 2.1** ([7], Lemma 1; [6], Lemma 1) Let \((X, \neq, \neq)\) be an anti-ordered and \( \tau \) is a quasi-antiorder on \( X \). Then, the relation \( q = \tau \cup \tau^{-1} \) is a coequality relation on \( X \), and \( X/q = \{aq : a \in X\} \) with the anti-order \( \theta \), defined by \((aq, bq) \in \theta \iff (a, b) \in \tau \) \((a, b) \in X\), is an anti-ordered set and \( \pi : X \longrightarrow X/q \), defined by \( \pi(a) = aq \), is an reverse isotone strongly extensional mapping from \( X \) onto \( X/q \).

**Lemma 2.2** ([6], Lemma 2; [9], Theorem 5) Let \((X, \neq, \neq, \alpha)\) and \((Y, \neq, \neq, \beta)\) be anti-ordered sets and \( \varphi : X \longrightarrow Y \) an reverse isotone strongly extensional mapping. Then,
\[
\varphi^{-1}(\beta) = \{(a, b) \in X \times X : (\varphi(a), \varphi(b)) \in \beta\}
\]
is a quasi-antiorder on $X$ with $\varphi^{-1}(\varphi) \cup (\varphi^{-1}(\varphi)) = \text{Coker}\varphi$, and $X/\text{Coker}\varphi \cong \text{Im}\varphi$ as anti-ordered sets.

**Lemma 2.3** ([9], Theorem 6) Let $(X, =, \neq, \alpha)$ and $(Y, =, \neq, \beta)$ be antiordered sets and $\varphi : X \to Y$ an reverse isotone strongly extensional mapping and $\rho$ a quasi-antiorder on $X$. Then, $\rho \supseteq \varphi^{-1}(\beta)$ if and only if there is a unique reverse isotone strongly extensional mapping $\psi$ from $X/\text{Coker}\varphi$ to $T$ such that $\varphi = \psi \circ \pi$. Moreover $\text{Im}\varphi = \text{Im}\psi$.

**Lemma 2.4** ([7], Theorem 1; [9], Theorem 8) Let $(X, =, \neq, \alpha)$ be a set, $\sigma$ a quasi-antiorder on $X$ such that $\sigma \subseteq \rho$. Then, the relation $\sigma/\rho$, defined by

$$
\sigma/\rho = \{(x(\rho \cup \rho^{-1}), y(\rho \cup \rho^{-1})) \in X/(\rho \cup \rho^{-1}) \times X/(\rho \cup \rho^{-1}) : (x, y) \in \sigma\},
$$

is a quasi-antiorder on $X/(\rho \cup \rho^{-1})$ and

$$(X/(\rho \cup \rho^{-1})/((\sigma/\rho) \cup (\sigma/\rho)^{-1}) \cong X/(\sigma \cup \sigma^{-1})$$

holds as anti-ordered sets.

**Lemma 2.5** ([6], Theorem 3; [7], Theorem 2) Let $(X, =, \neq)$ be a set with apartness, $\alpha$ a quasi-antiorder on $X$. Let $A = \{\tau : \tau$ is quasi-antiorder on $X$ such that $\tau \subseteq \sigma\}$. Let $B$ be the set of all quasi-antiorders on $X/q$, where $q = \sigma \cup \sigma^{-1}$. For $\tau \in A$, we define a relation $\tau'' = \{(aq, bq) \in X/q \times X/q : (a, b) \in \tau\}$. The mapping $\psi : A \to B$ defined by $\psi(\tau) = \tau''$ is strongly extensional, injective and surjective mapping from $A$ onto $B$ and for $\tau_1, \tau_2 \in A$ we have $\tau_1 \subseteq \tau_2$ if and only if $\psi(\tau_1) \subseteq \psi(\tau_2)$.

3 Definitions and basic properties

Let $(X, =, \neq)$ be a set with apartness, $q$ be a coequality relation on $X$ and $\alpha$ be an anti-order relation on $X$. With it is associated the following relative $((X/q, =_1, \neq_1), \theta)$ where $\theta = \pi \circ \alpha \circ \pi^{-1}$. In [8] giving an answer on question “When the relation $\theta$, defined above, is an anti-order relation on $X/q$?” we find necessary and sufficient conditions that the relation $\pi \circ \alpha \circ \pi^{-1}$ is an anti-order relation on $X/q$.

**Lemma 3.1** ([8], Theorem 4) Let $q$ be a coequality relation in anti-ordered set $(X, =, \neq, \alpha)$. Then, the relation $\theta = \pi \circ \alpha \circ \pi^{-1}$ is an anti-order relation on factor-set $X/q$ if and only if the relation $\tau = \text{Ker}\pi \circ \alpha \circ \text{Ker}\pi$ is a quasi-antiorder relation on $X$ such that $\tau \cup \tau^{-1} = q$. 
By definition, for a quasi-antiorder \( \rho \) on an anti-ordered set \((X, =, \neq, \alpha)\) holds \( \rho \subseteq \alpha \). Opposite inclusion does not hold, but result in Theorem 3.1 is a motive for introducing of the following new notion:

**Definition 1** Let \((X, =, \neq, \alpha)\) be an anti-ordered set. A quasi-antiorder \( \rho \) on \( X \) is called a *quotient quasi-antiorder* (abbreviated to Q-quasi-antiorder) on \( X \) if holds

\[
\alpha \subseteq \ker \pi \circ \rho \circ \ker \pi.
\]

Let \( \varphi : (X, =, \neq, \alpha) \rightarrow (Y, =, \neq, \beta) \) be a strongly extensional reverse isotone mapping between anti-ordered sets. Then, by Lemma 2.2, the relation \( \varphi^{-1}(\beta) \) is a quasi-antiorder on \( X \) with \( \varphi^{-1}(\varphi) \cup (\varphi^{-1}(\beta)) = \text{Coker} \varphi \), and \( X/\text{Coker} \varphi \cong \text{Im} \varphi \) as anti-ordered sets. Besides, holds \( \varphi^{-1}(\beta) \subseteq \alpha \) because \( \varphi \) is a reverse isotone mapping. A little generalization of notion introduced in the Definition 1 is the following notion:

**Definition 2** Let \((X, =, \neq, \alpha)\) and \((Y, =, \neq, \beta)\) be anti-ordered sets. A reverse isotone strongly extensional mapping \( \varphi : X \rightarrow Y \) is called a *quotient anti-ordered mapping* (abbreviated to QA-mapping) of \( X \) to \( Y \) if holds

\[
\alpha \subseteq \ker \pi \circ \varphi^{-1}(\beta) \circ \ker \pi.
\]

In the case when \( \varphi \) is onto, \( T \) is called a *quotient anti-ordered set* of \( S \).

In the following theorem a characteristic of Q-quasi-antiorder is present:

**Theorem 3.2** Let \((X, =, \neq)\) be an anti-ordered set and \( \rho \) a Q-quasi-antiorder on \( X \). Then \( \pi : X \rightarrow X/(\rho \cup \rho^{-1}) \) is a QA-mapping from \( X \) onto \( X/(\rho \cup \rho^{-1}) \). Thus, \( X/(\rho \cup \rho^{-1}) \) is a quotient anti-ordered set of \( X \).

**Proof:** Let \( \rho \) is a Q-quasi-antiorder relation on \( X \). Then \( q = \rho \cup \rho^{-1} \) is a coequality relation on \( X \) and \( \theta \) on \( X/q \), defined by \((aq, bq) \in \theta \leftrightarrow (a, b) \in \rho\), is an anti-order on \( X/q \) and the mapping \( \pi : X \rightarrow X/q \), defined by \( \pi(a) = aq \quad (a \in X) \), is a strongly extensional reverse isotone mapping from \( X \) onto \( X/q \) by Lemma 2.1. Since \( \rho \) is a Q-quasi-antiorder relation on \( X \), then the inclusion \( \alpha \subseteq \ker \pi \circ \rho \circ \ker \pi \) holds. Besides, since \( \rho = \theta^{-1}(\theta) \), we have \( \alpha \subseteq \ker \pi \circ \pi^{-1}(\theta) \circ \ker \pi \). Therefore, \( \pi \) is a QA-mapping from \( X \) onto \( X/q \). \( \square \)

In the next assertion we give a connection between QA-mappings and Q-quasi-antiorders on anti-ordered sets.

**Theorem 3.3** Let \((X, =, \neq, \alpha)\) and \((Y, =, \neq, \beta)\) be anti-ordered sets and \( \varphi : X \rightarrow Y \) a strongly extensional reverse isotone QA-mapping. Then,
\( \varphi^{-1}(\beta) \) is a Q-quasi-antiorder on \( X \) with \( \varphi^{-1}(\beta) \cup (\varphi^{-1}(\beta))^{-1} = \text{Coker}\varphi \).

**Proof:** Let \( \varphi : X \rightarrow Y \) be a strongly extensional reverse isotone QA-mapping. Then, by Lemma 2.2, \( \varphi^{-1}(\beta) \) is a quasi-antiorder on \( X \) such that \( \varphi^{-1}(\beta) \cup (\varphi^{-1}(\beta)) = \text{Coker}\varphi \). Since, by Definition 2, we have \( \alpha \subseteq \text{Ker}\varphi \circ \varphi^{-1}(\beta) \circ \text{Ker}\varphi \), then \( \varphi^{-1}(\beta) \) is Q-quasi-antiorder relation on \( X \). \( \square \)

## 4 Isomorphism theorems

In this section we present two isomorphism theorems on QA-mappings and Q-quasi-antiorder.

**Theorem 4.1** (First Isomorphism theorem) Let \( (X, =, \neq, \alpha) \) and \( (Y, =, \neq, \beta) \) be anti-ordered sets and \( \varphi : X \rightarrow Y \) a QA-mapping and \( \rho \) a Q-quasi-antiorder on \( X \). Then, \( \rho \supseteq \varphi^{-1}(\beta) \) if and only if there is a unique QA-mapping \( \psi : X/(\rho \cup \rho^{-1}) \rightarrow Y \) such that \( \varphi = \psi \circ \pi \). Moreover, \( \text{Im}\varphi = \text{Im}\psi \).

**Proof:**

\((\Rightarrow)\) Let \( (\alpha \supseteq) \rho \supseteq \varphi^{-1}(\beta) \). By Lemma 2.3, there exists a unique reverse isotone strongly extensional mapping \( \psi \) from \( X/\text{Coker}\varphi \) to \( T \) such that \( \varphi = \psi \circ \pi \) with \( \text{Im}\varphi = \text{Im}\psi \). Further on, \( q = \rho \cup \rho^{-1} \) is a coequality on \( X \) and \( \pi : X \rightarrow X/q \) is a strongly extensional reverse isotone QA-mapping from \( X \) onto \( (X/q, =_1, \neq_1, \theta) \) by Theorem 3.2. Besides, let \( (aq, bq) \in \theta \) be an arbitrary element. Then, by definition of \( \theta \), we have \( (a, b) \in \rho \). Since \( \rho \) is a Q-quasi-antiorder on \( X \) and \( \varphi \) is QA-mapping, the inclusion

\[ \rho \subseteq \alpha \subseteq \text{Ker}\varphi \circ \varphi^{-1}(\beta) \circ \text{Ker}\varphi \]

is valid. Thus, there exist elements \( x, y \in X \) such that \( (a, x) \in \text{Ker}\varphi \) and \( (y, b) \in \text{Ker}\varphi \), i.e. there exist elements \( x, y \in X \) such that

\[ \varphi(a) = \varphi(x) \land (x, y) \in \varphi^{-1}(\beta) \land \varphi(y) = \varphi(b) \]

i.e. we have element \( x, y \in X \) such that

\[ \psi(\pi(a)) = \psi(\pi(x)) \land (\pi(x), \pi(y)) \in \psi^{-1}(\beta) \land \psi(\pi(y)) = \psi(\pi(b)) \].

Finally, we have

\[ (\pi(a), \pi(b)) \in \text{Ker}\psi \circ \psi^{-1}(\beta) \circ \text{Ker}\psi \].

So, the inclusion

\[ \theta \subseteq \text{Ker}\psi \circ \psi^{-1}(\beta) \circ \text{Ker}\psi \]

is proved. Therefore, \( \psi : (X/q, =_1, \neq_1, \theta) \rightarrow (Y, =, \neq, \beta) \) is QA-mapping.

\((\Leftarrow)\) This part of proof immediately follows from Lemma 2.3. \( \square \)
Theorem 4.2 (Second Isomorphism Theorem) Let \((X,\neq,\neq,\alpha)\) be a set, \(\rho\) and \(\sigma\) \(Q\)-quasi-antiorders on \(X\) such that \(\sigma \subseteq \rho\). Then the relation \(\sigma/\rho\), defined by
\[
\sigma/\rho = \{(x(\rho \cup \rho^{-1}), y(\rho \cup \rho^{-1})) \in X/(\rho \cup \rho^{-1}) \times X/(\rho \cup \rho^{-1}) : (x, y) \in \sigma\},
\]
is a \(Q\)-quasi-antiorder on \(X/(\rho \cup \rho^{-1})\) and
\[
(X/(\rho \cup \rho^{-1}))/((\sigma/\rho)/(\sigma/\rho)^{-1}) \cong X/(\sigma \cup \sigma^{-1})
\]
holds as anti-ordered sets.

**Proof:** By Lemma 2.4, the relation \(\sigma/\rho\) is a quasi-antiorder on \((X/(\rho \cup \rho^{-1}),=_1,\neq_1,\theta)\) and
\[
(X/(\rho \cup \rho^{-1}))/((\sigma/\rho)/(\sigma/\rho)^{-1}) \cong X/(\sigma \cup \sigma^{-1})
\]
holds as anti-ordered sets. Let \(\pi_t\) be the natural strongly extensional reverse isotone mapping from \((X/(\rho \cup \rho^{-1})),(X/(\sigma \cup \sigma^{-1}))\) onto \((X/(\rho \cup \rho^{-1}))/((\sigma/\rho)/(\sigma/\rho)^{-1}).\) We need to prove only that \(\sigma/\rho\) is \(Q\)-quasi-antiorder on \(X/(\rho \cup \rho^{-1})\), i.e. we need to prove
\[
\theta \subseteq Ker\pi_t \circ \sigma/\rho \circ Ker\pi_t.
\]
Let \(q = \rho \cup \rho^{-1}, p = \sigma \cup \sigma^{-1}, t = (\sigma/\rho)/(\sigma/\rho)^{-1}\) and \((a_q, b_q)\) be an arbitrary element of \(\theta\). Then, by definition of \(\theta\), \((a, b) \in \rho \subseteq \alpha\). Since \(\sigma\) is a \(Q\)-quasi-antiorder on \(X\), we have
\[
(a, b) \in \rho \subseteq \alpha \subseteq Ker\pi_{\sigma} \circ \sigma \circ Ker\pi_{\sigma}.
\]
Thus, there exist elements \(x, y\) of \(X\) such that
\[
\pi_{\sigma}(a) = \pi_{\sigma}(x) \wedge (x, y) \in \sigma \wedge \pi_{\sigma}(y) = \pi_{\sigma}(b),
\]
i.e. such that
\[
(a, x) \precsim_p \sigma \wedge (xq, yq) \in \sigma/\rho \wedge (y, b) \precsim psupseteq \sigma.
\]
Further on, let \((uq, vq)\) be an arbitrary element of \(t\), i.e. let \((u, v)\) be an arbitrary element of \(\sigma\). Thus, we have \(((u, a) \in b\sigma \vee (a, x) \in \sigma \vee (x, v) \in \sigma)\).

So, we conclude
\[
((uq)t \neq_3 (xq)t) \wedge ((xq)t \neq_3 (vq)t)
\]
because the case \((a, x) \in \sigma \subseteq p\) is impossible. Therefore, we have \((aq, xq) \precsim t\).

Analogously, we have that \((yq, bq)\) in \(t\) also. Finally, we have
\[
(aq)t =_3 (xq)t \wedge (xq, yq) \in \sigma/\rho \wedge (yq)t =_3 (bq)t,
\]
i.e. we have \((aq, bq) \in Ker\pi_t \circ \sigma/\rho \circ Ker\pi_t. □

References


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