Topological Properties of the Composition of Polynomials of the Form $z(z^d + c_n)$

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Abstract

We consider the set $D_\infty := \{(c_n) \in K^N_\delta; J_{(c_n)}$ has infinity many components$\}$. The aim of this paper is to show that $D_\infty$ is of the second Baire category in $K^N_\delta$.

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1 Introduction

For a sequence $(c_k)$ of complex numbers, we consider the polynomials $f_{c_k}(z) = z(z^d + c_k)$, and the sequence $(F_k)$ of iterates $F_k := f_{c_k} \circ \cdots \circ f_1$. A polynomial $f_{c_k}(z) = z(z^d + c_k)$ has $d$ critical points (counting with multiplicity when $c = 0$):

$$\rho_k, \omega \rho_k, \cdots, \omega^{d-1} \rho_k,$$

where $\omega = e^{2\pi i/d}$ and $\rho_k$ is one of the solutions of $(d + 1)z^d + c_k = 0$, i.e.,

$\rho_k = \left(-\frac{c_k}{d+1}\right)^{1/d}$.

Let $C_{(c_k)} := \bigcup_{k=1}^{\infty} F_k^{-1}(\rho_{k+1})$, be the critical set of sequence $(F_k)$ and $J_{(c_k)}$ is the Julia set, according to the classical iteration theory.

In [?] we proved the following.

**Theorem** The Julia set $J_{(c_n)}$ is connected if and only if $C_{(c_n)}$ is bounded.

If $c_k = c$ for all $k$, the Mandelbrot set $M_d$ is defined as the set of all $c \in \mathbb{C}$ such that $J_c$ is connected. It is well known that $J_c$ is either connected or
totally disconnected depending on whether \( c \in M_d \) or not [?].

We consider the following sets:

\[
K_\delta^N := \overbrace{K_\delta \times \cdots \times K_\delta}^{\text{N times}}, \text{ such that } K_\delta := \{ z \in \mathbb{C}; \left| z \right| \leq \delta \} \text{ for some } \delta > 0,
\]

\[
\mathcal{D} := \{(c_n) \in K_\delta^N : \mathcal{J}_{(c_n)} \text{ is connected}\},
\]

\[
\mathcal{D}_N := \{(c_n) \in K_\delta^N : \mathcal{J}_{(c_n)} \text{ has more than N components}\},
\]

\[
\mathcal{D}_\infty := \{(c_n) \in K_\delta^N : \mathcal{J}_{(c_n)} \text{ has infinity many components}\},
\]

\[
\mathcal{I} := \{(c_n) \in K_\delta^N : \mathcal{J}_{(c_n)} \text{ is totally disconnected}\}.
\]

Obviously, we have \( \mathcal{I} \subset \mathcal{D}_\infty \subset \mathcal{D}_N \subset \mathcal{D} \).

Now, we equip \( K_\delta^N \) with the product topology, where \( K_\delta \) carries the usual topology induced from \( \mathbb{C} \). Note that \( K_\delta^N \) is a complete metric space, and a sequence \( \{(c_{mn})\}_{m=1}^\infty \) in \( K_\delta^N \) is convergent to some \( (c_0^n) \in K_\delta^N \) if and only if it is pointwise convergent, that is, \( c_{mn} \to c_0^n (m \to \infty) \) for all \( n \in \mathbb{N} \).

## 2 Main Results

In this section we study topological properties of the sets \( \mathcal{D}, \mathcal{D}_N, \mathcal{D}_\infty \) and \( \mathcal{I} \). Here, we consider \( \gamma \) such that \( \gamma^d + \gamma \leq 1 \).

The main goal is to show that \( \mathcal{D}_\infty \) is of the second Baire category in \( K_\delta^N \). This will be done in several steps. For that purpose we recall the notion of Baire category.

Let \( X \) be a topological space. A set \( A \subset X \) is nowhere dense in \( X \) if the closure \( \overline{A} \) has empty interior, and \( A \) is of the first Baire category in \( X \) if \( A \) is a countable union of nowhere dense sets. Otherwise \( A \) is of the second Baire category. From Baire’s category theorem we know that non-empty open subsets of a complete metric space \( X \) are of the second category in \( X \).

**Theorem 2.1** The set \( \mathcal{I} \) is dense in \( K_\delta^N \) provided that \( \delta > \gamma \).

**Proof.** Let \( (c_0^n) \in K_\delta^N \). We define a sequence \( \{(c_{mn})\}_{m=1}^\infty \) in \( K_\delta^N \) by

\[
c_{mn} := \begin{cases} c_0^n & \text{for } n = 1, \ldots, m, \\ c & \text{for } n > m, \end{cases}
\]

where \( c \in K_\delta^N - M_d \). Then \( (c_{mn}) \to (c_0^n) (m \to \infty) \) and \( \mathcal{J}_{(c_n)} \) is totally disconnected. Since \( \mathcal{J}_{(c_m)} = (f_{(c_0)}) \circ \cdots \circ f_{(c_1)}^{-1}(\mathcal{J}(f_c)) \) the Julia sets \( \mathcal{J}_{(c_m)} \) are also totally disconnected which means that \( (c_{mn}) \in \mathcal{I} \) for all \( m \in \mathbb{N} \).

**Theorem 2.2** The set \( \mathcal{D}_\infty \) has empty interior.

**Proof.** Let \( (c_0^n) \in \mathcal{D}_\infty \). We define a sequence \( \{(c_{mn})\}_{m=1}^\infty \) in \( K_\delta^N \) by

\[
c_{mn} := \begin{cases} c_0^n & \text{for } n = 1, \ldots, m, \\ c & \text{for } n > m, \end{cases}
\]

where \( c \in K_\delta^N - M_d \). Then \( (c_{mn}) \to (c_0^n) (m \to \infty) \) and \( \mathcal{J}_{(c_n)} \) is totally disconnected. Since \( \mathcal{J}_{(c_{mn})} = (f_{(c_0)}) \circ \cdots \circ f_{(c_1)}^{-1}(\mathcal{J}(f_c)) \), the Julia sets \( \mathcal{J}_{(c_m)} \) are also totally disconnected which means that \( (c_{mn}) \in \mathcal{I} \) for all \( m \in \mathbb{N} \). \( \square \)
where $c \in K_\delta^N \cap \mathcal{M}_d$. Then $(c_n^m) \to (c_0^0)(m \to \infty)$ and $\mathcal{J}(f_c)$ is connected. 
Since $\mathcal{J}(c_n^m) = (f_{c_n^m} \circ \cdots \circ f_{c_1^m})^{-1}(\mathcal{J}(f_c))$, the Julia sets $\mathcal{J}(c_n^m)$ have only finitely many components which means that $(c_n^m) \notin \mathcal{D}_\infty$ for all $m \in \mathbb{N}$. \hfill \Box

**Theorem 2.3** The set $\mathcal{D}$ is a dense open subset of $K_\delta^N$ provided that $\delta > \gamma$.

**Proof.** Let $(c_n^0) \in \mathcal{D}$. There exists $z_0 \in \mathbb{C}$ such that $F_m(z_0) = (f_{c_n^0} \circ \cdots \circ f_{c_1^0})(z_0) = \rho_{m+1} \in \mathcal{C}_{m+1}$ for some $m \in \mathbb{N}$ and $F_n(z_0) \to \infty(n \to \infty)$ which implies that $(f_{c_n^0} \circ \cdots \circ f_{c_1^0}^{-1}(\rho_{m+1})) \to \infty(n \to \infty)$. Therefore we may choose $R$ so large and $N \in \mathbb{N}$, $N > m$ such that $|(f_{c_n^0} \circ \cdots \circ f_{c_1^0})(\rho_{m+1})| > R$. Since $(f_{c_n^0} \circ \cdots \circ f_{c_1^0})$ depends continuously on $c_{m+1}, \ldots, c_N$ there exists a neighborhood $U = U_{m+1} \times \cdots \times U_N \subset K_\delta^{N-m}$ of $(c_{m+1}^0, \ldots, c_N^0)$ such that $|(f_{c_n^0} \circ \cdots \circ f_{c_1^0})(\rho_{m+1})| > R$ for all $(c_{m+1}, \ldots, c_N) \in U$. We set $\mathcal{U} := K_m^m \times U \times K_\delta^N$. Then $\mathcal{U}$ is a neighborhood of $(c_n^0)$ with respect to the product topology of $K_\delta^N$.

In order to show that $\mathcal{U} \subset \mathcal{D}$, let $(c_n) \in \mathcal{U}$. We choose $\zeta \in \mathbb{C}$ with $(f_{c_n} \circ \cdots \circ f_{c_1})(\zeta) = \rho_{m+1}$. We have $(c_{m+1}, \ldots, c_N) \in U$ and thus $|(f_{c_n} \circ \cdots \circ f_{c_1}(\rho_{m+1})| > R$ which means that $|(f_{c_n} \circ \cdots \circ f_{c_1})(\zeta)| > R$. This implies that $|(f_{c_n} \circ \cdots \circ f_{c_1})(\zeta)| \to \infty(n \to \infty)$ and thus $(c_n) \in \mathcal{D}$.

Finally, to show that $\mathcal{D}$ is dense in $K_\delta^N$, let $(c_n^0) \in K_\delta^N$. We define a sequence $(c_n^m)_{m=1}^\infty$ in $K_\delta^N$ by

$$c_n^m := \begin{cases} c_n^0 & \text{for } n = 1, \ldots, m, \\ c & \text{for } n > m, \end{cases}$$

where $c \in K_\delta^N \setminus \mathcal{M}_d$. Then $(c_n^m) \to (c_0^0)(m \to \infty)$ and $\mathcal{J}(f_c)$ is disconnected. Since $\mathcal{J}(c_n^m) = (f_{c_n^m} \circ \cdots \circ f_{c_1^m})^{-1}(\mathcal{J}(f_c))$, the Julia sets $\mathcal{J}(c_n^m)$ are also disconnected which means that $(c_n^m) \notin \mathcal{D}_\infty$ for all $m \in \mathbb{N}$.

**Lemma 2.4** Let $(c_n) \in K_\delta^N$, and assume that $\mathcal{J}(c_{n+1})$ has exactly $p$ components. Then $\mathcal{J}_n$ either has exactly $2p$ or $2p-1$ components.

**Proof.** Let $J_1, \ldots, J_p$ be the components of $\mathcal{J}(c_{n+1})$, and for $j \in \{1, \ldots, p\}$ let $U_j$ be the unbounded component of $\mathbb{C} - J_j$. We have $\mathcal{J}_n = f_{c_1}^{-1}(\mathcal{J}(c_{n+1}))$. If $c_1 \in U_j$ for all $j = 1, \ldots, p$, then $f_{c_1}^{-1}(J_j)$ has two components for all $j = 1, \ldots, p$, so that $\mathcal{J}_n$ has $2p$ components. Now, let $c_1 \notin U_{j_0}$ for some $j_0 \in \{1, \ldots, p\}$. Since $\mathcal{J}_n = \partial \mathcal{F}_n$ the maximum modulus principle implies that all bounded components of $\mathcal{F}_n$ are simply connected. This gives $c_1 \in U_{j_0}$ for some $j_0 \in \{1, \ldots, p\}, j \neq j_0$, thus $\mathcal{J}_n$ has $2p-1$ components. \hfill \Box

**Theorem 2.5** For $n \in \mathbb{N}$ the set $\mathcal{D}_N$ is a dense open subset of $K_\delta^N$ provided that $\delta > \gamma$.

**Proof.** we set $U := \{(c_n) \in K_\delta^N; \mathcal{J}_n \text{ has at most } N \text{ components}\}$,
and we show by induction that \( U_N \) is closed. For \( N = 1 \) this follows from theorem 2.3.

We assume that \( U_N \) is closed for all \( k = 1, \ldots, N \). Let \( \{(c_n^m)\}_{m=1}^{\infty} \) be a sequence in \( U_{N+1} \) which is convergent to some \((c_0^m) \in K_\delta^N\). We have \( J_{(c_0^m)} = f_{(c_0^m)}^{-1}(J_{(c_0^{m+1})}) \). If \( p_m \in \mathbb{N} \) denote the number of components of \( J_{(c_0^{m+1})} \), then by Lemma 2.4 we have \( 2p_m - 1 \leq N + 1 \), and thus \( p_m \leq N \). We first assume that \( 2p_m \leq N + 1 \) for all sufficiently large \( m \). Then by induction we get \( 2p_0 \leq N + 1 \). Therefore, by Lemma 2.4, \( J_{(c_0^m)} \) has at most \( N + 1 \) components which gives \((c_0^0) \in U_{N+1}\).

Now let \( 2p_m - 1 = N + 1 \) for infinitely many \( m \). By passing to a subsequence we may assume that this holds for all \( m \in \mathbb{N} \). For every \( m \) there exists a component \( J_m \) of \( J_{(c_0^{m+1})} \) such that \( f_{(c_0^{m+1})}^{-1}(J_m) \) is connected and yields \((c_0^m) \in K_{(c_0^{m+1})}\).

This means that \(|(f_{(c_0^m)} \circ \cdots \circ f_{(c_0^1)})(c_0^1)| \leq R \) for all \( m, n \in \mathbb{N} \). Fixing \( n \), and letting \( m \to \infty \), we obtain \(|(f_{(c_0^m)} \circ \cdots \circ f_{(c_0^1)})(c_0^1)| \leq R \) for all \( n \in \mathbb{N} \) and thus \((c_0^0) \in K_{(c_0^m+1)}\). By induction we have \( 2p_0 - 1 \leq N + 1 \). Therefore, by Lemma 2.4, \( J_{(c_0^m)} \) has at most \( N + 1 \) components which gives \((c_0^0) \in U_{N+1}\).

Finally, to show that \( D_N \) is dense in \( K_\delta^N \), let \((c_0^0) \in K_\delta^N \). We define a sequence \( \{(c_1^m)\}_{m=1}^{\infty} \) in \( K_\delta^N \) by

\[
c_1^m := \begin{cases} c_0^0 & \text{for } n = 1, \ldots, m, \\ c & \text{for } n > m, \end{cases}
\]

where \( c \in K_\delta^N - \mathcal{M}_d \). Then \((c_1^m) \to (c_0^0) (m \to \infty) \) and \( J_f(c) \) has more than \( N \) components. Since \( J_{(c_0^m)} = (f_{(c_0^m)} \circ \cdots \circ f_{(c_0^1)})(J_f(c)) \), the Julia sets \( J_{(c_0^m)} \) are also has more than \( N \) components which means that \((c_1^m) \in D_N \) for all \( m \in \mathbb{N} \).

\[\square\]

**Theorem 2.6** The set \( D_\infty \) is a countable intersection of dense open subset of \( K_\delta^N \) provided that \( \delta > \gamma \). In particular, is of the second Baire category in \( K_\delta^N \) while the component \( K_\delta^N - \mathcal{D}_\infty \) is of the first Baire category in \( K_\delta^N \).

**Proof.** The assertion immediately follows from theorem 2.5. \( \square \)

**References**


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