Int. J. Contemp. Math. Sciences, Vol. 3, 2008, no. 3, 131 - 143

On the Survival Time of a Warm Standby System

Edmond J. Vanderperre[1]

University of South Africa
Department of Decision Sciences
P.O. Box 392, Pretoria 0003, South Africa
van1939@gmail.com

Abstract

We analyse the survival function of a duplex system characterized by warm standby and attended by two repairmen. In order to obtain the survival function of the system, we employ a stochastic process endowed with time–dependent transition measures satisfying coupled partial differential equations. The solution procedure is based on the theory of sectionally holomorphic functions combined with the notion of dual transforms. As an example, we consider the particular case of deterministic repair.

Mathematics Subject Classification: 60K10, 35M10

Keywords: reliability, warm standby, failure rate, dual transform, functional equation, sectionally holomorphic function, transition measure, survival function

1 Introduction

Standby provides a powerful tool to enhance the reliability, availability, quality and safety of operational plants, e.g. [1,5,6,7]. A particular variant of Gaver's parallel system, e.g.[11] is Birolini's duplex system [2], henceforth called a **B**-system. The **B**-system consists of an active unit (called the **o**-unit) sustained by an identical unit in warm standby (called the **s**-unit). The **B**-system is attended by a single repairman. The notion of warm standby signifies that the failure-free time of the **s**-unit is stochastically larger [8] than the failure-free time of the **o**-unit. Note that the warm standby mode is often indispensable to perform an instantaneous switch from standby into the operative state,

---

[1] Ruzettelaan 183, Bus 158, B-8370 Blankenberge, Belgium

allowing continuous operation of a system upon failure of the on-line unit. Furthermore, any switch from standby to the operative state changes the failure rate of the **s**-unit into the failure rate of the **o**-unit. The **B**-system is down if, and only if, both units are down. Otherwise, the **B**-system is up. The **B**-system acts as a closed queueing system evolving in time, i.e. any failed unit goes immediately into repair provided that the repairman is idle. Otherwise, the failed **o**-unit has to wait for repair. On the other hand, any repaired unit lines-up in warm standby if the remaining unit is active. Otherwise, the repaired unit becomes instantaneously operative.

Industrial applications of warm standby systems have been presented in [9]. The analysis of duplex systems is far from complete! As a variant, we consider a **B**-system attended by *two* repairmen, henceforth called a **T**-system. Repairman R is skilled in repairing **o**-failures, whereas repairman R_s is an expert in repairing **s**-failures. Both repairmen are (jointly) busy if, and only if, the **o**-unit and the **s**-unit are down. Otherwise, at least one repairman is idle. Finally, note that any **o**-failure is always allocated to repairman R, whereas a **s**-failure is always directed to repairman R_s. Consequently, repairman R_s can be idle if the **T**-system is down.

In order to determine the *survival* function of the **T**-system, we introduce a stochastic process endowed with time–dependent transition measures satisfying coupled partial differential equations. The solution procedure is based on the theory of *sectionally* holomorphic functions, e.g. [4], combined with the notion of *dual* transforms. Finally, note that our time-dependent differential equations are generalizing the steady–state equations obtained in [13]. As an example, we consider the case of deterministic repair (replacement).

2 Formulation

Consider the **T**-system subjected to the following conditions :

- The **o**-unit has a general failure–free time distribution $F(\cdot), F(0) = 0$ and a general repair time distribution $R(\cdot), R(0) = 0$. The failure-free time and the repair time are respectively denoted by f and r. We assume that $F(\cdot)$ is Lebesgue–absolutely continuous with density function (in the Radon–Nikodym sense) of bounded variation on $[0, \infty)$.
- The **s**-unit has a constant failure rate λ_s and a general repair time distribution $R_s(\cdot), R_s(0) = 0$. The failure-free time and the repair time are respectively denoted by f_s and r_s.
- The random variables $f,\ f_s,\ r,\ r_s$ are supposed to be statistically independent with finite mean. In addition, we assume that r_s has a finite variance and that f_s is stochastically larger than f.

- λ_s changes into the failure rate of the **o**–unit upon switch from standby into the operative state.
- Any repair is perfect.
- Characteristic functions are formulated in terms of a complex transform variable. For instance,

  $$\mathbf{E}e^{i\omega r} = \int_0^\infty e^{i\omega x} dR(x), \text{ Im } \omega \geq 0.$$

  Note that

  $$\mathbf{E}e^{-i\omega r} = \int_{-\infty}^0 e^{i\omega x} d\left\{1 - R((-x)-)\right\}, \text{ Im } \omega \leq 0.$$

  The corresponding Fourier-Stieltjes-transforms are called *dual* transforms. Without loss of generality (See forthcoming Remarks 7.1), we may assume that R and R_s have density functions of bounded variation on $[0, \infty)$. Note that the bounded variation property implies that, for instance,

  $$\left|\mathbf{E}e^{i\tau f}\right| = O\big(\frac{1}{|\tau|}\big), \quad |\tau| \to \infty. \tag{2.1}$$

- The **T**-system is up (available) if, and only if, at least one unit is up. Otherwise, the the **T**-system is down. We recall that any **o**-failure is always allocated to repairman R, whereas a **s**-failure is always directed to repairman R_s. In order to describe the random behaviour of the **T**-system, we employ a stochastic process $\{N_t, \, t \geq 0\}$ with discrete state space $\{A, B, C, D\} \subset [0, \infty)$ characterized by the following mutually exclusive events:

  $\{N_t = A\}$: "The **T**-system is up and both repairmen are idle at time t."

  $\{N_t = B\}$: "The **T**-system is up and repairman R_s is busy at time t."

  $\{N_t = C\}$: "The **T**-system is up and repairman R is busy at time t."

  $\{N_t = D\}$: "The **T**-system is down at time t, i.e. repairman R is busy and repairman R_s is idle *or* both repairmen are busy at time t."

  State A is called the safe state. States B and C are called risky states and state D is called the system down state. The non–Markovian process $\{N_t\}$ is defined on a filtered probability space $\{\Omega, \, \mathcal{A}, \, \mathbf{P}, \, \mathcal{F}\}$ where the *history* $\mathcal{F} := \{\mathcal{F}_t, \; t \geq 0\}$ satisfies the Dellacherie conditions

  - $\mathcal{F}_0$ contains the **P**–null sets of $\mathcal{A}$,

- $\forall\, t \geq 0,\ \mathcal{F}_t = \bigcap_{u>t} \mathcal{F}_u$, i.e. the family $\mathcal{F}$ is right–continuous.

Consider the $\mathcal{F}$–stopping time

$$\theta := \inf \{t : N_t = D | N_0 = A,\ V_0 = 0\},$$

where V_t is the past failure–free time of the **o**–unit being operative at time t. We assume that the **T**–system starts functioning at some time origin $t = 0$ is state A, i.e. let $N_0 = A, V_0 = 0$, **P**-a.s. Thus, from $t = 0$ onwards, θ is the *survival* time (lifetime) of the **T**–system. The corresponding survival function is denoted by $\Re(\cdot)$. Clearly, $\Re(t) = \mathbf{P}\{\theta > t\}, t \geq 0$. A (vector) Markov characterization of the non–Markovian process $\{N_t,\ t \geq 0\}$ with absorbing state D, is piecewise and conditionally defined by:

$\{(N_t,\ U_t)\}$, if $N_t = A$ (i.e. if the event $\{N_t = A\}$ occurs), where U_t denotes the remaining failure-free time of the **o**-unit being up at time t.
$\{(N_t, U_t, Y_t)\}$, if $N_t = B$, where Y_t denotes the remaining repair time of the **s**–unit being under progressive repair at time t.
$\{(N_t, U_t, X_t)\}$ if $N_t = C$, where X_t denotes the remaining repair time of the **o**–unit being under progressive repair at time t.
$\{N_t\}$, if $N_t = D$ (the absorbing state).

The state space of the underlying Markov process, with absorbing state D, is given by

$$\{(A, u)\} \bigcup \{(B, u, y)\} \bigcup \{(C, u, x)\} \bigcup \{D\}, \quad u \geq 0, x \geq 0, y \geq 0.$$

For $K = A, B, C, D$, let $p_K(t) := \mathbf{P}\{N_t = K\}$, $t \geq 0$.

- Finally, we introduce the transition measures

$$\begin{aligned} p_A(t, u)du &:= \mathbf{P}\{N_t = A,\ U_t \in du\}, \\ p_B(t, u, y)du\, dy &:= \mathbf{P}\{N_t = B,\ U_t \in du,\ Y_t \in dy\}, \\ p_C(t, u, x)du\, dx &:= \mathbf{P}\{N_t = C,\ U_t \in du,\ X_t \in dx\}. \end{aligned}$$

Note that, for instance,

$$p_C(t) = \int_0^\infty\!\!\int_0^\infty d_u\, d_x\, \mathbf{P}\{N_t = C,\ U_t \leq u,\ X_t \leq x\} = \int_0^\infty\!\!\int_0^\infty p_C(t, u, x)du\, dx.$$

3 Notations

■ The indicator (function) of an event $\{N_t = K\}$ is denoted by $\mathbf{1}\{N_t = K\}$.
■ The complex plane and the real line are respectively denoted by **C** and **R** with obvious superscript notations such as $\mathbf{C}^+$ and $\mathbf{C}^-$. For instance, $\mathbf{C}^+ :=$

$\{\omega \in \mathbf{C} : \text{Im } \omega > 0\}$.

■ We frequently use the characteristic function

$$\gamma_s^+(\tau) := \begin{cases} \dfrac{\mathbf{E}e^{i\tau r_s} - 1}{i\tau \mathbf{E}r_s}, & \text{if} \quad \tau \neq 0, \\ \quad 1 \quad , & \text{if} \quad \tau = 0. \end{cases}$$

Note that

$$\gamma_s^+(\omega) = \frac{1}{\mathbf{E}r_s} \int_0^\infty e^{i\omega x}\big(1 - R_s(x)\big)dx, \ \text{Im } \omega \geq 0. \tag{3.1}$$

Property 3.1 [10, Appendix]
The function $1 + \lambda_s \mathbf{E}r_s \gamma_s^+(\omega)$, $\text{Im}\,\omega \geq 0$ *has no zeros in* $\mathbf{C}^+ \bigcup \mathbf{R}$.

■ The Heaviside unit–step function, with the unit–step at $t = t_0$, is denoted by $H_{t_0}(\cdot)$, i.e.

$$H_{t_0}(t) := \begin{cases} 1\,, & \text{if } t \geq t_0 > 0, \\ 0\,, & \text{if } t < t_0. \end{cases}$$

■ The greatest integer function is denoted by $[\cdot]$.

■ The Laplace–transform of any locally integrable and bounded function on $[0, \infty)$ is denoted by the corresponding character marked with an asterisk. For instance,

$$p_A^*(z) := \int_0^\infty e^{-zt} p_A(t)dt, \quad \text{Re } z > 0.$$

Observe that

$$\Re^*(z) = \frac{1 - \mathbf{E}e^{-z\theta}}{z}, \quad \text{Re } z > 0. \tag{3.2}$$

Moreover, by the product rule for Lebesgue–Stieltjes integrals, e.g. [3, Appendix],

$$zp_D^*(z) = \int_{0-}^\infty e^{-zt} dp_D(t) = \mathbf{E}e^{-z\theta}, \quad \text{Re } z > 0. \tag{3.3}$$

■ Let $\phi(\tau), \tau \in \mathbf{R}$ be a bounded and continuous function. $\phi(\cdot)$ is called Γ–integrable if

$$\lim_{\substack{T \to \infty \\ \epsilon \downarrow 0}} \int_{\Gamma_{T,\epsilon}} \phi(\tau) \frac{d\tau}{\tau - u}, \quad u \in \mathbf{R}$$

exists, where $\Gamma_{T,\epsilon} := (-T, u - \epsilon] \cup [u + \epsilon, T)$. The corresponding singular integral, denoted by

$$\frac{1}{2\pi i} \int_\Gamma \phi(\tau) \frac{d\tau}{\tau - u},$$

is called a Cauchy principal value in double sense.

4 Differential Equations

In order to derive a system of differential equations, we observe the random behaviour of the **T**–system in some time interval $(t, t+\Delta), \Delta \downarrow 0$. Grouping terms of $o(\Delta)$ and taking the absorbing state D into account, reveals that

$$p_A(t+\Delta, u-\Delta) = p_A(t,u)(1-\lambda_s\Delta) + p_B(t,u,0)\Delta + p_C(t,u,0)\Delta + o(\Delta),$$

$$p_B(t+\Delta, u-\Delta, y-\Delta) = p_B(t,u,y) + \lambda_s p_A(t,u)\tfrac{\mathrm{d}}{\mathrm{d}y}R_s(y)\Delta + o(\Delta),$$

$$p_C(t+\Delta, u-\Delta, x-\Delta) = p_C(t,u,x) + p_A(t,0)\tfrac{\mathrm{d}}{\mathrm{d}u}F(u)\tfrac{\mathrm{d}}{\mathrm{d}x}R(x)\Delta \ + o(\Delta),$$

$$p_D(t+\Delta) = p_D(t) + \textstyle\int_0^\infty p_B(t,0,y)dy\Delta + \int_0^\infty p_C(t,0,x)dx\Delta + o(\Delta).$$

Taking the definition of *directional* derivative into account, for instance,

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial u} - \frac{\partial}{\partial x}\right) p_C(t,u,x) := \lim_{\Delta\downarrow 0} \frac{p_C(t+\Delta, u-\Delta, x-\Delta) - p_C(t,u,x)}{\Delta},$$

entails that for $t > 0, u > 0, x > 0, y > 0$,

$$\left(\lambda_s + \frac{\partial}{\partial t} - \frac{\partial}{\partial u}\right) p_A(t,u) = p_B(t,u,0) + p_C(t,u,0), \tag{4.1}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial u} - \frac{\partial}{\partial y}\right) p_B(t,u,y) = \lambda_s p_A(t,u)\frac{\mathrm{d}}{\mathrm{d}y}R_s(y), \tag{4.2}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial u} - \frac{\partial}{\partial x}\right) p_C(t,u,x) = p_A(t,0)\frac{\mathrm{d}}{\mathrm{d}u}F(u)\frac{\mathrm{d}}{\mathrm{d}x}R(x), \tag{4.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}p_D(t) = \int_0^\infty p_B(t,0,y)dy + \int_0^\infty p_C(t,0,x)dx. \tag{4.4}$$

Note that the initial condition $N_0 = A, V_0 = 0$, **P**-a.s. implies that

$$p_A(0,u) = \frac{\mathrm{d}}{\mathrm{d}u}F(u), \quad u > 0.$$

Finally, observe that the equations (4.1)–(4.4) are consistent with the probability law $\sum_K p_K(t) = 1$ and that $p_A(0) = 1$.

5 Functional Equation

First, we remark that our system of differential equations is well–adapted to a Laplace–Fourier transformation. As a matter of fact, the transition functions are bounded on their appropriate regions and locally integrable with respect to t. Consequently, each Laplace–transform exists for Re $z > 0$. Moreover, the integrability of the density functions and the transition functions with regard to u, x, y also implies the integrability of the corresponding partial derivatives. Applying a Laplace–Fourier–transform technique to the equations (4.1)–(4.4) and taking the initial condition into account, reveals that for $\mathrm{Re}\ z > 0, \mathrm{Im}\ \omega \geq 0, \mathrm{Im}\ \eta \geq 0, \mathrm{Im}\ \zeta \geq 0$,

$$(z + \lambda_s + i\omega) \int_0^\infty e^{-zt} \mathbf{E}(e^{i\omega U_t} \mathbf{1}\{N_t = A\}) dt + p_A^*(z, 0) =$$

$$\mathbf{E}e^{i\omega f} + \int_0^\infty e^{i\omega u} p_B^*(z, u, 0) du + \int_0^\infty e^{i\omega u} p_C^*(z, u, 0) du, \tag{5.1}$$

$$(z + i\omega + i\eta) \int_0^\infty e^{-zt} \mathbf{E}(e^{i\omega U_t} e^{i\eta Y_t} \mathbf{1}\{N_t = B\}) dt + \int_0^\infty e^{i\omega u} p_B^*(z, u, 0) du +$$

$$\int_0^\infty e^{i\eta y} p_B^*(z, 0, y) dy = \lambda_s \mathbf{E} e^{i\eta r_s} \int_0^\infty e^{-zt} \mathbf{E}(e^{i\omega U_t} \mathbf{1}\{N_t = A\}) dt, \tag{5.2}$$

$$(z + i\omega + i\zeta) \int_0^\infty e^{-zt} \mathbf{E}(e^{i\omega U_t} e^{i\zeta X_t} \mathbf{1}\{N_t = C\}) dt + \int_0^\infty e^{i\omega u} p_C^*(z, u, 0) du +$$

$$\int_0^\infty e^{i\zeta x} p_C^*(z, 0, x) dx = p_A^*(z, 0) \mathbf{E} e^{i\omega f} \mathbf{E} e^{i\zeta r}, \tag{5.3}$$

$$z p_D^*(z) = \int_0^\infty p_B^*(z, 0, y) dy + \int_0^\infty p_C^*(z, 0, x) dx. \tag{5.4}$$

Adding the equations (5.1)–(5.3) yields the functional equation

$$\begin{aligned}
&(z + \lambda_s(1 - \mathbf{E}e^{i\eta r_s}) + i\omega) \int_0^\infty e^{-zt} \mathbf{E}(e^{i\omega U_t} \mathbf{1}\{N_t = A\}) dt \ + \\
&\qquad p_A^*(z, 0)(1 - \mathbf{E}e^{i\omega f} \mathbf{E}e^{i\zeta r}) \ + \\
&(z + i\omega + i\eta) \int_0^\infty e^{-zt} \mathbf{E}(e^{i\omega U_t} e^{i\eta Y_t} \mathbf{1}\{N_t = B\}) dt \ + \\
&(z + i\omega + i\zeta) \int_0^\infty e^{-zt} \mathbf{E}(e^{i\omega U_t} e^{i\zeta X_t} \mathbf{1}\{N_t = C\}) dt \ + \\
&\int_0^\infty e^{i\eta y} p_B^*(z, 0, y) dy \ + \int_0^\infty e^{i\zeta x} p_C^*(z, 0, x) dx \quad = \quad \mathbf{E}e^{i\omega f}\ .
\end{aligned} \tag{5.5}$$

6 Survival function

In order to obtain the Laplace–transform of the *survival* function, we first remark that by equations (5.4) and (3.3)

$$\mathbf{E}e^{-z\theta} = \int_0^\infty e^{i\eta y} p_B^*(z, 0, y) dy \bigg|_{\eta=0} + \int_0^\infty e^{i\zeta x} p_C^*(z, 0, x) dx \bigg|_{\zeta=0}.$$

On the other hand, substituting $\zeta = \eta = 0, \omega = iz, \mathrm{Re}\ z > 0$ into the equation (5.5) yields

$$p_A^*(z,0)(1 - \mathbf{E}e^{-zf}) + \mathbf{E}e^{-z\theta} = \mathbf{E}e^{-zf}.$$

Hence, by equation (3.2),

$$\Re^*(z) = \frac{1 - \mathbf{E}e^{-zf}}{z}(1 + p_A^*(z,0)). \tag{6.1}$$

Consequently, we only have to determine $p_A^*(z,0)$.

7 Solution procedure

In order to derive the unknown $p_A^*(z,0)$, we first eliminate the functions

$$\int_0^\infty e^{-zt}\mathbf{E}(e^{i\omega U_t}e^{i\eta Y_t}\mathbf{1}\{N_t = B\})dt$$

and

$$\int_0^\infty e^{-zt}\mathbf{E}(e^{i\omega U_t}e^{i\zeta X_t}\mathbf{1}\{N_t = C\})dt.$$

Substituting $\omega = -\tau + iz, \zeta = \eta = \tau, \tau \in \mathbf{R}, \mathrm{Re}\ z > 0$ into equation (5.5), noting that $z + i\omega + i\eta = z + i\omega + i\zeta = 0$, yields

$$\begin{aligned}-(\lambda_s(\mathbf{E}e^{i\tau r_s} - 1) + i\tau)\int_0^\infty e^{-zt}\mathbf{E}(e^{-i(\tau - iz)U_t}\mathbf{1}\{N_t = A\})dt\ +\\ p_A^*(z,0)(1 - \mathbf{E}e^{-i(\tau-iz)f}\mathbf{E}e^{i\tau r})\ +\\ \int_0^\infty e^{i\tau x}(p_B^*(z,0,x)\ + p_C^*(z,0,x))dx \quad = \quad \mathbf{E}e^{-i(\tau-iz)f}.\end{aligned}$$

Taking Lemma 3.1 into account, entails that for $\tau \in \mathbf{R}, \mathrm{Re}\ z > 0$,

$$\psi^+(z,\tau) - \psi^-(z,\tau) = \varphi(z,\tau), \tag{7.1}$$

where

$$\psi^+(z,\omega) := \frac{\phi^+(z,\omega)}{1 + \lambda_s\mathbf{E}r_s\gamma_s^+(\omega)},$$

$$\phi^+(z,\omega) := \int_0^\infty e^{i\omega x}(p_B^*(z,0,x)\ + p_C^*(z,0,x))dx - \lambda_s\mathbf{E}r_s\gamma_s^+(\omega)p_A^*(z,0), \quad \mathrm{Im}\ \omega \geq 0.$$

$$\psi^-(z,\omega) := i\omega\int_0^\infty e^{-zt}\mathbf{E}(e^{-i(\omega-iz)U_t}\mathbf{1}\{N_t = A\})dt - p_A^*(z,0), \quad \mathrm{Im}\ \omega \leq 0.$$

$$\varphi(z,\tau) := \varphi_a(z,\tau) + p_A^*(z,0)\varphi_b(z,\tau),$$

$$\varphi_a(z,\tau) := \frac{\mathbf{E}e^{-i(\tau-iz)f}}{1+\lambda_s\mathbf{E}r_s\gamma_s^+(\tau)},$$

$$\varphi_b(z,\tau) := \varphi_a(z,\tau)\mathbf{E}e^{i\tau r}.$$

In order to apply the theory of sectionally holomorphic functions, e.g. [4], we introduce the following

Lemma 7.1

The functions $\varphi_a(z,\tau)$ and $\varphi_b(z,\tau)$ are Lipschitz–continuous on $\mathbf{R}$ and at infinity.

Proof

Note that Lemma 3.1 implies that $\sup_{\tau\in\mathbf{R}}|1+\lambda_s\mathbf{E}r_s\gamma_s^+(\tau)|^{-1} < \infty$. Hence, the existence of $\mathbf{E}f, \mathbf{E}r, \mathbf{E}r_s$ and $\mathbf{E}r_s^2$ entails that

$$\sup_{\tau\in\mathbf{R}}\left|\frac{\partial}{\partial\tau}\varphi_p(z,\tau)\right| < \infty,$$

where (by convention) $p = a, b$. Consequently, by the mean value theorem, there exist constants K_a and K_b such that for all $\tau_1, \tau_2 \in \mathbf{R}$

$$|\varphi_p(z,\tau_1) - \varphi_p(z,\tau_2)| \leq K_p|\tau_1 - \tau_2|.$$

Hence, $\varphi_a(z,\tau)$ and $\varphi_b(z,\tau)$ are Lipschitz–continuous on $\mathbf{R}$.

Finally, note that the Lipschitz–continuity of $\varphi_a(z,\tau)$ and $\varphi_b(z,\tau)$ at infinity follows from the inequality $|\varphi_b(z,\tau)| \leq |\varphi_a(z,\tau)|$ and the order relation (2.1).

Corollary 7.1

The function

$$\mathcal{K}_p(z,\omega) := \frac{1}{2\pi i}\int_\Gamma \varphi_p(z,\tau)\frac{\mathrm{d}\tau}{\tau-\omega}, \quad \omega \in \mathbf{C}$$

is sectionally holomorphic and regular.

In order to proceed with the solution procedure, let $\mathcal{K}_p^-(z,\omega)$ denote the *restriction* of $\mathcal{K}_p(z,\omega)$ to $\mathbf{C}^-$. Note that $\mathcal{K}_p^-(z,\omega)$ is continuous from the right, i.e. for $u \in \mathbf{R}$,

$$\mathcal{K}_p^-(z,u) = \lim_{\substack{\omega\to u\\ \omega\in\mathbf{C}^-}} \mathcal{K}_p^-(z,\omega)$$

and that

$$\mathcal{K}_p^-(z,\omega) = \frac{1}{2\pi i}\int_\Gamma \varphi_p(z,\tau)\frac{\mathrm{d}\tau}{\tau-\omega}, \quad \omega \in \mathbf{C}^-. \tag{7.2}$$

Applying the Sokhotski–Plemelj formula for the region $\mathbf{C}^-$, reveals that

$$\mathcal{K}_p^-(z,u) = -\frac{1}{2}\varphi_p(z,u) + \frac{1}{2\pi i}\int_\Gamma \varphi_p(z,\tau)\frac{\mathrm{d}\tau}{\tau-u}. \tag{7.3}$$

On the other hand, we have by equation (7.1)

$$\psi^-(z,\omega) = \frac{1}{2\pi i}\int_\Gamma \varphi_a(z,\tau)\frac{\mathrm{d}\tau}{\tau-\omega} + p_A^*(z,0)\frac{1}{2\pi i}\int_\Gamma \varphi_b(z,\tau)\frac{\mathrm{d}\tau}{\tau-\omega}, \quad \omega \in \mathbf{C}^-. \tag{7.4}$$

However,

$$\lim_{\substack{\omega\to 0\\ \omega\in\mathbf{C}^-}} \psi^-(z,\omega) = -p_A^*(z,0). \tag{7.5}$$

From equations (7.2)–(7.5) we obtain the relation

$$-p_A^*(z,0) = \mathcal{K}_a^-(z,0) + p_A^*(z,0)\mathcal{K}_b^-(z,0), \tag{7.6}$$

where

$$\mathcal{K}_p^-(z,0) = -\frac{1}{2}\varphi_p(z,0) + \frac{1}{2\pi i}\int_\Gamma \varphi_p(z,\tau)\frac{\mathrm{d}\tau}{\tau}.$$

Consequently, $\Re^*(z)$ is completely determined by equations (7.6) and (6.1). We summarize the following explicit result.

Property 7.1

$$\Re^*(z) = \frac{1-\mathbf{E}e^{-zf}}{z}\Big(1 - \frac{\mathcal{K}_a^-(z,0)}{1+\mathcal{K}_b^-(z,0)}\Big),$$

where

$$\mathcal{K}_a^-(z,0) = -\frac{1}{2}\frac{\mathbf{E}e^{-zf}}{1+\lambda_s\mathbf{E}r_s} + \frac{1}{2\pi i}\int_\Gamma \frac{\mathbf{E}e^{-i(\tau-iz)f}}{1+\lambda_s\mathbf{E}r_s\gamma_s^+(\tau)}\frac{\mathrm{d}\tau}{\tau},$$

$$\mathcal{K}_b^-(z,0) = -\frac{1}{2}\frac{\mathbf{E}e^{-zf}}{1+\lambda_s\mathbf{E}r_s} + \frac{1}{2\pi i}\int_\Gamma \frac{\mathbf{E}e^{-i(\tau-iz)f}\mathbf{E}e^{i\tau r}}{1+\lambda_s\mathbf{E}r_s\gamma_s^+(\tau)}\frac{\mathrm{d}\tau}{\tau},$$

$$\gamma_s^+(\tau) := \begin{cases} \dfrac{\mathbf{E}e^{i\tau r_s}-1}{i\tau\mathbf{E}r_s}, & \text{if} \quad \tau \neq 0, \\ \quad 1 \quad , & \text{if} \quad \tau = 0. \end{cases}$$

Remarks 7.1

It should be noted that Property 3.1 holds for a *general* R_s with finite mean. Clearly, the existence of a moment does not depend on the canonical structure (Lebesgue decomposition) of the underlying distribution. Therefore, Lemma 7.1 also holds for general R and R_s. The requirement of a finite variance $\sigma_{r_s}^2$ is extremely mild. In fact, the current repair time distribution functions of interest to Statistical Reliability Engineering, even have moments of *any* order. Therefore, our initial assumption concerning the existence of repair time density functions is totally superfluous to ensure the existence of the integral

$$\frac{1}{2\pi i}\int_\Gamma \varphi_p(z,\tau)\frac{\mathrm{d}\tau}{\tau-\omega}, \quad \omega \in \mathbf{C}.$$

Example

Let $F(u) = 1 - e^{-\lambda u}, \lambda > \lambda_s$. Note that f_s is stochastically larger than f. Furthermore, let R and R_s be general. We have

$$\mathbf{E}e^{-i(\tau - iz)f} = \frac{-i\lambda}{\tau - i(\lambda + z)} \,.$$

Equation (7.2) yields

$$\mathcal{K}_a^-(z,\omega) = -\frac{1}{2\pi i}\int_\Gamma \frac{i\lambda(\tau-\omega)^{-1}(\tau - i(\lambda+z))^{-1}}{1+\lambda_s \mathbf{E} r_s \gamma_s^+(\tau)}\,\mathrm{d}\tau \,.$$

For $\omega \in \mathbf{C}^-, \mathrm{Re}\ z \geq 0$, the integrand represents a *meromorphic* function in $\mathbf{C}^+$ with single pole $i(\lambda + z)$. Moreover, the function vanishes at infinity in $\mathbf{C}^+ \cup \mathbf{R}$. An application of the residue theorem entails that

$$\mathcal{K}_a^-(z,\omega) = \frac{i\lambda}{\omega - i(\lambda+z)}\,\frac{\lambda + z}{\lambda + z + \lambda_s(1 - \mathbf{E}e^{-(\lambda+z)r_s})} \,.$$

Hence, by continuity,

$$\mathcal{K}_a^-(z,0) = -\frac{\lambda}{\lambda + z + \lambda_s(1 - \mathbf{E}e^{-(\lambda+z)r_s})} \,.$$

In a similar way,

$$\mathcal{K}_b^-(z,0) = -\frac{\lambda \mathbf{E}e^{-(\lambda+z)r}}{\lambda + z + \lambda_s(1 - \mathbf{E}e^{-(\lambda+z)r_s})} \,.$$

Hence, by Property (7.1),

$$\Re^*(z) = \frac{1}{\lambda+z}\Big(1 + \frac{\lambda}{z + \lambda_s(1-\mathbf{E}e^{-(\lambda+z)r_s}) + \lambda(1 - \mathbf{E}e^{-(\lambda+z)r})}\Big) \,.$$

Note that this formula holds for *arbitrary* R and R_s and that

$$\mathbf{E}\theta = \frac{1}{\lambda} + \frac{1}{\lambda_s(1-\mathbf{E}e^{-\lambda r_s}) + \lambda(1-\mathbf{E}e^{-\lambda r})} \,.$$

Finally, we consider the case of *deterministic* repair (replacement), i.e. let $R(\cdot) = R_s(\cdot) = H_{t_0}(\cdot)$, where we take t_0 as time unit. Clearly, $\mathbf{E}e^{-zr} = \mathbf{E}e^{-zr_s} = e^{-z}$. Hence, for $\mathrm{Re}\ z \geq 0$,

$$\Re^*(z) = \frac{1}{\lambda+z}\,\frac{\lambda}{z + \lambda_s + \lambda - (\lambda_s + \lambda)e^{-(z+\lambda)}} \,.$$

Applying the inversion technology presented in [12], yields the *exact* survival function

$$\Re(t) = e^{-\lambda t}\Big(1 + \frac{\lambda}{\lambda_s}\sum_{k=0}^{[t]}\Big(\frac{\lambda}{\lambda+\lambda_s}\Big)^k\big(1 - e^{-\lambda_s(t-k)}\sum_{j=0}^{k}\frac{(\lambda_s(t-k))^j}{j!}\big)\Big) \,.$$

Figure 1 displays the graph of $\Re(t), 0 \leq t \leq 10, \lambda = 0.5, \lambda_s = 0.25$.

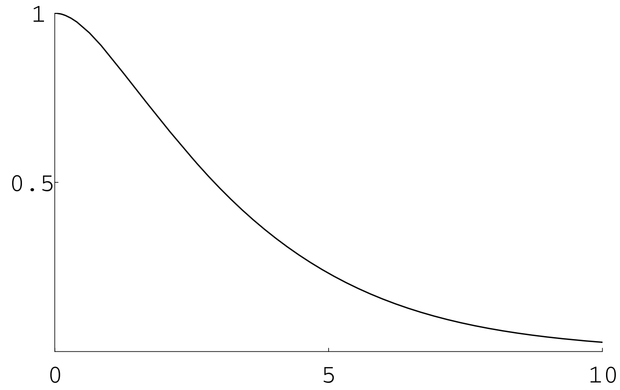

Figure 1: Graph of $\Re(t), 0 \leq t \leq 10, \lambda = 0.5, \lambda_s = 0.25$

References

[1] A. Birolini, *Quality and Reliability of Technical Systems*, Springer–Verlag, Berlin, 1994.

[2] A. Birolini, On the use of stochastic processes in modeling reliability problems, *Lecture notes in Economics and Mathematical Systems* **252**, Springer–Verlag Berlin, 1985.

[3] P. Brémaud, *Point Processes and Queues*, Springer Series in Statistics, Springer–Verlag, Berlin, 1991.

[4] F.D. Gakhov, *Boundary Value Problems*, Pergamon Press, Oxford, 1996.

[5] L. Gertsbakh, *Statistical Reliability Theory*, Marcel Dekker, New York, 2001.

[6] B. Gnedenko and I.A. Ushakov, *Probabilistic Reliability Engineering*, John Wiley & Sons, New York, 1995.

[7] H. Pham, *Recent Advances in Reliability and Quality Engineering*, World Scientific Pub. Co., Singapore, 2001.

[8] M. Shaked and I.G. Shanthikumar, Reliability and Maintainability, In: *Handbook in Operations Research and Management Science* **2**, Heyman, D.P. and Sobel, M.J. (Eds.), North-Holland Pub. Co., Amsterdam, 1996.

[9] J. Shao and L.R. Lamberson, Impact of BIT design parameters on systems RAM, *Reliability Engineering & System Safety* **23** (1988), 219-246.

[10] E.J. Vanderperre, A Sokhotski–Plemelj problem related to a robot–safety device, *Operations Research Letters* **27** (2000), 67–71.

[11] E.J. Vanderperre, V.S.S. Yadavalli and S.S. Makhanov, On Gaver's parallel system, *South African Journal of Industrial Engineering* **15** (2004), 141-147.

[12] E.J. Vanderperre, On the idle time of a repairman attending a renewable cold standby system, *International Journal of Mathematical Analysis* **1** (2007), 463–469.

[13] E.J. Vanderperre, Long–run availability of a warm standby system, To appear in : *Mathematical Notes* (2007).

Received: August 17, 2007