

A Study on Some Integer Sequences

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Abstract. In this note, some integer sequences are investigated and the relationships between them are obtained. Especially, many similarities between enumeration of generalized Fibonacci sequences and Padovan sequence are realized. By this similarities some new sequences are obtained. Also, recurrence relations that satisfied by the sequences are studied.

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1. Introduction

Edouard Lucas(1842–1891) made a deep study of sequences which is called generalized Fibonacci sequences that begin the sum of the preceding two. The simplest such series; 0,1,1,2,3,5,8,13,21,... is called the Fibonacci sequences by Lucas. The next simplest series, 2,1,3,4,7,11,18,... is then called the Lucas numbers in his honor. The Fibonacci rule of adding the latest two to get the next is kept, but here we begin with 2 and 1. The position of each number in this sequences is traditionally indicated by a subscript, so that

$$F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2 \quad (1)$$

and so on. Thus, Fibonacci sequence can be defined as follows

$$F_n = F_{n-1} + F_{n-2}; \quad n \geq 2 \quad (2)$$

which two boundary conditions; $F_0 = 0, F_1 = 1$. The Lucas sequence is defined as follows where we write its members as $L_n = L_{n-1} + L_{n-2}$, $n > 1$ which two boundary conditions; $L_0 = 2, L_1 = 1$. Lucas numbers have lots of properties similar to those of Fibonacci numbers.

If F_n nth Fibonacci number, then $\frac{F_{n+1}}{F_n}$ is equal to a continued fraction. Indeed, we apply the Euclidean algorithm to F_{n+1} and F_n . In effect we get a sequence of equations: $\frac{F_{n+1}}{F_n} = [1; 1, 1, \dots]$ If the value of continued fraction $[1; 1, 1, \dots]$ is called δ , and the n th convergence of this fraction is $C_n = [1; 1, \dots, 1]$ then, $\delta = \lim_{n \rightarrow \infty} [1; 1, \dots, 1] = \lim_{n \rightarrow \infty} C_n = \lim_{n \rightarrow \infty} \frac{F_{n+1}}{F_n} = \lim_{n \rightarrow \infty} \frac{F_n + F_{n-1}}{F_n} = 1 + \frac{1}{\delta}$, the positive root of this equation is $\frac{1+\sqrt{5}}{2} = [1; 1, \dots]$ which is known as golden ratio [4].

For any positive real number k , the k -Fibonacci sequence, say $\{F_{k,n}\}_{n \in \mathbb{N}}$ is defined recurrently by

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1}, \quad n \geq 1 \quad (3)$$

with initial conditions $F_{k,0} = 0$; $F_{k,1} = 1$. If $k=1$, the classical Fibonacci sequence is obtained and if $k=2$, the classical Pell sequence appears.

Note that if k is a real variable x then $F_{k,n} = F_{x,n}$ and they correspond to the Fibonacci polynomials defined by

$$F_{n+1}(x) = \begin{cases} 1 & \text{if } n=0, \\ x & \text{if } n=1, \\ xF_n(x) + F_{n-1}(x) & \text{if } n > 1 \end{cases} \quad (4)$$

2. Some Properties of Fibonacci Sequence

Theorem 2.1: If $L_1 = 1$, $L_2 = 3$ and for $n \geq 3$, $L_n = L_{n-1} + L_{n-2}$ and $\alpha = \frac{1+\sqrt{5}}{2}$ ve $\beta = \frac{1-\sqrt{5}}{2}$ then

$$L_n = \alpha^n + \beta^n \quad (5)$$

Proof: The proof by induction is easy.

Definition 2.1 (Binet's Formula) Alternatively we can define the Fibonacci numbers for integer n by

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{6}$$

where α and β are two solutions of $p(z) = z^2 - z - 1$. Note that $\alpha + \beta = 1$, $\alpha\beta = -1$, $\alpha + 1 = \alpha^2$, $\beta + 1 = \beta^2$. Then

$$\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} + \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\alpha^n(\alpha + 1) - \beta^n(\beta + 1)}{\alpha - \beta} = \frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta}.$$

Here $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ the formula indirectly implies that, for $n \geq 0$, $F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n$ which is a repeated correlation [3].

The nth k- Fibonacci number is given by

$$F_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2} \tag{7}$$

where r_1, r_2 are the roots of the characteristic equation $r^2 = kr + 1$ and $r_1 > r_2$.

Linear Algebra. Define matrices

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, P = \begin{bmatrix} 1 & 1 \\ \alpha & \beta \end{bmatrix}, D = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \tag{8}$$

And for integer $n \geq 1$, define

$$v_n = \begin{bmatrix} F_{n-1} \\ F_n \end{bmatrix} \tag{9}$$

Standard computations show that

$$P^{-1} = \frac{1}{\beta - \alpha} \begin{bmatrix} \beta & -1 \\ -\alpha & 1 \end{bmatrix}, D^n = \begin{bmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{bmatrix} \tag{10}$$

The characteristic polynomial of M is $p(z)$ and a diagonal decomposition of M is $M = PDP^{-1}$ [3]. Then $Mv_n = v_{n+1}$, and two routine inductions show that $v_n = M^{n-1}v_1$ and $M^n = PD^nP^{-1}$. We conclude that $v_n = PD^{n-1}P^{-1}v_1$ yields following equation in vector form:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

Theorem 2.2: For $n \geq 1$, the consecutive Lucas numbers are prime numbers in themselves, that means $(L_{n+1}, L_n) = 1$ [4].

Definition 2.2: If $p(x) = x^k - a_{k-1}x^{k-1} - a_{k-2}x^{k-2} - \dots - a_1x - a_0$ polynomial is a k degree monic polynomial, then the companion matrix is defined such as [2]:

$$A_{p(x)} = \begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_{k-2} & a_{k-1} \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix} \tag{11}$$

Theorem 2.3 : Let $L_1=1, L_0=2$, then $\begin{bmatrix} L_{n+1} \\ L_n \end{bmatrix} = A^n \begin{bmatrix} L_1 \\ L_0 \end{bmatrix}$

With the help of this theorem, each required Lucas number can easily be found. For example 6 th Lucas number is:

$$\begin{bmatrix} L_6 \\ L_5 \end{bmatrix} = A^5 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 18 \\ 11 \end{bmatrix}$$

Definition 2.3: The Padovan sequence is the sequence of integers $P(n)$ defined by the initial values

$$p(0)=p(1)=p(2)=1 \tag{12}$$

and the recurrence relation

$$P(n)=P(n-2)+P(n-3) \tag{13}$$

The first few values of $P(n)$ are 1,1,1,2,2,3,4,5,7,9,12,16,21,28,... [1].

When the table written for Lucas sequence, is again rewritten for Fibonacci sequence, the sum of the diagonal elements sums up to Padovan sequence.

Now consider the infinite dimensional matrix A_{ij} where we define recursively each elements a_{ij} as follows for $i, j \geq 1$, [5];

$$A_{ij} = a_{ij} = 0 \begin{cases} a_{ij} = 1 & \text{if, } i = j \\ & \text{if, } i > j \text{ or } j \geq 2i+1 \\ a_{ij} = a_{i-1,j-2} + a_{i-1,j-1} & \text{other cases.} \end{cases} \tag{14}$$

Thus, we obtain the following table:

Table 1. The elements of the A_{ij} matrix

i / j	1	2	3	4	5	6	7	8	9	10...
1	1	2	0	0	0	0	0	0	0	0
2	0	1	3	2	0	0	0	0	0	0
3	0	0	1	4	5	2	0	0	0	0
4	0	0	0	1	5	9	7	2	0	0
5	0	0	0	0	1	6	14	16	9	2
6	0	0	0	0	0	1	7	20	30	25
7	0	0	0	0	0	0	1	8	27	50
8	0	0	0	0	0	0	0	1	9	35
⋮										

3. The Results

According to the table above, the following results could be written:

Corollary 3.1: (a) For $i, j \geq 1$; $\sum_{i=1}^j a_{i,j} = L_{j+1}$

(b) The sum of the diagonal elements on the table are the elements of $S(n)$:

$$S(n) = \sum_{k=1}^n a_{n-k+1,k} . \tag{15}$$

The first few values of $S(n)$ are, 1,2,1,3,3,4,6,7,10,13,17,23,...

Note that the $S(n)$ sequence also satisfies the recurrence relations;

$$\begin{aligned} S(n) &= P(n) + P(n - 3) & ; & & P(0) = P(1) = P(2) = 1 \\ S(n) &= P(n-2) + 2.P(n-3) & & & (16) \\ S(n) &= 2P(n) - P(n - 2) & ; & & S(0) = S(2) = 1, S(1) = 2 \end{aligned}$$

(c) For $n \geq 1$, if the rows of the table are shown by $R(n)$, then the sum of the elements of $R(n)$ could be found according to this formula:

$$R(n) = 2^{n-1}.3 \tag{17}$$

The elements of this sequence are the elements of Pisot sequence.

Corollary 3.2: For sequences $F_n = F_{n-1} + F_{n+1}$ and $L_n = L_{n-1} + L_{n-2}$ if $n > 4$, then $L_n = F_n - F_{n-4}$.

Proof: This could be verified by induction. If $n = 5$, $L_5 = 4$, $F_5 = 5$, $F_{5-4} = F_1 = 1$ then $L_5 = F_5 - F_1$.

Let $4 < k < n$ be true. Namely, $L_n = F_n - F_{n-4}$. We will show that $L_{n+1} = F_{n+1} - F_{n-3}$. Since $L_{n+1} = L_n + L_{n-1}$ and from hypothesis with the help of equations $L_n = F_n - F_{n-4}$ and $L_{n-1} = F_{n-1} - F_{n-4}$, it is seen that

$$L_n + L_{n-1} = F_{n+1} - F_{n-3}.$$

Corollary 3.3: Let's consider three different integer sequences whose a few first terms are as following:

$$P: 1, 1, 1, 3, 4, 7, 11, 18, \dots \quad ; \text{ where } P(n) = P(n-2) + P(n-3)$$

$$P_r: 3, 0, 2, 3, 2, 5, 5, 7, 10, 12, 17, \dots; \text{ where } P_r(n) = P_r(n-2) + P_r(n-3) \quad (18)$$

$$S: 1, 2, 1, 3, 3, 4, 6, 7, 10, 13, 17, 23, \dots; \text{ where } S(n) = S(n-2) + S(n-3)$$

Between these sequences, for $n > 12$,

$$S(n) = P_r(n) + P(n-12)$$

is valid. Here $P_r(n)$ is a Padovan sequence the initial terms of which are different.

Proof: Let's assume $n=3$ in order to prove by induction. Since, $S(13)=30$, $P_r(13) = 29$, $P(13-12) = P(1) = 1$

From equation $S(13) = P_r(13) + P(1)$ the validity of the equation is obvious. Let $12 < k \leq n$ be true. That's for $12 < k \leq n$

$$S(n) = P_r(n) + P(n-12),$$

$$S(n+1) = P_r(n+1) + P(n-11). \quad (19)$$

The validity of this equation should be shown. Since $S(n+1) = S(n-1) + S(n-2)$ from hypothesis

$$S(n-1) = P_r(n-1) + P(n-13).$$

With the help of $S(n-2) = P_r(n-2) + P(n-14)$ the validity of this equation is obvious. With the help of these tables several integer sequences could be established. Moreover, the relations between the properties of the sequences could be understood easily.

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