On New Subclass of Analytic
P-valent Function

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Abstract

In this paper, the authors introduce and study a new subclass of normalized analytic p-valent function. The results concern basically on the coefficient bounds, and in fact generalised some known results by other authors.

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1 Introduction and Motivation

Let $A(p)$ be the class of functions $f$ normalized by

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p}z^{n+p}, \quad (1.1)$$

$p \in N$, which are analytic in the open unit disk

$$U = \{ z \in \mathbb{C} : |z| < 1 \}.$$

As usual, we denote by $S(p)$ the subclass of $A$, consisting of functions which are also p-valent in $U$.

Let $S_w(p)$ denote the class of function $f(z)$ which analytic and $p$-valent in the open unit disk $U = \{ z : |z| < 1 \}$, of the form

$$f(z) = \frac{1}{(z-w)^p} + \sum_{n=1}^{\infty} a_{n+p}(z-w)^{n+p} \quad (a_{n+p} \geq 0) \quad (1.2)$$
where \( w \) is a fixed point in \( U \).

The function \( f(z) \in S_w(p) \) is said to be \( p\)-valent starlike of order \( \beta \) and \( p\)-valent convex of order \( \beta \) in \( U \) if they satisfy:

\[
\text{Re} \left( \frac{(z - w)f'(z)}{f(z)} \right) > \beta \quad (z \in U; 0 \leq \beta < p; p \in N) \quad (1.3)
\]

\[
1 + \text{Re} \left( \frac{(z - w)f''(z)}{f'(z)} \right) > \beta \quad (z \in U; 0 \leq \beta < p; p \in N) \quad (1.4)
\]

respectively. Let \( S_w(1) = S_w \) which were studied by Acu and Owa [4] and also studied by Kanas and Ronning [6].

Generally, the classes of all \( p\)-valent convex functions of order \( \beta \) and \( p\)-valent starlike functions of order \( \beta \) \((0 \leq \beta < p; p \in N)\) in \( U \), respectively, are denote by \( S_w^c(p, \beta) \) and \( S_w^s(p, \beta) \) in the literature.(see for their details [1],[5],[7] and see also [3]).

Now for the function \( f(z) \) in the class \( S_w(p) \), we define

\[
D^0 f(z) = f(z),
\]

\[
D^1 f(z) = (z - w)f'(z) + \frac{1 + p}{(z - w)^p}
\]

\[
D^2 f(z) = (z - w) (D^1 f(z))' + \frac{1 + p}{(z - w)^p}
\]

and for \( k = 1, 2, 3, \ldots \)

\[
D^k f(z) = (z - w) (D^{k-1} f(z))' + \frac{p + 1}{z - w} = \frac{1}{(z - w)^p} + \sum_{n=1}^{\infty} (n + p)^k a_{n+p}(z - w)^{n+p}
\]

For \( p = 1 \) and \( w = 0 \) the differential operator \( D^k \) reduced to Frasin and Darus [2].

With the help of the differential operator \( D^k \), we define the class \( S_w^s(p, k, \beta) \) as follow:
Definition: The function \( f(z) \in S_w(p) \) is said to be a member of the class \( S_w^*(p, k, \beta) \) if it satisfies
\[
\left| \frac{(z-w)(D^k f(z))'}{D^k f(z)} + 1 \right| < \left| \frac{(z-w)(D^k f(z))'}{D^k f(z)} + 2\beta - 1 \right|
\]
\((k \in N_0 = N \cup 0)\)
for some \( \beta(0 \leq \beta < 1) \) and for all \( 0 \leq z < 1 \) in \( U \).

It is easy to check that \( S_w^*(p, 0, \beta) \) is the class of \( p \)-valent starlike functions of order \( \beta \) and \( S_w^*(p, 0, 0) \) is the class of \( p \)-valent starlike functions for all \( z \in U \).

Let us write
\[
S_w^*(p, k, \beta) = S_w^*(p, k, \beta) \cap S_w^*(p) \quad (1.5)
\]
where \( S_w^*(p) \) is the class of the form (1.1) that are analytic and \( p \)-valent in \( U \).

In this paper, we will give coefficient estimates and results on distortion theorem for the classes \( S_w^*(p, k, \beta) \) and \( S_w^*(p, k, \beta) \).

2 Coefficient Estimates

Our first result provides a sufficient condition for a function, \( p \)-valent and analytic in \( U \), to be in \( S_w^*(p, k, \beta) \).

Theorem 2.1: Let the function \( f(z) \) be defined by (1.2). If
\[
\sum_{n=1}^{\infty} (n+p)^k ((n+p)+\beta) |a_{n+p}| \leq (1 - \beta) \quad (2.6)
\]
where \( 0 \leq \beta < 1 \), then \( f(z) \in S_w^*(p, k, \beta) \).

Proof:
Suppose that (2.1) holds true for \( 0 \leq \beta < 1 \). Consider the expression
\[
M(f, f') = \left| (z-w)(D^k f(z))' + D^k f(z) \right| - \left| (z-w)(D^k f(z))' + (2\beta - 1) D^k f(z) \right|
\]
then for \( 0 < |z-w| = r < 1 \) we have
\[
M(f, f') = \left| \sum_{n=1}^{\infty} (n+p)((n+p)+1)a_{n+p} (z-w)^{n+p} \right|
\]
\[
- \left| \frac{2(\beta - 1)}{(z-w)^p} + \sum_{n=1}^{\infty} (n+p)^k ((n+p) + 2\beta - 1) a_{n+p} (z-w)^{n+p} \right|
\]
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\[ rM(f, f') \leq \sum_{n=1}^{\infty} (n + p)^k \left((n + p) + 1\right) a_{n+p} r^{n+p+1} \]

\[ 2(1 - \beta) + \sum_{n=1}^{\infty} (n + p)^k \left((n + p) + 2\beta - 1\right) a_{n+p} r^{n+p+1} \]

\[ \leq \sum_{n=1}^{\infty} 2(n + p)^k \left((n + p) + \beta\right) a_{n+p} r^{n+p+1} - 2(1 - \beta). \quad (2.7) \]

The inequality in (2.2) holds true for all \( r(0 \leq r < 1) \). Therefore, letting \( r \to 1 \) in \( (2.2) \), we obtain

\[ M(f, f') \leq \sum_{n=1}^{\infty} 2(n + p)^k \left((n + p) + \beta\right) a_{n+p} - 2(1 - \beta) \]

by the hypothesis \((2.1)\). Hence it follows that

\[ \left| \frac{(z - w)(D^k f(z))'}{D^k f(z)} + 1 \right| < \left| \frac{(z - w)(D^k f(z))'}{D^k f(z)} + 2\beta - 1 \right|, \]

so that \( f(z) \in S_w^* (p, k, \beta) \).

Hence the theorem.

**Corollary 2.2:** let \( k = \beta = 0 \) in the Theorem 2.1, then we have \( \sum_{n=1}^{\infty} n |a_{n+p}| \leq 1 \), therefore \( f \) is starlike \( p \)-valent in all \( z \in U \).

**Corollary 2.3:** let \( k = 1 \) and \( \beta = 0 \) in the Theorem 2.1, then we have \( \sum_{n=1}^{\infty} n^2 |a_{n+p}| \leq 1 \), therefore \( f \) is convex \( p \)-valent in all \( z \in U \).

Next, we give a necessary and sufficient condition for a function \( f \in S_w \) to be in the class \( S_w^*(k, \beta) \)

**Theorem 2.4:** Let the function \( f(z) \) be defined by \((1.2)\) and let \( f(z) \in S_w(p) \). Then \( f(z) \in S_w^*(p, k, \beta) \) if and only if \((2.1)\) is satisfied. The result \((2.1)\) is sharp.

**Proof:** In view of Theorem 2.1, it sufficient to show that the 'only if ' part. Assume that \( f \in S_w^*(p, k, \beta) \). Then

\[ \left| \frac{(z - w)(D^k f(z))'}{D^k f(z)} + 1 \right| = \left| \frac{(z - w)(D^k f(z))'}{D^k f(z)} + 2\beta - 1 \right| \]
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\[ \left| \sum_{n=1}^{\infty} (n+p)^k ((n+p) + 1) a_{n+p}(z-w)^{n+p} \right| < 1 \quad (z \in U) \quad (2.8) \]

Since $Re(z) \leq |z|$ for all $z$, it follows from (2.3) that

\[ \text{Re} \left\{ \sum_{n=1}^{\infty} (n+p)^k ((n+p) + 1) a_{n+p}(z-w)^{n+p} \right\} < 1 \quad (z \in U) \quad (2.9) \]

we now choose the values $f(z)$ on the real axis so that $\frac{(z-w)(D^k f(z))'}{D^k f(z)}$ is real. Upon clearing the denominator in (2.4) and letting $(z-w) \to 1$ through real values, we obtain

\[ \sum_{n=1}^{\infty} (n+p)^k ((n+p) + 1) a_{n+p} \leq 2(1 - \beta) - \sum_{n=1}^{\infty} (n+p)^k ((n+p) + 2\beta - 1) a_{n+p}, \quad (2.10) \]

\[ \sum_{n=1}^{\infty} 2(n+p)^k ((n+p) + \beta) a_{n+p} \leq 2(1 - \beta) \]

which immediately yields the required condition (2.3).

Our assertion in theorem 2.2 is sharp for functions of the form

\[ f_n(z) = \frac{1}{(z-w)^p} + \frac{(1 - \beta)}{(n+p)^k ((n+p) + \beta)} (z-w)^{n+p} \quad (n \geq 1; k \in \mathbb{N}_0) \quad (2.11) \]

**Corollary 2.5:** Let the function $f(z)$ be defined by (1.2) and let $f(z) \in S_w(p)$. If $f \in S_s^p (p, k, \beta)$, then

\[ a_{n+p} \leq \frac{1 - \beta}{(n+p)^k ((n+p) + \beta)} \quad (2.12) \]

The result (2.7) is sharp for functions $f_n(z)$ given by (2.6).

**Corollary 2.6:** Let the function $f(z)$ be defined by (1.2) and let $f(z) \in S_w(p)$. If $f \in S_s^p (p, k, \beta)$, then

\[ (n+p) a_{n+p} \leq \frac{1 - \beta}{(n+p)^{k-1} ((n+p) + \beta)} \quad (2.13) \]

The result (2.7) is sharp for functions $f_n(z)$ given by (2.6).
A distortion property for functions in the class \( S^*_W(p, k, \beta) \) is contained in

**Theorem 2.7:** If the function \( f(z) \) be defined by (1.2) is in the class \( S^*_W(p, k, \beta) \), then for \( 0 < |z - w| = r < 1 \) we have

\[
\frac{1}{r^p} - \frac{(1 - \beta)}{(p + 1 + \beta)^r} \leq |f(z)| \leq \frac{1}{r^p} + \frac{(1 - \beta)}{(p + 1 + \beta)^r}
\]

with equality for

\[
f_1(z) = \frac{1}{(z-w)^p} + \frac{(1-\beta)}{(p+1+\beta)}(z-w)^{p+1}
\]

\( (z = ir, r) \)

and

\[
\frac{p}{r^2} - \frac{(1 - \beta)}{(p + 1 + \beta)^r} \leq |f'(z)| \leq \frac{\alpha}{r^2} + \frac{(1 - \beta)}{(p + 1 + \beta)^r}
\]

with equality for

\[
f_1(z) = \frac{1}{(z-w)^p} + \frac{(1-\beta)}{(p+1+\beta)}(z-w)^{p+1}
\]

\( (z = \pm ir, \pm r) \)

**Proof** Since \( f \in S^*_W(p, k, \beta) \), Corollary 2.5 readily yields the inequality

\[
\sum_{n=1}^{\infty} a_{n+p} \leq \frac{(1 - \beta)}{(n + p)^k ((n + p) + \beta)}
\]

Thus, for \( 0 < |z - w| = r < 1 \), and making use of (2.10) we have

\[
|f(z)| \leq \left| \frac{1}{(z-w)^p} \right| + \sum_{n=1}^{\infty} a_{n+p} |z-w|^{n+p} \leq \frac{1}{r^p} + \frac{(1 - \beta)}{(p + 1 + \beta)^r}
\]

\[
\left| \frac{1}{r^p} \right| + r^p \sum_{n=1}^{\infty} a_{n+p} \leq \left| \frac{1}{r^p} \right| + \frac{(1 - \beta)}{(p + 1 + \beta)^r} (2.15)
\]

and

\[
|f(z)| \geq \left| \frac{1}{(z-w)^p} \right| - \sum_{n=1}^{\infty} a_{n+p} |z-w|^{n+p} \geq \left| \frac{1}{r^p} \right| - \frac{\alpha}{r^2} \left( \frac{(1 - \beta)}{(p + 1 + \beta)^r} \right)^{r+1} (2.16)
\]
Also from Corollary 2.6, it follows that
\[ \sum_{n=1}^{\infty} na_{n+p} \leq \frac{(1 - \beta)}{(n + p)^{k-1} ((n + p) + \beta)}. \] (2.17)

Hence
\[ |f'(z)| \leq \left| \frac{-p}{(z - w)^{p+1}} \right| + \sum_{n=1}^{\infty} (n + p) a_{n+p} |z - w|^{n+p-1} \leq \left| \frac{p}{r^{p+1}} \right| + \frac{(1 - \beta)}{(p + 1 + \beta)} r^p \] (2.18)

and
\[ |f'(z)| \geq \left| \frac{-p}{(z - w)^{p+1}} \right| - \sum_{n=1}^{\infty} (n + p) a_{n+p} |z - w|^{n+p-1} \geq \left| \frac{p}{r^{p+1}} \right| - \frac{(1 - \beta)}{(p + 1 + \beta)} r^p. \] (2.19)

This completes the proof of 2.7.

### 3 Radii Of Starlikeness and Convexity

The radii of Starlikeness and convexity for the class for the class \( S^*_W(p, k, \beta) \) is given by the following theorems.

**Theorem 3.1:** If the function \( f(z) \) be defined by (1.2) is in the class \( S^*_W(p, k, \beta) \) then \( f(z) \) is starlike of order \( \delta \) \((0 \leq \delta < 1)\) in \(|z - w| < r_1\), where
\[ r_1 = r_1(p, k, \beta, \delta) = \inf \left\{ \frac{(n + p)^k ((n + p) + \beta) (1 - \delta)}{((n + p) + 2 - \delta) (1 - \beta)} \right\} \frac{1}{n + p + 1}. \] (3.20)

The result is sharp for the function \( f_n(z) \) given by (2.6).

**Proof:**
It sufficient to prove that
\[ \left| \frac{(z - w)f'(z)}{f(z)} + 1 \right| \] (3.21)
for \( |z - w| < r_1 \). We have

\[
\left| \frac{(z - w)f'(z)}{f(z)} + 1 \right| = \left| \sum_{n=1}^{\infty} ((n + p) + 1) a_{n+p}(z - w)^{n+p} \right|
\]

\[
\frac{\sum_{n=1}^{\infty} ((n + p) + 1) a_{n+p}(z - w)^{n+p+1}}{1 + \sum_{n=1}^{\infty} a_{n+p}(z - w)^{p+n+1}}
\]

\[
\leq \sum_{n=1}^{\infty} ((n + p) + 1) a_{n+p} |z - w|^{n+p+1}
\]

\[
\frac{1 - \sum_{n=1}^{\infty} a_{n+p} |z - w|^{n+p+1}}{1 - \sum_{n=1}^{\infty} a_{n+p} |z - w|^{n+p+1}}. \tag{3.22}
\]

Hence (3.3) holds true

\[
\sum_{n=1}^{\infty} ((n + p) + 1) a_{n+p} |z - w|^{n+p+1} \leq (1 - \delta) 1 - \sum_{n=1}^{\infty} a_{n+p} |z - w|^{n+p+1} \tag{3.23}
\]

or

\[
\sum_{n=1}^{\infty} ((n + p) + 2 - \delta) a_{n+p} |z - w|^{n+p+1}
\]

\[
\frac{1}{(1 - \delta)} \leq 1 \tag{3.24}
\]

with the aid of (2.1), (3.5) is true if

\[
\sum_{n=1}^{\infty} ((n + p) + 2 - \delta) a_{n+p} |z - w|^{n+p+1}
\]

\[
\frac{(1 - \beta)}{(n + p)^k ((n + p) + \beta)} \quad (n \geq 1) \tag{3.25}
\]

Solving (3.6) for \( |z - w| \), we obtain

\[
|z - w| < \left\{ \frac{(n + p)^k ((n + p) + \beta) (1 - \delta)}{(1 - \beta) ((n + p) + 2 - \delta)} \right\}^{1/(n+p+1)}. \tag{3.26}
\]
This completes the proof of Theorem 3.1.

**Theorem 3.2:** If the function \( f(z) \) be defined by (1.2) is in the class \( S_{W}^{p}(p, k, \beta) \), then \( f(z) \) is convex of order \( \delta (0 \leq \delta < 1) \) in \( |z - w| < r_2 \), where

\[
r_2 = r_2(p, k, \beta, \delta) = \inf \left\{ \frac{(n + p)^{k-1}((n + p) + \beta)(1 - \delta)}{(n + p)(2 - \delta)(1 - \beta)} \right\}^{\frac{1}{n+p+1}}. \tag{3.27}
\]

The result is sharp for the functions \( f_n(z) \) given by (2.6).

**Proof:** By using the technique employed in the proof of Theorem 3.1, we can show that

\[
\left| \frac{(z-w)f''(z)}{f'(z)} + 1 \right| \leq (1 - \delta). \tag{3.28}
\]

for \( |z - w| < r_2 \), with the aid of Theorem 2.1. Thus we have the assertion of Theorem 3.2.

### 4 Convex Linear Combinations

Our next result involves linear combinations of several functions of the type (2.6).

**Theorem 4.1:** Let

\[
f_0(z) = \frac{1}{(z-w)^p} \tag{4.29}
\]

and

\[
f_n(z) = \frac{1}{(z-w)^p} + \frac{(1-\beta)}{(n+p)^k((n+p)+\beta)}(z-w)^{n+p} \tag{4.30}
\]

\((n \geq 1; k \in N_0)\)

Then \( f(z) \in S_{W}^{p}(p, k, \beta) \) if and only if it can be expressed in the form

\[
f(z) = \sum_{n=0}^{\infty} \lambda_nf_n(z) \tag{4.31}
\]

where \( \lambda_{n+p} \geq 0 \) and \( \sum_{n=0}^{\infty} \lambda_{n+p} = 1 \).

**Proof:** From (4.1),(4.2),and (4.3), it is easily seen that

\[
f(z) = \sum_{n=0}^{\infty} \lambda_nf_n(z)
\]
\[ f(z) = \frac{1}{(z - w)^p} + \frac{(1 - \beta)}{(n + p)^k ((n + p) + \beta)} \lambda_{n+p}(z - w)^{n+p} \]  

(4.32)

Since
\[ \sum_{n=1}^{\infty} \frac{(n + p)^k ((n + p) + \beta)}{(1 - \beta)(n + p)^k ((n + p) + \beta)} \lambda_{n+p} = \sum_{n=1}^{\infty} \lambda_{n+p} = 1 - \lambda_0 \leq 1. \]

it follows from Theorem 2.2 that the function \( f(z) \in S_w^*(p, k, \beta) \).

Conversely, let us suppose that \( f(z) \in S_w^*(p, k, \beta) \). Since
\[ a_n \leq \frac{(1 - \beta)}{(n + p)^k ((n + p) + \beta)} \]  

\( (n \geq 1; k \in N_0) \)

setting
\[ \lambda_n = \frac{(n + p)^k ((n + p) + \beta)}{(1 - \beta)} a_{n+p}, \]  

\( (n \geq 1; k \in N_0) \) and \( \lambda_0 = 1 - \sum_{n=1}^{\infty} \lambda_n \)

it follows that \( f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z) \). This completes the proof of theorem.

Finally, we prove

**Theorem 4.2:** The class \( S_w^*(p, k, \beta) \) is closed under convex linear combinations.

**Proof:** Suppose that the function \( f_1(z) \) and \( f_2(z) \) defined by
\[ f_j(z) = \frac{1}{(z - w)^p} + \sum_{n=1}^{\infty} a_{n+p,j}(z - w)^{n+p} \]  

\( (j = 1, 2; z \in U) \)  

(4.33)

are in the class \( S_w^*(p, k, \beta) \)

Setting
\[ f(z) = \mu f_1(z) + (1 - \mu) f_2(z) \]  

\( (0 \leq \mu < 1) \)  

(4.34)

we find from (4.5) that
\[ f(z) = \frac{1}{(z - w)^p} + \sum_{n=1}^{\infty} \{\mu a_{n+p,1} + (1 - \mu) a_{n+p,2}\} (z - w)^{n+p} \]  

(4.35)

\( ((0 \leq \mu < 1); z \in U) \)

In view of Theorem 2.2, we have
\[ \sum_{n=1}^{\infty} \left[ (n + p)^k ((n + p) + \beta) \right] (\mu a_{n+p,1} + (1 - \mu) a_{n+p,2}) \]
\[= \mu \sum_{n=1}^{\infty} \left[ (n+p)^k ((n+p) + \beta) \right] a_{n+p,1} + (1 - \mu) \sum_{n=1}^{\infty} \left[ (n+p)^k ((n+p) + \beta) \right] a_{n+p,2}\]

\[\leq \mu (1 - \beta) + (1 - \mu)(1 - \beta) = (1 - \beta)\]

which shows that \(f(z) \in S^{\mu}_{(p,k,\beta)}.\) Hence the theorem.

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**References**


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