

# Eigenvalues of Differential Equations with Singularities and Turning Points

A. Neamaty

Department of Mathematics  
University of Mazandaran, Babolsar, Iran  
namaty@umz.ac.ir

A. Dabbaghian

Islamic Azad University, Neka Branch, Neka, Iran  
a.dabbaghian@umz.ac.ir

S. Mosazadeh

Department of Mathematics  
University of Mazandaran, Babolsar, Iran

## Abstract

A differential systems having a finite number of arbitrary order singularities and turning points is investigated. We establish properties of the spectral characteristics, find the asymptotic forms of the solution of the system (1) and we calculate the eigenvalues.

**Mathematics Subject Classification:** 34B24 34B05

**Keywords:** Singularities and turning points, Asymptotic form, Initial condition

## 1 Introduction

We consider the following system of differential equations

$$\frac{dy_1}{dx} = i\rho R(x)y_2, \frac{dy_2}{dx} = i\rho \frac{1}{R(x)}y_1, x \in [0, T] \quad (1)$$

with the initial conditions  $y_1(T, \rho) = -1, y_2(T, \rho) = 1$ . Here  $\rho = \sigma + i\tau$  is the spectral parameter, and  $R$  is a real function which is called the wave resistance. System (1) is a canonical form for many problems in natural sciences.

For example, for a wide class of problems describing the propagation of electromagnetic waves in a stratified medium, Maxwell's equations can be reduced to the canonical form (1). System (1) often appears in optics, spectroscopy and acoustic problems. System (1) also appears for the design of directional couplers for heterogeneous electronic lines, which constitute one of the important classes of radio physical synthesis problems (see [1],[2]). Some aspects of synthesis problems for system (1) with  $R > 0$  (i.e.  $N=0$ ) were studied in [7]. Inverse problem for system (1) with the initial conditions  $y_1(0, \rho) = 1, y_2(0, \rho) = -1$  and with  $R \geq 0$  were studied in [3],[4]. In [5] studied the eigenvalues and derived a formula for the asymptotic distribution of the eigenvalues in the case when the equation has singularities or/and turnings at the end-points of a finite interval. In [8] studied the asymptotic estimates for a special fundamental system of solutions of the corresponding differential equation and determined the asymptotic distribution of the eigenvalues with several singularities or/and turning points inside the interval  $[0,1]$  by EBERHARD and FREILING and WILCKEN.

The main purpose of this paper is to study the behavior asymptotic solution and eigenvalues in the case when the system has arbitrary order singularities and turning points inside the interval.

We will suppose that

$$R(x) = \prod_{j=1}^N |x - \gamma_j|^{p_j-1} R_0(x) \quad (2)$$

where  $0 < \gamma_1 < \gamma_2 < \dots < \gamma_N < T$ ,  $p_j$  are real numbers,  $R_0(x) \in W_2^1[0, T]$ ,  $R_0(x) > 0$ ,

$$R(T) = 1, R'(T) = 0. \quad (3)$$

In section 2, at the main spectral characteristic we introduce the characteristic function

$$\Delta(\rho) = \frac{1}{\sqrt{a}} y_1(0, \rho); a := R(0). \quad (4)$$

In section 3 we calculate eigenvalues (i.e. zeros of  $\Delta(\rho)$ ) in the case when  $R$  has two singularities and turning points.

## 2 Properties of spectral characteristics

2.1. We consider the replacement

$$y_1(x, \rho) = \sqrt{R(x)} u(x, \rho), y_2(x, \rho) = \frac{1}{\sqrt{R(x)}} v(x, \rho). \quad (5)$$

Let

$$h(x) = \frac{R'(x)}{2R(x)}, \quad (6)$$

then (1) transform to the system

$$u' + h(x)u = i\rho v, v' - h(x)v = i\rho u \quad ; \quad x \in [0, T] \quad (7)$$

with the conditions  $u(T, \rho) = -1, v(T, \rho) = 1$ .

System (7) after elimination of  $v$  reduces to the equation

$$-u'' + q(x)u = \lambda u, \quad \lambda = \rho^2, \quad (8)$$

and the conditions

$$u(T, \rho) = -1, u'(T, \rho) = i\rho, \quad (9)$$

where

$$q(x) = h^2(x) - h'(x). \quad (10)$$

It follows from (2), (6) and (10) that  $q(x)$  has the form

$$q(x) = \sum_{j=1}^N \frac{a_j}{(x - \gamma_j)^2} + q_0(x),$$

where

$$a_j := \frac{p_j^2 - 1}{4}, \quad p_j \neq 2k; k \in Z, \quad q_0(x) \cdot \prod_{j=1}^N |x - \gamma_j|^{1-|p_j|} \in L(0, T),$$

and  $q_0(x)$  has the form

$$q_0(x) = \frac{3}{4} \left( \frac{R'_0(x)}{R_0(x)} \right)^2 - \frac{R''_0(x)}{2R_0(x)} + \frac{R'_0(x)}{2R_0(x)} \sum_{j=1}^N \frac{p_j - 1}{x - \gamma_j} + \frac{1}{2} \sum_{i=1}^{N-1} \sum_{j=i+1}^N \frac{(p_i - 1)(p_j - 1)}{(x - \gamma_i)(x - \gamma_j)}.$$

It follows from (3),(4) and (5) that the characteristic function  $\Delta$  has the form:

$$\Delta(\rho) = u(0, \rho). \quad (11)$$

**2.2.** Let

$$v_j = \frac{|p_j|}{2}, \mu_{kj} = (-1)^k v_j + \frac{1}{2}, \omega_j = (\gamma_j, \gamma_{j+1}), \gamma_0 := 0, \gamma_{N+1} := T.$$

Denote:  $C_{kj}(x, \lambda) = (x - \gamma_j)^{\mu_{kj}} \sum_{m=0}^{\infty} C_{kmj}(\rho(x - \gamma_j))^{2m}; k = 1, 2, j = \overline{1, N}$ ,

where

$$C_{10j}C_{20j} = (2v_j)^{-1}, C_{kmj} = (-1)^m C_{k0j}(\prod_{s=1}^m ((2s + \mu_{kj})(2s + \mu_{kj} - 1) - a_j))^{-1}.$$

Here and in the sequel  $z^\mu = \exp(\mu(\ln|z| + i \arg z))$ ,  $\arg z \in (-\pi, \pi]$ .

For  $x \in \omega_j \cup \omega_{j-1}$ , the functions  $C_{kj}(x, \lambda)$  are solutions of the equation

$$-y'' + \frac{a_j}{(x - \gamma_j)^2}y = \lambda y,$$

and  $\det[C_{kj}^{(m-1)}(x, \lambda)]_{k,m=1,2} = 1$ .

Let  $S_{kj}(x, \lambda)$ ,  $j = \overline{1, N}, k = 1, 2$  be solutions of the following integral equation

$$S_{kj}(x, \lambda) = C_{kj}(x, \lambda) + \int_{\gamma_j}^x g_j(x, t, \lambda) \left( q(t) - \frac{a_j}{(t - \gamma_j)^2} \right) S_{kj}(t, \lambda) dt; x \in \omega_j \cup \omega_{j-1},$$

where  $g_j(x, t, \lambda) = C_{1j}(t, \lambda)C_{2j}(x, \lambda) - C_{2j}(t, \lambda)C_{1j}(x, \lambda)$ .

The functions  $S_{kj}(x, \lambda)$  are entire in  $\lambda$  of order  $\frac{1}{2}$ , and they form a FSS of equation (8). Moreover

$$\det[S_{kj}^{(m-1)}(x, \lambda)]_{k,m=1,2} = 1.$$

In [6] asymptotic properties of  $S_{kj}(x, \lambda)$  are investigated. Denote

$$\Omega := \{(\rho, x) : |\rho(x - \gamma_j)| \geq 1; j = \overline{1, N}\}.$$

From [3], we have the following Lemma:

**Lemma 1** . For  $x \in \omega_s$ ,  $(\rho, x) \in \Omega$ :

$$\begin{aligned} S_{kj}^{(m)}(x, \lambda) &= \beta_{kj} \rho^{-\mu_{kj}} ((-i\rho)^m \exp(-i\rho(x - \gamma_j)) [1]_\gamma \\ &\quad + (i\rho)^m \exp(-i\pi\mu_{kj}) \exp(i\rho(x - \gamma_j)) [1]_\gamma \\ &\quad - 2i(i\rho)^m \sum_{p=j+1}^s \cos\pi v_p \exp(i\rho(x + \gamma_j - 2\gamma_p)) [1]_\gamma), s \geq j, \end{aligned} \quad (12)$$

$$\begin{aligned} S_{kj}^{(m)}(x, \lambda) &= \beta_{kj} \rho^{-\mu_{kj}} ((-i\rho)^m \exp(-i\rho(x - \gamma_j)) [1]_\gamma \\ &\quad + (i\rho)^m \exp(i\pi\mu_{kj}) \exp(i\rho(x - \gamma_j)) [1]_\gamma \\ &\quad + 2i(i\rho)^m \sum_{p=s+1}^{j-1} \cos\pi v_p \exp(i\rho(x + \gamma_j - 2\gamma_p)) [1]_\gamma), s < j, \end{aligned} \quad (13)$$

where  $[1]_\gamma = 1 + \sum_{j=1}^N O((\rho(x - \gamma_j))^{-1})$ ,  $\beta_{1j}\beta_{2j} = (-4i \sin \pi v_j)^{-1}$ .

Denote

$$\psi_k(x, \lambda) = (-1)^{k-1} (S_{2j}^{(2-k)}(T, \lambda) S_{1j}(x, \lambda) - S_{1j}^{(2-k)}(T, \lambda) S_{2j}(x, \lambda)), k = 1, 2. \quad (14)$$

The functions  $\psi_k(x, \lambda)$  are solutions of (8) and

$$\psi_k^{(m-1)}(x, \lambda) = \delta_{k,m}, k, m = 1, 2, \quad (15)$$

( $\delta_{k,m}$  is the Kronecker delta). Moreover

$$\langle \psi_1(x, \lambda), \psi_2(x, \lambda) \rangle = 1, \text{ where } \langle y, z \rangle := yz' - zy'.$$

Using (12)-(13) we get for  $x \in \omega_s$ ,  $(\rho, x) \in \Omega$  :

$$\begin{aligned} \psi_k^{(m-1)}(x, \lambda) &= \frac{1}{2} (i\rho)^{m-k} ((-1)^{m-k} \exp(i\rho(T-x)) [1]_\gamma + \exp(-i\rho(T-x)) [1]_\gamma \\ &+ (-1)^{k-1} 2i \sum_{j=s+1}^N \cos \pi v_j \exp(i\rho(x+T-2\gamma_j)) [1]_\gamma); \quad |\rho| \rightarrow \infty, k, m = 1, 2. \end{aligned} \quad (16)$$

In view of (9), we have

$$u(x, \rho) = -\psi_1(x, \lambda) + i\rho\psi_2(x, \lambda). \quad (17)$$

**Theorem 1** . For  $|\rho| \rightarrow \infty$ ,  $k, m = 1, 2$ :

$$u^{(m)}(x, \rho) = (-i\rho)^m \exp(i\rho(T-x)) [1]_\gamma$$

$$- 2i (i\rho)^m \sum_{j=s+1}^N \cos \pi v_j \exp(i\rho(x+T-2\gamma_j)) [1]_\gamma, \quad (18)$$

$$\Delta(\rho) = -\exp(i\rho T) [1] - 2i \sum_{j=1}^N \cos \pi v_j \exp(i\rho(T-2\gamma_j)) [1], \quad (19)$$

where  $[1] = 1 + O(\rho^{-1})$ .

**Proof.** Using (16), (17) we get for  $|\rho| \rightarrow \infty$ ,  $k, m = 1, 2$ ,

$$\begin{aligned} u^{(m)}(x, \rho) &= -\psi_1^{(m)}(x, \lambda) + i\rho\psi_2^{(m)}(x, \lambda) \\ &= -\left[ \frac{1}{2} (i\rho)^m ((-1)^m \exp(i\rho(T-x)) [1]_\gamma \right. \\ &\quad \left. + \exp(-i\rho(T-x)) [1]_\gamma + 2i \sum_{j=s+1}^N \cos \pi v_j \exp(i\rho(x+T-2\gamma_j)) [1]_\gamma \right] \end{aligned}$$

$$\begin{aligned}
& + i\rho\left[\frac{1}{2}(i\rho)^{m-1}((-1)^{m-1}\exp(i\rho(T-x)))[1]_\gamma\right. \\
& + \left.\exp(-i\rho(T-x))[1]_\gamma - 2i\sum_{j=s+1}^N \cos\pi v_j \exp(i\rho(x+T-2\gamma_j))[1]_\gamma\right] \\
& = -\frac{1}{2}(-i\rho)^m \exp(i\rho(T-x))[1]_\gamma - \frac{1}{2}(i\rho)^m \exp(-i\rho(T-x))[1]_\gamma \\
& - \frac{1}{2}(i\rho)^m 2i \sum_{j=s+1}^N \cos\pi v_j \exp(i\rho(x+T-2\gamma_j))[1]_\gamma \\
& - \frac{1}{2}(-i\rho)^m \exp(i\rho(T-x))[1]_\gamma + \frac{1}{2}(i\rho)^m \exp(-i\rho(T-x))[1]_\gamma \\
& - \frac{1}{2}(i\rho)^m 2i \sum_{j=s+1}^N \cos\pi v_j \exp(i\rho(x+T-2\gamma_j))[1]_\gamma.
\end{aligned}$$

Using (11) and (18) we arrive at (19).

### 3 Distribution of the eigenvalues

The main result of this paper is the following theorem on the asymptotic behavior of the eigenvalues of system differential equations when the system has two singularities and turning points  $\gamma_1$  and  $\gamma_2$  (i.e.  $N=2$ ).

**Theorem 2 .** The system of differential equations has a countable set of eigenvalues

$$\rho_{kj} = \frac{k\pi}{\gamma_1} + \frac{LnA_j}{2i\gamma_1}; k \in Z, j = 1, 2,$$

where

$$A_j = \frac{a^2 - b^2 + [1] + (-1)^j \sqrt{(b^2 - a^2 + [1])^2 - 4a^2[1]}}{2a[1]}, a = 2i\cos\pi v_1, b = 2i\cos\pi v_2.$$

**Proof.** According to (19) from  $\Delta(\rho) = 0$  we have,

$$-\exp(i\rho T)[1] - 2i \sum_{j=1}^2 \cos\pi v_j \exp(i\rho(T-2\gamma_j))[1] = 0$$

$$2i \sum_{j=1}^2 \cos\pi v_j \exp(-2i\rho\gamma_j) = [1],$$

$$\begin{cases} 2i\cos\pi v_1 \exp(-2i\rho\gamma_1) + 2i\cos\pi v_2 \exp(-2i\rho\gamma_2) = [1], \\ 2i\cos\pi v_1 \exp(2i\rho\gamma_1) + 2i\cos\pi v_2 \exp(2i\rho\gamma_2) = [1]. \end{cases} \quad (20)$$

Now consider

$$a = 2i\cos\pi v_1, b = 2i\cos\pi v_2, X = \exp(2i\rho\gamma_1), Y = \exp(2i\rho\gamma_2). \quad (21)$$

Substituting (21) into (20), we obtain the following system with respect to X and Y :

$$aX + bY = [1], \quad (22)$$

$$\frac{a}{X} + \frac{b}{Y} = [1]. \quad (23)$$

In view of (23), we have

$$Y = \frac{bX}{[1]X - a}.$$

Substituting Y into (22),

$$X = \frac{a^2 - b^2 + [1] \pm \sqrt{(b^2 - a^2 + [1])^2 - 4a^2[1]}}{2a[1]}.$$

According to (21), we have

$$\exp(2i\rho_{kj}\gamma_1 - 2k\pi i) = A_j; j = 1, 2, k \in Z,$$

where

$$A_j = \frac{a^2 - b^2 + [1] + (-1)^j \sqrt{(b^2 - a^2 + [1])^2 - 4a^2[1]}}{2a[1]}.$$

Conclude,

$$\rho_{kj} = \frac{k\pi}{\gamma_1} + \frac{LnA_j}{2i\gamma_1}.$$

## References

- [1] V. P. Meshanov and A. L. Feldstein, *Automatic Design of Directional Couplers*, Moscow: Sviza, (1980) (in Russian).
- [2] A. G. Sveshnikov and A. S. Il'inskii, *Design problems in electrodynamics*, Dokl. Akad. Nauk SSSR. **204**(1972), 5, 1077-1080.
- [3] G. Freiling and V. Yurko, *On the determination of differential equations with singularities and turning points*, Results in Mathematics. **41**(2002) 275-290.
- [4] G. Freiling and V. Yurko, *inverse Sturm-Liouville problems and their applications*, Nova Science Publishers, Inc, Huntington, New York,(2001).
- [5] G. Freiling and V. Yurko, *Inverse problems for differential operators with singular boundary conditions*, Math. Nachrichten, to appear.
- [6] V. A. Yurko, *Inverse problem for differential equations with a singularity*, Differ Uravneniya **28**(1992), no. 8, 1355-1362 (in Russian); English transl. in Differ Equations **28**(1992), 1100-1107.
- [7] O. N. Litvinenko and V. I. Soshnikov, *The Theory of Heterogeneous Lines and their Applications in Radio Engineering*, Moscow: Radio, (1946) (in Russian).
- [8] W. Eberhard and G. Freiling and K. Wilcken, *Indefinite Eigenvalue problems with Several Singular Points and Turning Points*, Math. Nachr. **229**(2001), 51-71.

**Received: August 14, 2007**