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On Some Fractional Integro-Differential Equations

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Abstract

In this paper we study a class of fractional integro-differential equations considered in an arbitrary Banach space. Mild solutions for the considered class are studied. The existence and uniqueness of the considered problem is also studied. We also give an application for partial differential equations of fractional order.

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1 Introduction

Let E denotes a Banach space and let J denotes the closure of the interval $[t_0, T]$,
$t_0 < T \leq \infty$. In this paper , we consider the fractional integro-differential equation in a Banach space E :

$$\frac{d^\alpha u(t)}{dt^\alpha} + Au(t) = f(t, u(t)) + K(u)(t), \quad t \geq t_0,\ u(t_0) = u_0 \tag{1.1}$$

where $0 < \alpha \leq 1$ and

$$K(u)(t) = \int_{t_0}^{t} a(t-s)g(s, u(s))ds$$

Let $-A$ is the infinitesimal generator of an analytic semigroup $S(t)$, $t \geq 0$ in E, the function a is real-values and locally integrable on $[0, \infty)$, and the nonlinear maps f and g are defined on $[0, \infty) \times E$ into E. We note that if $-A$ is the

infinitesimal generator of an analytic semigroup then $-(A+\lambda I)$ is invertible and generates a bounded analytic semigroup for
$\lambda > 0$ large enough. This allows us to reduce the general case in which $-A$ is the infinitesimal generator of an analytic semigroup to the case in which the semigroup is bounded and the generator is invertible. hence for convenience, we suppose that $\| S(t) \| \leq M$ for $t \geq 0$ and $0 \in \rho(-A)$, where $\rho(-A)$ is the resolvent of $-A$. It follows that for $0 \leq \lambda \leq 1$, A^λ can be defined as a closed linear invertible operator with its domain $D(A^\lambda)$ being dense in E. We denote by E_λ the Banach space $D(A^\lambda)$ equipped with norm

$$\| x \|_\lambda = \| A^\lambda x \|_\lambda$$

which is equivalent to the graph norm of A^λ. We have $E_\beta \to E_\lambda$ for $0 < \lambda < \beta$ and the embedding is continuous. By a classical solution to (1.1) on J, we mean a function $u \in G(J,E) \cap C^1(J \backslash \{t_0\}; E)$ satisfing (1.1) on J. By a classical solution to (1.1) on J we mean there exist a T_0, $t_0 < T_0 < T$, and a function u defined from $J_0 = [t_0, T_0]$ into E such that u is a classical solution to (1.1) on J_0.
Integro-differential equations are studied in many papers (see [1], [2], [3], [4], [5]). In this paper we establish the existence of a unique classical solution to (1.1), we shall require the following assumption on the maps f and g.
Assumption 1: Let U be an open subset of $[0,\infty) \times E_\lambda$. For every $(t,x) \in U$ there exist a neighborhood $V \subset U$ of (t,x) and constants $L \geq 0$, $0 < \gamma < 1$ such that

$$\| f(\theta_1, u) - f(\theta_2, v) \| \leq L[|\theta_1 - \theta_2|^\gamma + \| u - v \|_\lambda] \tag{1.2}$$

for all (θ_1, u) and (θ_2, v) in V.
By a mild solution to (1.1) on J we mean a continuous function u defined from J into E satisfying the integral equation

$$u(t) = \int_0^\infty \xi_\alpha(\theta) S((t-t_0)^\alpha \theta) u_0$$

$$+\alpha \int_{t_0}^t \int_0^\infty \theta (t-\eta)^{\alpha-1} \xi_\alpha(\theta) S((t-\eta)^\alpha \theta)[f(\eta, u(\eta)) + K(u)(\eta)] d\theta d\eta, \ t \in J, \tag{1.3}$$

where $\xi_\alpha(\theta)$ is a probability density function defined on $(0,\infty)$,

$$\int_0^\infty \xi_\alpha(\theta) d\theta = 1 \ \ (see \ [6], \ [7], \ [8], \ [9], \ [10]).$$

We say that (1.1) has a local mild solution if there exist a T_0, $0 < T_0 < T$ and a continuous function u defined from $J_0 = [t_0, T_0]$ into E such that u is a mild to (1.1) on J_0.

To establish the existence of a unique local mild solution, we only need the following assumption on f and g.

Assumption 2 : Let U be an open subset of $[0, \infty) \times E_\lambda$. For every $(t, x) \in U$ there exist a neghborhood $V \subset U$ of (t, x) and a constant $L_0 > 0$ such that

$$\| f(\theta, u) - f(\theta, v) \| \leq L_0 \| u - v \|_\lambda \qquad (1.4)$$

for all (θ, u) and (θ, v) in V.

2 Local Existence of Mild Solutions

As pointed out earlier, we may suppose without loss of generality that the analytic semigroup generated by $-A$ is bounded and that $-A$ is invertible. Furthermore, we assume that $0 < T < \infty$ to establish local existence. We can therfore state the following theorem

Theorem 2.1 : Suppose that the operator $-A$ generates the analytic semigroup $S(t)$ with $\| S(t) \| \leq M,\ t \geq 0$ and that $0 \in \rho(-A)$. If the maps f and g satisfy Assumption 2 and the real valued map a is integrable on J, then (1.1) has a unique local mild solution for every $u_0 \in E_\lambda$.

Proof: We sall use the notions and notations introduced in the preceding section. We fix a point (t_0, u_0) in the open subset U of $[0, \infty) \times E_\lambda$ and choose $t_1' > t_0$ and $\delta > 0$ such that (1.4), with some constant $L_0 > 0$ holds for the functions f and g on the set

$$V = \{(t, x) \in U : t_0 \leq t \leq t_1', \ \ \| x - u_0 \|_\lambda \leq \delta\}. \qquad (2.1)$$

Let

$$B_1 = \sup_{t_0 \leq t \leq t_1'} \| f(t, u_0) \|$$

and Let

$$B_2 = \sup_{t_0 \leq t \leq t_1'} \| g(t, u_0) \| .$$

Choose $t_1 > t_0$ such that

$$\| S(t - t_0) - I \| \| A^\lambda u_0 \| \leq \frac{1}{2}\delta. \ for\ t_0 \leq t \leq t_1 \qquad (2.2)$$

and

$$t_1 - t_0 < min\{t_1' - t_0, [\frac{\delta}{2} C_\lambda^{-1} \alpha^2 (1-\lambda) \{(L_0\delta + B_1) + a_T(L_0\delta + B_2)\}^{-1}]^{\frac{1}{\alpha(1-\lambda)}}\} \qquad (2.3)$$

where C_λ is a positive constant depending on λ satisfying

$$\| A^\lambda S(t) \| \leq C_\lambda t^{-\lambda}, \ for\ t > t_0, \quad (2.4)$$

and

$$a_T = \int_0^T |a(s)|ds. \qquad (2.5)$$

Let $Y = C([t_0, t_1]; E)$ be endowed with the supermum norm

$$\| y \|_Y = \sup_{t_0 \le t \le t} \| y(t) \| .$$

Then Y is a Banach space. We define a map on Y by $Fy = \tilde{y}$ where $\tilde{y}$ is given by

$$\tilde{y} = \int_0^\infty \xi_\alpha(\theta) S((t-t_0)^\alpha \theta) A^\lambda u_0 d\theta$$

$$+\alpha \int_{t_0}^t \int_0^\infty \theta (t-\eta)^{\alpha-1} \xi_\alpha(\theta) A^\lambda S((t-\eta)^\alpha \theta) f(\eta, A^{-\lambda} y(\eta)) d\theta d\eta$$

$$+\alpha \int_{t_0}^t \int_0^\infty \theta (t-\eta)^{\alpha-1} \xi_\alpha(\theta) A^\lambda S((t-\eta)^\alpha \theta) [\int_{t_0}^\eta a(\eta-\tau) g(\tau, A^{-\lambda} y(\tau)) d\tau] d\theta d\eta.$$

Now, for every $y \in Y$, $Fy(t_0) = A^\lambda u_0$ and for $t_0 \le \theta \le t \le t_1$ we have

$$Fy(t) - Fy(s) = \int_0^\infty [\xi_\alpha(\theta) S((t-t_0)^\alpha \theta) - \xi_\alpha(\theta) S((s-t_0)^\alpha \theta)] A^\lambda u_0 d\theta +$$

$$\alpha \int_s^t \int_0^\infty \theta (t-\eta)^{\alpha-1} \xi_\alpha(\theta) A^\lambda S((t-\eta)^\alpha \theta) [f(\eta, A^{-\lambda} y(\eta))$$

$$+ \int_{t_0}^\eta a(\eta-\tau) g(\tau, A^{-\lambda} y(\tau)) d\tau] d\theta d\eta +$$

$$\alpha \int_{t_0}^s \int_0^\infty \theta (t-\eta)^{\alpha-1} \xi_\alpha(\theta) A^\lambda [S((t-\eta)^\alpha \theta) - S((s-\eta)^\alpha \theta)] [f(\eta, A^{-\lambda} y(\eta))$$

$$+ \int_{t_0}^\eta a(\eta-\tau) g(\tau, A^{-\lambda} y(\tau)) d\tau] d\theta d\eta.$$

It follows from Assumption 2 on the functions f and g, (2.4) and (2.5) that $F : Y \to Y$.

Let Z be the nonempty closed and bounded set given by

$$Z = \{y \in Y : y(t_0) = A^\lambda u_0, \| y(t) - A^\lambda u_0 \| \le \delta\}. \qquad (2.6)$$

Then for $y \in Z$ we have

$$\| Fy(t) - A^\lambda u_0 \| \le \| \int_0^\infty \xi_\alpha(\theta) (S((t-t_0)^\alpha \theta) - I) A^\lambda u_0 d\theta \|$$

$$+ \| \alpha \int_{t_0}^t \int_0^\infty \theta (t-\eta)^{\alpha-1} \xi_\alpha(\theta) A^\lambda S((t-\eta)^\alpha \theta) [f(\eta, A^{-\lambda} y(\eta)) - f(\eta, u_0)] d\theta d\eta] \|$$

$$+ \| \alpha \int_{t_0}^t \int_0^\infty \theta (t-\eta)^{\alpha-1} \xi_\alpha(\theta) A^\lambda S((t-\eta)^\alpha \theta) f(\eta, u_0) d\theta d\eta \|$$

$$+\parallel \alpha\int_{t_0}^{t}\int_0^\infty \theta(t-\eta)^{\alpha-1}\xi_\alpha(\theta)A^\lambda S((t-\eta)^\alpha\theta)[\int_{t_0}^{\eta} a(\eta-\tau)(g(\tau,A^{-\lambda}y(\tau))-g(\tau,u_0))d\tau]d\theta d\eta \parallel$$

$$+\parallel \alpha\int_{t_0}^{t}\int_0^\infty \theta(t-\eta)^{\alpha-1}\xi_\alpha(\theta)A^\lambda S((t-\eta)^\alpha\theta)[\int_{t_0}^{\eta} a(\eta-\tau)g(\tau,u_0)d\tau]d\theta d\eta \parallel$$

$$\leq \int_0^\infty \xi_\alpha(\theta)\parallel S((t-t_0)^\alpha\theta)-I\parallel\parallel A^\lambda u_0\parallel d\theta$$

$$+\alpha\int_{t_0}^{t}(t-\eta)^{\alpha(1-\lambda)-1}\parallel f(\eta,A^{-\lambda}y(\eta))-f(\eta,u_0)\parallel d\eta$$

$$+\alpha\int_{t_0}^{t}(t-\eta)^{\alpha(1-\lambda)-1}[\int_{t_0}^{\eta}|a(\eta-\tau)|\parallel g(\tau,A^{-\lambda}y(\tau))-g(\tau,u_0)\parallel d\tau]d\eta$$

$$+\alpha\int_{t_0}^{t}(t-\eta)^{\alpha(1-\lambda)-1}[\int_{t_0}^{\eta}|a(\eta-\tau)|\parallel g(\tau,u_0)\parallel d\tau]d\eta$$

$$\leq \frac{1}{2}\delta+\alpha C_\lambda[L_0\delta+B_1+a_TL_0\delta+a_TB_2]\int_{t_0}^{t_1}(t_1-\eta)^{\alpha(1-\lambda)-1}d\eta$$

$$=\frac{1}{2}\delta+\alpha C_\lambda[L_0\delta+B_1+a_T(L_0\delta+B_2)](\alpha(1-\lambda)^{-1}(t_1-t_0)^{\alpha(1-\lambda)})\leq\delta \quad (2.7)$$

where the last two inequalities follow from (2.2) and (2.3). Thus, we have that $F : Z \to Z$. Now we show that F is a strict contraction on Z which will ensure the existence of a unique continuous function satisfying equation (1.3). Let y and z in Z; then

$$\parallel Fy(t)-Fz(t)\parallel=\parallel \tilde{y}(t)-\tilde{z}(t)\parallel$$

$$\leq \alpha\int_{t_0}^{t}(t-\eta)^{\alpha(1-\lambda)-1}\parallel f(\eta,A^{-\lambda}y(\eta))-f(\eta,A^{-\lambda}z(\eta))\parallel d\eta$$

$$+\alpha\int_{t_0}^{t}(t-\eta)^{\alpha(1-\lambda)-1}[\int_{t_0}^{\eta}|a(\eta-\tau)|\parallel g(\tau,A^{-\lambda}y(\tau))-g(\tau,A^{-\lambda}z(\tau))\parallel d\tau]d\eta. \quad (2.8)$$

Using Assumption 2 on f and g and (2.4), (2.5), we get

$$\parallel Fy(t)-Fz(t)\parallel\leq \alpha L_0[(1+a_T)\int_{t_0}^{t}(t-\eta)^{\alpha(1-\lambda)-1}d\eta]\parallel y-z\parallel_Y$$

$$\leq \alpha L_0(1+a_T)C_\lambda(\alpha(1-\lambda))^{-1}(t_1-t_0)^{\alpha(1-\lambda)}\parallel y-z\parallel_Y$$

$$\leq \frac{\alpha}{\delta}L_0\delta(1+a_T)C_\lambda(\alpha(1-\lambda))^{-1}(t_1-t_0)^{\alpha(1-\lambda)}\parallel y-z\parallel_Y$$

$$\leq \frac{1}{\delta}[L_0\delta+B_1+a_T(L_0\delta+B_2)]\alpha C_\lambda(\alpha(1-\lambda))^{-1}(t_1-t_0)^{\alpha(1-\lambda)}\parallel y-z\parallel_Y$$

$$\leq \frac{1}{2}\parallel y-z\parallel_Y, \quad (2.9)$$

using (2.3) in the last inequality. Thus F is a strict contraction map from Z into Z and therefore by the Banach contraction principle there exist a unique fixed point y of F in Z, i.e., there is a unique $y \in Z$ such that $Fy = y = \tilde{y}$. Let $u = A^{-\lambda}y$. Then for $t \in [t_0, t_1]$, we have

$$u(t) = A^{-\lambda}y(t)$$
$$= \int_0^\infty \xi_\alpha(\theta)S((t-t_0)^\alpha\theta)u_0 d\theta$$
$$+\alpha\int_{t_0}^t\int_0^\infty \theta(t-\eta)^{\alpha-1}\xi_\alpha(\theta)S((t-\eta)^\alpha\theta)[f(\eta,u(\eta))+K(u)(\eta)]d\theta d\eta.$$

Hence u is a unique local mild solution.

3 Application

Let $\Omega \subset R^n$ be a bounded with smooth boundary $\partial\Omega$. Consider the linear partial differential operator

$$A(x,D) = \sum_{|\beta|\le 2m} a_\beta(x)D^\beta, \qquad (3.1)$$

where $a_\beta(x)$ is a real or complex valued function defined on $\overline{\Omega}$ for each multi-index β. We assume that $A(x,D)$ is strongly elliptic, i.e., there exists a constant $c > 0$ such that

$$\sum_{|\beta|=2m} a_\beta(x)\zeta^\beta \ge c|\zeta|^{2m} \qquad (3.2)$$

for all $x \in \overline{\Omega}$ and $\zeta \in R^n$. Consider the parabolic integrodifferential equation

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + A(x,D)u(x,t) = f(x,t,u(x,t),Du(x,t),...,D^{2m-1}u(x,t))$$
$$+\int_{t_0}^t a(t-s)g(x,s,u(x,s),Du(x,s),...,D^{2m-1}u(x,s))ds,$$
$$x \in \Omega,\ t > t_0,\ 0 < \alpha \le 1 \qquad (3.3)$$

$$u(x,t_0) = u_0(x), x \in \Omega$$
$$u(x,t) = 0, x \in \Omega, t \in [t_0,T), t_0 < T \le \infty,$$

where D^j stands for any j-th order derivative. We assume that f and g are continuously differentiable functions of all their variables, except possibly in x.

The parabolic integrodifferential equation (3.3) can be reformulated as the following abstract integrodifferential equation in $E = L^p(\Omega)$:

$$\frac{d^\alpha u(t)}{dt^\alpha} + A_p u(t) = F(t,u(t)) + \int_{t_0}^t a(t-s)G(s,u(s))ds,\ t \ge t_0,\ u(t_0) = u_0, \quad (3.4)$$

where $A_p : D(A_p) \subset E \to E$ given by

$$D(A_p) = W^{2m,p}(\Omega) \cap W_0^{2m,p}, \; A_p u = A(x, D)u + \mu u \;\; for \; u \in D(A_p), \; \mu > 0$$

and $F, G : [t_0, T) \times D(A_p) \to E$ are given by

$$F(t, u)(x) = f(x, t, u(x, t), Du(x, t), ..., D^{2m-1}u(x, t)) \quad (3.5)$$

$$G(t, u)(x) = g(x, t, u(x, t), Du(x, t), ..., D^{2m-1}u(x, t)) \quad (3.6)$$

where we assume the usual sufficient Caratheodary and grorowth conditions on the functions f and g in (3.5) and (3.6) to be well defined . Here we assume that μ is large enough so that A_p is invertible. It follows that $-A_p$ is the infinitesimal generator of an analytic semigroup on E. Also, from imbeding theorems it follows that E_λ is continuously imbedded in $C^{2m-1}(\overline{\Omega})$ for $1 - \frac{1}{2m} < \lambda < 1$ and p large enough. It can be verfied that the Assumption1 is satisfied by F and G. Consequently theorem 2.1 can be applied for the equation (3.3) under suitable assumptions on the kernel a.

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