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High-Order Quadrature Rules for Acoustic Scattering Calculations

Juan C. Aguilar[1]

Intituto Tecnológico Autónomo de México
Departamento de Matemáticas, México, D.F. 01000, México
aguilar@itam.mx

Yu Chen

Courant Institute of Mathematical Sciences
New York University, New York, NY 10012, USA
yuchen@cims.nyu.edu

Abstract

In this report we construct corrected trapezoidal quadrature rules up to order 40 to evaluate 2-dimensional integrals of the form $\int_D v(x,y)\log(\sqrt{x^2+y^2})dxdy$, where the domain D is a square containing the point of singularity $(0,0)$ and v is a C^∞ function of compact support contained in D. The procedure we use is a modification of the method constructed in [1]. These quadratures are particularly useful in acoustic scattering calculations with large wave numbers. We describe how to extend the procedure to calculate other 2-dimensional integrals with different singularities.

Mathematics Subject Classification: 65D30

Keywords: Quadrature rules, Corrected trapezoidal rule, Correction coefficients, Logarithmic singularity

1 Introduction

Acoustic scattering calculations involve the evaluation of integrals of singular functions. For the 2-dimensional case, integrals of the form

$$J(v) = \int_D v(x,y)\log(\sqrt{x^2+y^2})dxdy, \tag{1}$$

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where v is a function of compact support contained in D, are encountered (see [2], [4]). In [2] it is argued the necessity of higher-order quadrature rules in acoustic scattering calculations when large wave numbers are used.

In [1] it is described a $20th$ order quadrature rule to evaluate integrals of the form (1). In this paper we construct a $40th$ order quadrature for the calculation of (1). The procedure we use is a modification of the approach described in [1]. This new approach makes simpler to obtain quadratures of higher order in 2-D.

The paper is organized as follows: Section 2 contains the notation and main definitions. Sections 3 and 4 describe how to construct the new quadrature rules for integrals of the form (1). In Section 5 we describe how to obtain quadratures for integrals with other singularities.

2 Definitions and notation

The construction of the quadratures rules to evaluate integrals of the form (1) can be described using most of the notation and definitions described in [1]. For completeness of this paper we include part of the notation.

Let $D = [a_1, b_1] \times [a_2, b_2]$ be a square that contains the point $(0, 0)$ and $v : \mathbb{R}^2 \to \mathbb{R}$ a C^∞ function. Define the function f as

$$f(x, y) = v(x, y) \log(\sqrt{x^2 + y^2}) \quad \text{if } (x, y) \neq (0, 0), \tag{2}$$

and let $\tilde{f}$ be given by

$$\tilde{f}(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases} \tag{3}$$

Let n be a positive integer, and $h = (b_1 - a_1)/(n - 1)$. Discretize the square D with n^2 grid points distributed uniformly:

$$P_{i,j} = (a_1 + ih, a_2 + jh), \qquad i, j = 0, \ldots, n - 1. \tag{4}$$

We assume that $(0, 0)$ is one of the grid points. There are 3 main parts involved in the construction of the quadratures:

- The trapezoidal rule
- Boundary correction to the trapezoidal rule
- Correction near $(0, 0)$ due to the singularity.

The *trapezoidal* rule to approximate an integral $\int_D g(x,y)dxdy$ with respect to the grid points $\{P_{i,j}\}$ is defined as

$$T_h(g) = h^2 \left(\sum_{j=1}^{n-2} S_j + \frac{1}{2}(S_0 + S_{n-1}) \right), \tag{5}$$

where

$$S_j = \sum_{i=1}^{n-2} g(P_{i,j}) + \frac{1}{2}(g(P_{0,j}) + g(P_{n-1,j})) \quad \text{for } j = 0, \ldots, n-1. \tag{6}$$

The *boundary corrected trapezoidal rule* applied to a function g of two variables consists of the trapezoidal rule plus a weighted sum of values of g evaluated at grid points close to the boundary of the square D : following the notation of [5], let m be a positive odd integer and β_k^m, $k = 1, \ldots, (m-1)/2$ be the $(m-1)/2$ coefficients for boundary correction (see [5] for their numerical values). If $\{P_{i,j}\}$ is the grid defined in (4) used to discretize the square D and $g : \mathbb{R}^2 \to \mathbb{R}$ is a function, then the *boundary corrected trapezoidal rule* applied to g with respect to the n^2 grid points $\{P_{i,j}\}$ is denoted by $T^n_{\beta^m}(g)$, and is given by the formula

$$T^n_{\beta^m}(g) =$$

$$h^2 \left(\sum_{j=1}^{n-2} S_j + \frac{1}{2}(S_0 + S_{n-1}) \right) + h^2 \sum_{k=1}^{\frac{m-1}{2}} \left(-S_{-k} + S_k + S_{n-1-k} - S_{n-1+k} \right) \beta_k^m \tag{7}$$

where

$$\begin{aligned} S_j \quad = \quad & \sum_{i=1}^{n-2} g(P_{i,j}) + \frac{1}{2}(g(P_{0,j}) + g(P_{n-1,j})) \\ & + \sum_{k=1}^{\frac{m-1}{2}} \left(-g(P_{-k,j}) + g(P_{k,j}) + g(P_{n-1-k,j}) - g(P_{n-1+k,j}) \right) \beta_k^m, \end{aligned} \tag{8}$$

for $j = -(m-1)/2, \ldots, n-1+(m-1)/2$.

If the function g has $m+1$ continuous derivatives then (see [5])

$$\int_D g(x,y)dxdy - T^n_{\beta^m}(g) = O(h^{m+1}). \tag{9}$$

A correction near $(0,0)$ due to the logarithmic singularity associated to a smooth function v and to the grid $G^h = \{(hp, hq) | p, q \in \mathbb{Z}\}$ is a weighted sum of the form

$$h^2 v(0,0)\log(h) + h^2 \left(\sum_{r=1}^{k} c_r \sum_{(ph,qh)\in G_r} v(ph, qh) \right),$$

where for each positive integer r the set G_r is a finite subset of the grid G^h, and with the properties that $G_i \bigcap G_j = \{\}$ if $i \neq j$, and $\bigcup_{r=1}^{\infty} G_r = G^h$. The numbers c_r are called *correction coefficients* associated to the sets G_r, and they are independent of h and of v. The quadratures of [1] and the ones in this paper take the form

$$\begin{aligned}\int_D v(x,y)\log(\sqrt{x^2+y^2})dxdy \quad &\approx \quad T^n_{\beta^m}(\tilde{f}) + h^2 v(0,0)\log(h) \\ &+ \quad h^2 \left(\sum_{r=1}^{k} c_r \sum_{(ph,qh)\in G_r} v(ph, qh) \right), \end{aligned} \tag{10}$$

and when v has compact support contained in D the quadratures reduce to

$$\int_D v(x,y)\log(\sqrt{x^2+y^2})dxdy$$

$$\approx T_h(\tilde{f}) + h^2 v(0,0)\log(h) + h^2 \left(\sum_{r=1}^{k} c_r \sum_{(ph,qh)\in G_r} v(ph, qh) \right). \tag{11}$$

In [1] it was found a particular distribution of the sets G_r such that when the first $k = s(s+1)/2 + 1$ of them are used, there are associated correction coefficients c_r such that if v has compact support then

$$\begin{aligned}\int_D v(x,y)\log(\sqrt{x^2+y^2})dxdy \quad &= \quad T_h(\tilde{f}) + h^2 v(0,0)\log(h) \\ &+ \quad h^2 \left(\sum_{r=1}^{k} c_r \sum_{(ph,qh)\in G_r} v(ph, qh) \right) + O(h^{4+2s}), \end{aligned} \tag{12}$$

In the next section we describe a different way to select the distribution of the sets G_r which has the advantage of reducing the number of correction coefficients (roughly by a factor of 2) involved in the approximation (11) while keeping the same order of accuracy. The procedure can be extended to 3 dimensions where according to our preliminary results the savings in the number of correction coefficients is roughly by a factor of 6 when compared to the quadratures obtained in [3].

3 Selection of the sets G_r for logarithmic correction

We describe in this section a different way to select the sets G_r than the method of [1]. For any positive integer r the set G_r containing grid points is determined by a single grid point. If $(hp, hq) \in G_r$ then G_r contains the points of the form $(\pm ph, \pm qh)$, $(\pm qh, \pm ph)$. Therefore any set G_r contains at most 8 grid points. So to define a set G_r we will specify its grid point (ph, qh) that satisfies $0 \leq q \leq p$. Figure 3 shows a point $\mathbf{P}_r$ that belongs to G_r for $r = 1, ..., 16$. For instance, $\mathbf{P}_1 = (0,0)$ belongs to G_1, therefore $G_1 = \{(0,0)\}$; $\mathbf{P}_{10} = (3h, 2h)$ belongs to G_{10}, hence $G_{10} = \{(3h,2h),(3h,-2h),(2h,3h),(-2h,3h),(-3h,2h),(-3h,-2h),$ $(2h,-3h),(-2h,-3h)\}$. In general, given a grid point (ph, qh), this point belongs to the set G_r were r is given as follows: if $t = |p| + |q|$, then

$$r = \begin{cases} \dfrac{(t+2)^2}{4} - \min(|p|,|q|), & \text{if } t \text{ is even,} \\ \dfrac{(t+1)(t+3)}{4} - \min(|p|,|q|), & \text{if } t \text{ is odd.} \end{cases} \tag{13}$$

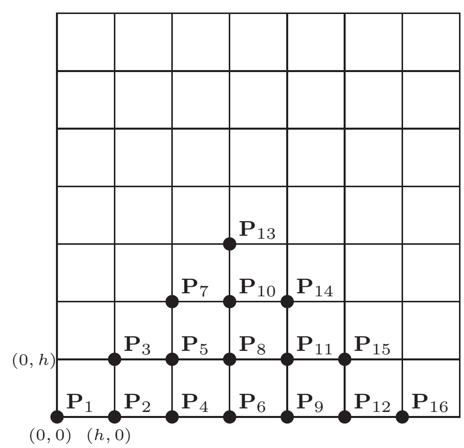

Figure 1: Points $\mathbf{P}_r$ belong to the set G_r. If $\mathbf{P}_r = (a, b)$ then all points in G_r are of the form $(\pm a, \pm b)$, $(\pm b, \pm a)$.

We now describe how to calculate the correction coefficients associated to the first k sets G_1,...,G_k: given any integer $k \geq 1$, for each integer $r \in [1, k]$ let (ph, qh) be in G_r. Define the monomial function

$$v_r(x, y) = x^{2|p|} y^{2|q|}, \tag{14}$$

let $f_r(x,y) = v_r(x,y)\log(\sqrt{x^2+y^2})$, and let $\tilde{f}_r$ be defined by (3). Following the technique of central corrections for 1-D (see [5]), let $D = [-1,1]\times[-1,1]$, and $h = 1/40$ be the distance between sampling points in the discretization of D given by (4). We set up a linear system of k equations with unknowns $c_1, c_2, ..., c_k$

$$\begin{aligned}\int_D v_r(x,y)\log(\sqrt{x^2+y^2})dxdy &= T^n_{\beta m}(\tilde{f}_r) + h^2 v(0,0)\log(h) \\ &+ h^2\left(\sum_{r=1}^{k} c_r \sum_{(ph,qh)\in G_r} v_r(ph,qh)\right), \quad (15)\end{aligned}$$

for $r = 1, ..., k$. In (15) the integrals $\int_D v_r(x,y)\log(\sqrt{x^2+y^2})dxdy$ were calculated analytically, and m was set as 41. The system of equations was solved numerically with a floating point arithmetic of 100 decimal digits. This is due to the high condition number of the matrix that defines the system of equations (15). Although the solution vector $\boldsymbol{c^k} = (c_1, ..., c_k)$ depends on h, our numerical experiments indicate that such dependence is very weak in the sense that if $h \le 1/40$ then any two solution vectors $\boldsymbol{c^k} = (c_1, ..., c_k)$ and $\tilde{\boldsymbol{c}}^{\boldsymbol{k}} = (\tilde{c}_1, ..., \tilde{c}_k)$ calcultaded with different values of h satisfy $|c_r - \tilde{c}_r|/|c_r| \le 10^{-20}$ for $r = 1, ..., k$ if $k \le 100$. Therefore, in the widely used floating point arithmetic of 16 decimal places, the coefficients $\boldsymbol{c^k} = (c_1, ..., c_k)$ can be regarded as independent of h. Given any $k \le 100$, the coefficients $\boldsymbol{c^k} = (c_1, ..., c_k)$ are calculated only once, and they are used in Formula (11) to approximate integrals of the form $\int_D v(x,y)\log(\sqrt{x^2+y^2})dxdy$, where D is a square, and v has compact support contained on D.

If the number of coefficients k is of the form $k = \dfrac{(s+2)^2}{4}$ if s is even, or $k = \dfrac{(s+1)(s+3)}{4}$ if s is odd, then according to our numerical experiments we obtain a quadrature of order h^{4+2s},

$$\begin{aligned}\int_D v(x,y)\log(\sqrt{x^2+y^2})dxdy &= T_h(\tilde{f}) + h^2 v(0,0)\log(h) \quad (16) \\ &+ h^2\left(\sum_{r=1}^{k} c_r \sum_{(ph,qh)\in G_r} v(ph,qh)\right) + O(h^{4+2s}),\end{aligned}$$

provided v is a C^∞ function with compact support contained in D. Table 2 in the appendix shows the correction coefficients to obtain quadratures of orders 4, 6, 14, 20, and 40.

4 Numerical experiments

We tested the quadratures defined in (16) using v as a linear combination of 25 Gaussian functions:

$$v(x, y) = 0.4 \sum_{i,j=2}^{6} f_{i,j}(x, y), \tag{17}$$

such that for $i, j \in \{1, 2, ..., 7\}$, $f_{i,j}$ is an exponential function of the form

$$f_{i,j}(x, y) = e^{-w((x-c_i)^2+(y-c_j)^2)},$$

whose center $(c_i, c_j) \in [-1, 1]\times[-1, 1]$ has coordinates $(c_i, c_j) = (-1+i\Delta_c, -1+j\Delta_c)$, where $\Delta_c = 2/8$. The constant w defined as $w = -25/(\Delta_c)^2$. The functions $f_{i,j}$ are not included in (17) when i or $j \in \{1, 7\}$; this is to ensure that q is close to zero near the boudary of $D = [-1, 1] \times [-1, 1]$. The graph of v is shown in Figure 2. As domain of integration we used the square $D =$

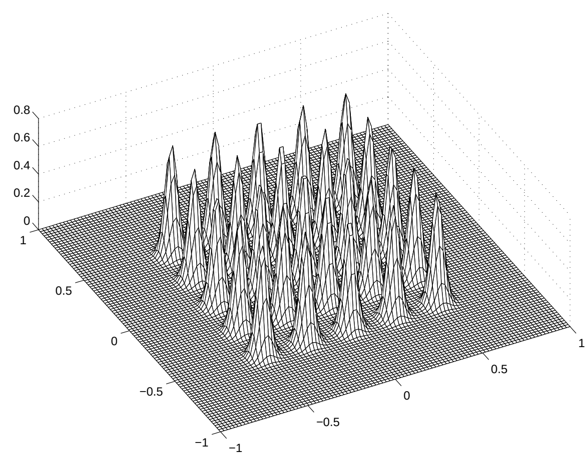

Figure 2: Test function v formed by 25 Gaussian functions.

$[-1, 1] \times [-1, 1]$. Table 1 shows the relative errors obtained for different values of the number n of mesh points in each direction and for several values of the number k of coefficients. Using k = 0, 1, 2, 16, 25, and 100 we obtain quadratures of orders 2, 4, 6, 16, 20, and 40 respectively.

k	Order	$n = 81$, Relative Error	$n = 161$, Relative Error
0	2	4.9×10^{-3}	1.2×10^{-3}
1	4	1×10^{-4}	5.8×10^{-6}
2	6	1.6×10^{-5}	2.4×10^{-7}
12	14	8.4×10^{-7}	1.5×10^{-10}
25	20	3.18×10^{-7}	4.8×10^{-12}
100	40	7.7×10^{-8}	5.8×10^{-15}

Table 1: Relative errors produced by applying the quadratures (16) to the function $f(x, y) = v(x, y) \log(\sqrt{x^2 + y^2})$, with v defined in (17). The domain of integration is $D = [-1, 1] \times [-1, 1]$, and n^2 is the number of equally spaced grid points used to discretize D.

5 Quadratures for other singularities

We describe in this section how to apply the method of central correction for logarithmic singularities to obtain quadratures for singular integrals of the form

$$\int_D \frac{v(x, y)}{\sqrt{x^2 + y^2}} dxdy. \tag{18}$$

A 3-dimensional version of this integral appears in acoustic scattering calculations.

Let

$$f(x, y) = \frac{v(x, y)}{\sqrt{x^2 + y^2}} \quad \text{if } (x, y) \neq (0, 0), \tag{19}$$

and let $\tilde{f}$ be given by (3).

Take G_r as the sets of grid points defined in Section 3. Due to the singularity of $1/\sqrt{x^2 + y^2}$, a correction near $(0, 0)$ associated to a smooth function v and to the grid $G^h = \{(hp, hq)|p, q \in \mathbb{Z}\}$ takes the form

$$h \left(\sum_{r=1}^{k} c_r \sum_{(ph,qh) \in G_r} v(ph, qh) \right).$$

In this case it can be found correction coefficients $c_1, ..., c_k$ such that

$$\int_D \frac{v(x, y)}{\sqrt{x^2 + y^2}} dxdy = T_h(\tilde{f}) + h \left(\sum_{r=1}^{k} c_r \sum_{(ph,qh) \in G_r} v(ph, qh) \right) + O(h^{3+2s}), \tag{20}$$

if the number k of correction coefficients is of the form $k = \frac{(s+2)^2}{4}$ if s is even, or $k = \frac{(s+1)(s+3)}{4}$ if s is odd, and if v is a C^∞ function with compact support contained in D.

The procedure to find such coefficients is similar to the case of logarithmic singularities: for $r \in [1,k]$ let v_r be defined as in (14) and let $f_r(x,y) = \frac{v_r(x,y)}{\sqrt{x^2+y^2}}$. As before let $D = [-1,1] \times [-1,1]$, and $h = 1/40$ be the distance between sampling points in the discretization of D given by (4). We now solve the linear system of k equations with unknowns $c_1, c_2, ..., c_k$

$$\int_D \frac{v_r(x,y)}{\sqrt{x^2+y^2}} dxdy = T^n_{\beta^m}(\tilde{f}_r) + h \left(\sum_{r=1}^{k} c_r \sum_{(ph,qh)\in G_r} v_r(ph,qh) \right), \tag{21}$$

for $r = 1, ..., k$. Once the coefficients $c_1, ..., c_k$ are calculated they are used to approximate integrals of the form (18) by means of (20), which requires the classical trapezoidal rule plus a correction near the singularity at $(0,0)$. Table 3 shows correction coefficients to build quadratures of orders 3, 5, 15, 19, and 39.

6 Conclusions

We have constructed a 40th order quadrature rule for functions with a logarithmic singularity in two dimensions. This is achieved in a way similar to the work described in [1] but with a redistribution of the correction coefficients near the singularity. This new distribution allows to save in the number of correction coefficients and a more efficient way to build higher-order quadratures than the procedure described in [1]. The approach can be applied to construct quadratures for other singularities as shown in Section 5 for the Coulomb potential in 2-D.

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A Appendix

Tables 2 and 3 contain the correction coefficients used by the quadratures defined in (16) and (20) respectively.

$k = 1$, order 4	$k = 100$, order 40	Continuation, $k = 100$
-1.3105329259115095d0	-1.147330038140724d0	4.411823778173170d-10
	-4.094242835567297d-2	4.714979162547670d-10
$k = 2$, order 6	-1.448250839356992d-2	1.045541707089613d-9
-1.2133459579012365d0	1.430700659566972d-2	-1.964390760299861d-9
-2.4296742002568231d-2	2.824555575112553d-3	1.664108872830426d-8
	-5.177159251464851d-3	-2.471264455285524d-8
$k = 12$, order 14	-1.727218464256620d-5	-5.033326954397135d-11
-1.164629288157180d0	-1.140381781599775d-3	-5.181414119955031d-11
-3.594734349583470d-2	2.069440891330856d-3	-5.734646899033475d-11
-9.460118308887952d-3	1.865588708695156d-5	-6.035502990691202d-11
8.483476811768447d-3	4.505155751092005d-4	-1.520470849414210d-10
1.073366365238887d-3	-8.294048682554011d-4	3.234023291838046d-10
-1.609170031202822d-3	-1.835190957658235d-5	-2.396995896342711d-9
1.399092220983857d-6	-9.824571986071811d-7	3.440446426728432d-9
-1.625538455258746d-4	-1.750390017836740d-4	4.959131425018262d-12
2.261395613383293d-4	3.197114614243662d-4	5.219203555400925d-12
-1.019228443128018d-7	5.298363694117128d-6	5.904281273101352d-12
1.231529870356198d-5	-1.054509608651015d-6	6.030061669291561d-12
-1.634485912910005d-5	6.440221527035792d-5	1.754116775561700d-11
	-1.154283304399700d-4	-4.124043225399116d-11
$k = 25$, order 20	-1.243919218049841d-6	2.710365160311957d-10
-1.156374652558495d0	-1.626656896253728d-6	-3.760855317724745d-10
-3.819341364937093d-2	7.697399917645283d-7	-3.562542457300719d-13
-1.184917180472643d-2	-2.191751542504351d-5	-3.644027565873941d-13
1.099087271719868d-2	3.829191312533984d-5	-3.900927349224434d-13
1.805215796395918d-3	3.230199928591095d-7	-4.503204488581828d-13
-2.920563366803897d-3	4.674635320627944d-7	-4.382889429599770d-13
1.929689867296183d-6	-3.540762193076131d-7	-1.503103464516699d-12
-4.811330467015661d-4	6.772150930618065d-6	3.828083746653677d-12
7.357659186023864d-4	-1.148745495463553d-5	-2.254409101850990d-11
1.117394390314601d-6	-7.439265960459875d-8	3.025240873507058d-11
1.042841466732581d-4	-7.659023399347638d-8	1.697665784249113d-14
-1.548887700160065d-4	-1.229568063879984d-7	1.767607426952084d-14
-1.565467488263558d-6	1.285347505586520d-7	1.917397940987002d-14
7.003525143135724d-7	-1.868482573897108d-6	2.258585239899537d-14
-1.737015204620112d-5	3.070081917891172d-6	2.045323745324865d-14
2.470615383675196d-5	1.536440286019172d-8	8.495156090584972d-14
1.313080983017341d-7	1.627392136717323d-8	-2.303226344701126d-13
-1.528047202869342d-7	2.892221518733069d-8	1.225826055197402d-12
1.925411718076950d-6	-3.852626457645658d-8	-1.591546158694369d-12
-2.608819208804261d-6	4.527087658628963d-7	-3.937232928658755d-16
-4.334736547380545d-9	-7.195616109441111d-7	-4.007722718203044d-16
-6.450687240663935d-9	-2.689161232905659d-9	-4.233051774728503d-16
1.214838067967549d-8	-2.811647498592149d-9	-4.638805864953331d-16
-1.051651762129754d-7	-3.011305868545575d-9	-5.583904889719232d-16
1.355169136304195d-7	-5.951744851603889d-9	-4.548627626525295d-16
	9.593524251463836d-9	-2.372980891521527d-15
	-9.454562818403270d-8	6.746218112125880d-15
	1.452738146715521d-7	-3.267767202364451d-14
	4.088974825592030d-10	4.107001031588488d-14

Table 2: Correction coefficients $\mathbf{c}^k$ for a logarithmic singularity, $k = 1, 2, 12, 25, 100$.

$k=1$, order 3	$k=100$, order 39	Continuation, $k=100$
3.900264920001955d0	3.540437522493017d0	-7.666621492400688d-10
	9.324819586235752d-2	-8.667303986147378d-10
$k=2$, order 5	2.309389470414421d-2	-1.489327126605283d-9
3.671440609624736 9d0	-2.701986556870812d-2	9.810066963007772d-10
5.720607759430473 8d-2	-4.314102371702882d-3	-2.165178871911305d-8
	9.476105987597231d-3	4.185674930824761d-8
$k=16$, order 15	1.682588462897042d-4	8.739561251717323d-11
3.565757208521664d0	1.636720902075454d-3	9.010757151303481d-11
8.491939583712636d-2	-3.722278786881266d-3	9.956004922942362d-11
1.731132598595179d-2	-6.435511866703907d-5	1.131261848240956d-10
-1.832383761447747d-2	-6.323101465302855d-4	2.091945202619092d-10
-2.138279429486452d-3	1.476952736919040d-3	-1.855731851412067d-10
3.892926255618095d-3	3.047074524422283d-5	3.093257113433026d-9
2.805080889595222d-5	1.631177600508720d-5	-5.784091601130451d-9
3.994459078361247d-4	2.420135633053800d-4	-8.606839792564844d-12
-7.040056252680720d-4	-5.652054634207617d-4	-9.082193316545172d-12
-2.827941969660998d-6	-8.888613054181732d-6	-1.023576532432052d-11
-5.166073627308246d-5	-4.046964253575547d-6	-1.160097556662474d-11
8.810187850359157d-5	-8.805853308713688d-5	-2.326290231405148d-11
3.354948627967006d-7	2.028200306564257d-4	2.600071882144023d-11
1.358296354749591d-9	2.211815976342365d-6	-3.468277127963853d-10
3.400010875298589d-6	2.658893447282638d-6	6.272695960722751d-10
-5.523489697456951d-6	8.436606595118633d-7	6.178102264304521d-13
	2.969011567927738d-5	6.324458842985698d-13
$k=25$, order 19	-6.690139100097411d-5	6.795254416029268d-13
3.556876889280941d0	-5.713386131894902d-7	7.788943618612823d-13
8.769798752517432d-2	-7.465744229512058d-7	8.736173666852306d-13
1.933749949176409d-2	-1.160364558393795d-7	1.918395023950248d-12
-2.107122983820080d-2	-9.096615874313519d-6	-2.582071684157443d-12
-2.808968443839473d-3	1.995710724729911d-5	2.859830367918715d-11
5.404595880935471d-3	1.296504336158667d-7	-5.003034700860440d-11
5.611179842759117d-5	1.361950299788207d-7	-2.943205742632584d-14
7.023565852164995d-4	1.915193519529290d-7	-3.068764970357914d-14
-1.331172647761100d-3	-5.065882857248364d-9	-3.344763728928269d-14
-1.042058122406572d-5	2.489723443610326d-6	-3.892684439426480d-14
-1.478498060480158d-4	-5.302670384392650d-6	-4.282344280089762d-14
2.753704237765883d-4	-2.674925590761907d-8	-1.042352274970292d-13
2.652949472545816d-6	-2.916403899062138d-8	1.633521934137191d-13
5.256912678303280d-7	-4.383839507291626d-8	-1.541092914823847d-12
2.407492528462363d-5	9.857320504255599d-9	2.608217517466266d-12
-4.319718543562500d-5	-5.984825951067055d-7	6.821981832035646d-16
-2.221214851959172d-7	1.235247256676803d-6	6.948502557124812d-16
3.995589467653911d-8	4.679675204069028d-9	7.352814384672345d-16
-2.616195976172027d-6	4.889723883686582d-9	8.106822869578431d-16
4.482355118302548d-6	5.455385957698601d-9	9.572945587603390d-16
8.006062657136040d-9	8.756579664360134d-9	1.021835707571005d-15
1.034430204755506d-8	-3.824031080502379d-9	2.797957157778911d-15
-6.887821908546119d-9	1.240047786750355d-7	-4.970938458373152d-15
1.402224734963032d-7	-2.477701259108480d-7	4.070076967817479d-14
-2.284551690365670d-7	-7.110123409743356d-10	-6.665489630490816d-14

Table 3: Correction coefficients $\mathbf{c}^k$ used in (20), $k = 1, 2, 16, 25, 100$.