On Areas of Regions Bounded by Closed Lorentz Spherical Curves

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Abstract. In Euclidean 3-space, some properties and theorems have been given (Pottmann 1987 [1] and Muller 1962 [2]) circular curves and the projections of these areas. In this study, generalizations in Lorentz space of some theorems and the results obtained in Euclidean space are given.

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1. Preliminaries

A 3-dimensional vector space \( L = L^3_1 \) with scalar product \( \langle , \rangle_L \) of index 1 is called a Lorentz vector space. A vector \( X \) of \( L^3_1 \) is said to be space-like if \( \langle X, X \rangle_L > 0 \), time-like if \( \langle X, X \rangle_L < 0 \) and light-like or null if \( \langle X, X \rangle_L = 0 \) and \( X \neq 0 \).

A curve in \( L^3_1 \) is called space-like (time-like or null, respectively) if its tangent vector is space-like (time-like or null, respectively).

Let \( X = (X_i) \) and \( Y = (Y_i) \) be the vectors in a 3-dimensional lorentz vector space \( L^3_1 \), then the scalar product of \( X \) and \( Y \) is defined by

\[
\langle X, Y \rangle_L = X_1Y_1 + X_2Y_2 - X_3Y_3,
\]

which is called a Lorentz product. Furthermore, a Lorentz cross product \( X \wedge_L Y \) is given by

\[
X \wedge_L Y = (-X_2Y_3 + X_3Y_2, X_3Y_1 - X_1Y_3, X_1Y_2 - X_2Y_1).
\]

For \( X \in L^3_1 \), the norm of \( X \) is defined by \( \|X\|_L = \sqrt{\langle X, X \rangle_L} \), and \( X \) is called a unit vector if \( \|X\|_L = 1 \).
2. Introduction

Take a curve which is drawn by the constant point $X$ on $K$, moving Lorentz sphere and consider the parallel projecting the curve to any plane at any direction during $B' = K/K'$ closed Lorentz motion. In the paper, the projection area of a plane region made by parallel projecting of the curve is given.

Let the orthonormal system be $\{O, \overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}\}$ and $E_1, E_2$ and $E_3$ be the end points on Lorentz sphere of this orthonormal frame. The points $E_1, E_2, E_3$ draw the closed Lorentz spherical curves $c_1, c_2$ and $c_3$, consequently, on the $K'$ fixed lorentz sphere during the one parameter motion $B' = K/K'$ of closed Lorentz sphere. The formula of area of region bounded by these closed Lorentz spherical curves on the Lorentz sphere with $K'$ unit is

$$F_{E_i} = 2\pi + \Lambda_{E_i}, i = 1, 2, 3$$

[3].

3. Some Theorems and Results

The position vector of the constant point $X \in K$ can be written in terms of orthonormal basis vectors $\overrightarrow{e_1}, \overrightarrow{e_2}$ and $\overrightarrow{e_3}$ as

$$\overrightarrow{X}(t) = x_1\overrightarrow{e_1}(t) + x_2\overrightarrow{e_2}(t) + x_3\overrightarrow{e_3}(t).$$

Then

$$\left[\begin{array}{c}
\frac{d\overrightarrow{e_1}}{dt} \\
\frac{d\overrightarrow{e_2}}{dt} \\
\frac{d\overrightarrow{e_3}}{dt}
\end{array}\right] = \left[\begin{array}{ccc}
0 & \lambda & 0 \\
-\lambda & 0 & \mu \\
0 & \mu & 0
\end{array}\right] \left[\begin{array}{c}
\overrightarrow{e_1} \\
\overrightarrow{e_2} \\
\overrightarrow{e_3}
\end{array}\right],$$

(3.1)

$$\overrightarrow{f}(c_i) = \oint \overrightarrow{e_i} \wedge d\overrightarrow{e_i}$$

(3.2)

$$\overrightarrow{f}(c_i, c_j) = \frac{1}{2} \oint (\overrightarrow{e_i} \wedge d\overrightarrow{e_j} + \overrightarrow{e_j} \wedge d\overrightarrow{e_i})$$

and

$$\oint d\overrightarrow{e_i} = 0.$$

The area vector of the orbit of $c(X)$ during the one parameter closed Lorentz motion $B' = K/K'$ is

$$\overrightarrow{f}(c(X)) = \oint \overrightarrow{x} \wedge d\overrightarrow{x}$$

$$\overrightarrow{f}(c(X)) = (x_1^2 + x_2^2) \overrightarrow{f}(c_1) + (x_3^2 + x_2^2) \overrightarrow{f}(c_3) + 2x_1x_3 \overrightarrow{f}(c_1, c_3)$$
or

\[ \vec{f}(c(X)) = \sum_{i=1}^{3} x_i^2 \vec{f}(c_i) + 2 \sum_{i,j=1,i<j}^{3} x_i x_j \vec{f}(c_i, c_j). \]  

(3.3)

Here \( \vec{f}(c_i) \) is called the area vector of the Lorentz spherical curve \( c_i \) and \( \vec{f}(c_i, c_j) \) is called the mixed area vector of the Lorentz spherical curve \( c_j \) with the original Lorentz spherical curve \( c_i \).

In addition to this, the area of the planar regions obtained by the projecting those curves on a plane at the direction of \( \vec{n} \) (\( \|\vec{n}\|_L = 1 \)) is

\[ F(c_i^n) = \frac{1}{2} \left\langle \vec{n}, \vec{f}(c_i) \right\rangle_L, i = 1, 2, 3 \]

\[ F(c_i^n, c_j^n) = \frac{1}{2} \left\langle \vec{n}, \vec{f}(c_i, c_j) \right\rangle_L, (i < j) = 1, 2, 3. \]

If we move from the normal projection to any parallel projection, the projection area will change by the amount of cosine hyperbole of the angle between the normals of a plane whose normals are timelike.

**Theorem 1.** Let \( c(X) \) be the orbit of a point \( X \) taken from the \( K \) moving Lorentz sphere during the one parameter closed Lorentzian spherical motion \( B' = K/K' \). The orientated area \( F(c(X)^P) \) of the planar region made by parallel projection of this orbit to any planar, in terms of the parallel projection area of the closed Lorentz \( c_1, c_2 \) and \( c_3 \) spherical curves is

\[ F(c(X)^P) = \sum_{i=1}^{3} x_i^2 F(c_i^P) + 2 \sum_{i,j=1,i>j}^{3} x_i x_j F(c_i^P, c_j^P). \]  

(3.4)

**Theorem 2.** Let \( M, N, X \) and \( Y \) be the four different points on the \( \hat{MN} \) segment of the arc of the \( K \) moving Lorentz sphere. While \( M \) and \( N \) draw the same \( (\gamma) \) curve during the one parameter closed Lorentz spherical motion \( B' = K/K' \), let the points \( X \) and \( Y \) draw the \( c(X) \) and \( c(Y) \) curves. Let the projection area of these curves be \( F(\gamma^P) \), \( F(c(X)^P) \) and \( F(c(Y)^P) \). Then,

\[ F = F(\gamma^P) - F(c(X)^P) \]

and

\[ F' = F(\gamma^P) - F(c(Y)^P). \]

**Proof.** Let \( M \) and \( N \) be the two constant points on the \( \hat{MN} \) arc of the big Lorentz circle on the \( K \) moving Lorentz sphere. The position vectors of the constant points \( \vec{M} \) and \( \vec{N} \) can be written in terms of the orthonormal basis vectors \( \vec{e}_1, \vec{e}_2 \) and \( \vec{e}_3 \) of \( K \) moving Lorentz sphere

\[ \vec{M} = \sum_{i=1}^{3} m_i \vec{e}_i(t) \]
The constant points $M$ and $N$ on $K$ draw closed spherical curves $c(M)$ and $c(N)$ consequently on the constant sphere $K'$ during the one parameter closed Lorentz spherical motion $B' = K/K'$. The area vectors of these curves are, from (3.3)

$$\vec{f}(c(M)) = \sum_{i=1}^{3} m_i^2 \vec{f}(c_i) + 2 \sum_{i,j=1,i<j}^{3} m_im_j \vec{f}(c_i, c_j)$$

and

$$\vec{f}(c(N)) = \sum_{i=1}^{3} n_i^2 \vec{f}(c_i) + 2 \sum_{i,j=1,i<j}^{3} n_in_j \vec{f}(c_i, c_j).$$

(3.5)

The areas of the plane region made by the parallel projection of the curves $c(M)$ and $c(N)$ on any plane are, from (3.4)

$$F(c(M)^P) = \sum_{i=1}^{3} m_i^2 F(c_i^P) + 2 \sum_{i,j=1,i<j}^{3} m_im_j F(c_i^P, c_j^P)$$

and

$$F(c(N)^P) = \sum_{i=1}^{3} n_i^2 F(c_i^P) + 2 \sum_{i,j=1,i<j}^{3} n_in_j F(c_i^P, c_j^P).$$

(3.6)

Take a point $Y, Y \neq X$, on $\widetilde{MN}$ arc. The position vector $\vec{Y}(t)$ of this point, can be written in terms of the base vectors $\vec{e}_1, \vec{e}_2$ and $\vec{e}_3$ of $K$,

$$\vec{Y}(t) = y_1 \vec{e}_1(t) + y_2 \vec{e}_2(t) + y_3 \vec{e}_3(t).$$

The area vector of $c(Y)$ which is drown by this point during the same movement is;

$$\vec{f}(c(Y)) = \sum_{i=1}^{3} y_i^2 \vec{f}(c_i) + 2 \sum_{i,j=1,i<j}^{3} y_iy_j \vec{f}(c_i, c_j).$$

The oriantated projection area of the region made by the parallel projecting of $c(Y)$ on a plane is

$$F(c(Y)^P) = \sum_{i=1}^{3} y_i^2 F(c_i^P) + 2 \sum_{i,j=1,i<j}^{3} y_iy_j F(c_i^P, c_j^P).$$

(3.7)
After the selection of proper coordinates i.e. with an appropriate rotation from (3.4), (3.6) and (3.7), we get

\[
F (c(X)^P) = \sum_{i=1}^{3} x_i'^2 F(c_i^P)
\]

\[
F (c(M)^P) = \sum_{i=1}^{3} m_i'^2 F(c_i^P)
\]

\[
F (c(N)^P) = \sum_{i=1}^{3} n_i'^2 F(c_i^P)
\]

\[
F (c(Y)^P) = \sum_{i=1}^{3} y_i'^2 F(c_i^P).
\]

While \(M\) and \(N\) on \(\hat{MN}\) arc are drawing the same orbit curve \((\gamma)\) during \(B' = K/K'\) motion. Let \(X\) and \(Y\) draw the different curves respectively, \(c(X)\) and \(c(Y)\), during the same motion. If the area of the region between the curve which is the projection of \((\gamma)\) and \(c(X)\) on the plane, is \(F\) and the area of the region between the curves which is the projection of \((\gamma)\) and \(c(Y)\) on the plane is \(F'\), then

\[
F = F(\gamma^P) - F(c(X)^P) = \sum_{i=1}^{3} \left( m_i'^2 - x_i'^2 \right) F(c_i^P)
\]

and

\[
F' = F(\gamma^P) - F(c(Y)^P) = \sum_{i=1}^{3} \left( m_i'^2 - y_i'^2 \right) F(c_i^P)
\]

are obtained. \(\blacksquare\)

**Theorem 3.** Let \(M, N, X\) and \(Y\) be the four different points on \(\hat{MN}\) arc of K moving Lorentz sphere. While \(M\) and \(N\) draw the same curve \((\gamma)\) during one parameter the closed Lorentz spherical motion \(B' = K/K'\), let the points \(X\) and \(Y\) draw the curves \(c(X)\) and \(c(Y)\). Let the projection area of these curves are \(F(\gamma^P)\), \(F(c(X)^P)\) and \(F(c(Y)^P)\). In this case,

\[
F = F(\gamma^P) - F(c(X)^P)
\]

and

\[
F' = F(\gamma^P) - F(c(Y)^P).
\]

At it is seen, the ratio \(\frac{F'}{F}\) is independent from the motion, actually it depends on the double ratio of the four points on \(\hat{MN}\) arc.
Proof. From (3.8) it can be easily seen that
\[
\frac{F}{F'} = \frac{m_i^2 - x_i^2}{m_i'^2 - x_i'^2} = \text{const.}
\]

Let’s consider \(\tilde{M}N\), which is a piece of arc on a big Lorentz circle on \(K\) moving Lorentz sphere and let \(X\) be another point on \(\tilde{M}\tilde{N}\) arc. Let the position vectors of these points be \(\tilde{M}, \tilde{N}\) and \(\tilde{X}\). \(\theta_1, \theta_2\) and \(\theta\) are the center angles of \(\tilde{M}X, X\tilde{N}\) and \(\tilde{M}\tilde{N}\) respectively. Then we can write
\[
\tilde{X} = \frac{\sinh \theta_2}{\sinh \theta} \tilde{M} + \frac{\sinh \theta_1}{\sinh \theta} \tilde{N}.
\]

The area vector of \(c(X)\) which is drawn by the point \(X\) during the Lorentz motion \(B' = K/K'\), from (3.3) we obtain
\[
\vec{f}(c(X)) = \frac{\sinh^2 \theta_2}{\sinh^2 \theta} \vec{f}(c(M))
\]
\[
+ \frac{\sinh^2 \theta_1}{\sinh^2 \theta} \vec{f}(c(N)) + 2 \frac{\sinh \theta_1 \sinh \theta_2}{\sinh^2 \theta} \vec{f}(c(M), c(N))
\]
(3.9)

If \(M\) and \(N\) points draw the same (\(\gamma\)) curve during the one parameter closed Lorentz spherical motion \(B' = K/K'\), from (3.9) we have
\[
\vec{f}(c(M)) = \vec{f}(c(N)) = \vec{f}(c(M), c(N))
\]
so it becomes
(3.10)
\[
\vec{f}(c(X)) = \left(\frac{\sinh \theta_1 + \sinh \theta_2}{\sinh \theta}\right)^2 \vec{f}(c(M))
\]

In the same way, let’s consider a point \(Y (Y \neq X)\), on the same \(\tilde{M}\tilde{N}\) arc. Let \(\theta_1, \theta_2\) and \(\theta\) be the centre angles of \(\tilde{M}Y, Y\tilde{N}\) and \(\tilde{M}\tilde{N}\) respectively, and \(M\) and \(N\) draw the same curve (\(\gamma\)), therefore from (3.10) it can be written as
\[
\vec{f}(c(Y)) = \left(\frac{\sinh \theta_1 + \sinh \theta_2}{\sinh \theta}\right)^2 \vec{f}(c(M))
\]

If closed Lorentz curves (\(\gamma\)), \(c(X)\) and \(c(Y)\) are projected on a plane, the orientated projection areas of planar regions are
\[
\overline{f}(c(X)^P) = \left(\frac{\sinh \theta_1 + \sinh \theta_2}{\sinh \theta}\right)^2 \overline{f}(c(M)^P)
\]
and then
\[
\overline{f}(c(Y)^P) = \left(\frac{\sinh \theta_1 + \sinh \theta_2}{\sinh \theta}\right)^2 \overline{f}(c(M)^P).
\]
It the difference between the projected areas of (γ) and c(X) is F, and the difference between projected areas of (γ) and c(Y) is F', then we have

\[ \frac{F}{F'} = \frac{\sinh^2 \theta (\sinh \theta_1 + \sinh \theta_2)^2}{\sinh^2 \theta (\sinh \theta_1 + \sinh \theta_2)^2} = \text{const.} \]

\[ \text{Corollary 4.} \quad \frac{F}{F'} \text{ depends on the arc length of } \widetilde{MN}, \widetilde{XN}, \widetilde{MY}, \widetilde{YN}, \widetilde{MN} \text{ of the points } M, X, Y \text{ and } N \text{ on } \widetilde{MN} \text{ arc}, \text{ in other words, it is independent from the motion}.

\[ \text{Theorem 5.} \quad \text{Let } M, N \text{ be two different constant points on } K \text{ moving Lorentz sphere and } X \text{ be another constant point on } \widetilde{MN} \text{ arc. Let } \theta_1, \theta_2 \text{ and } \theta \text{ be the angles which are equal arc length of } \widetilde{MX}, \widetilde{XN} \text{ and } \widetilde{MN} \text{ respectively. While the points } M \text{ and } N \text{ draw the same curve } (\gamma) \text{ during motion } B' = K/K', \text{ let } X \text{ draws } c(X). \text{ In this case, the mixed orientated areas of the parallel projections of these curves are}

\[ F(c(M)^P, c(X)^P) = \frac{\sinh \theta_1 + \sinh \theta_2}{\sinh \theta} F(c(M)^P). \]

\[ \text{Proof.} \quad \text{The mixed area vectors of } c(M), c(X) \text{ and } c(N) \text{ curves are}

\[ (3.11) \quad \begin{align*}
&\overrightarrow{f} (c(M), c(X)) = \frac{\sinh \theta_1}{\sinh \theta} \overrightarrow{f} (c(M)) + \frac{\sinh \theta_2}{\sinh \theta} \overrightarrow{f} (c(M), c(N)) \\
&\overrightarrow{f} (c(X), c(N)) = \frac{\sinh \theta_1}{\sinh \theta} \overrightarrow{f} (c(M), c(N)) + \frac{\sinh \theta_2}{\sinh \theta} \overrightarrow{f} (c(X)) \\
&\overrightarrow{f} (c(M), c(N)) = \sum_{i=1}^{3} m_i n_i \overrightarrow{f} (c_i) + \sum_{i,j=1}^{3} (m_i n_j + m_j n_i) \overrightarrow{f} (c_i, c_j)
\end{align*} \]

If the points M and N draw the same (γ) curves during the one parameter closed Lorentz spherical motion B' = K/K', then from (3.11)

\[ \overrightarrow{f} (c(M), c(X)) = \overrightarrow{f} (c(X), c(N)) \]

and

\[ F(c(M)^P, c(X)^P) = \frac{\sinh \theta_1 + \sinh \theta_2}{\sinh \theta} F(c(M)^P). \]

\[ \text{Corollary 6.} \quad \text{Let the points } M \text{ and } N \text{ on } \widetilde{MN} \text{ arc of } K \text{ moving Lorentz sphere draw the same curve } (\gamma) \text{ and } X, \text{ which is different from the points } M \text{ and } N \text{ taken from on } \widetilde{MN} \text{ arc draw the curve } c(X). \text{ Then, for } \theta_1, \theta_2 \text{ and } \theta \text{ with } \theta = \theta_1 + \theta_2 \text{ are being the angles equal to the arc lengths of } MX, XN \text{ and } MN; \]

\[ \frac{\sinh \theta_1 + \sinh \theta_2}{\sinh \theta} \neq 1. \]
Therefore $M$ and $X$ or $N$ and $X$ never draw the same curve.

References


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