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On Areas of Regions Bounded by Closed Lorentz Spherical Curves

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Abstract. In Euclidean 3-space, some properties and theorems have been given (Pottmann 1987 [1] and Muller 1962 [2]) circular curves and the projections of these areas.In this study, generalizations in Lorentz space of some theorems and the results obtained in Euclidean space are given.

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1. Preliminaries

A 3−dimensional vector space $L = L_1^3$ with scalar product $\langle , \rangle_L$ of index 1 is called a Lorentz vector space.A vector X of L_1^3 is said to be space-like if $\langle X, X\rangle_L > 0$, time-like if $\langle X, X\rangle_L < 0$ and light-like or null if $\langle X, X\rangle_L = 0$ and $X \neq 0$.

A curve in L_1^3 is called space-like (time-like or null, respectively) if its tangent vector is space-like (time-like or null, respectively).

Let $X = (X_i)$ and $Y = (Y_i)$ be the vectors in a 3−dimensional lorentz vector space L_1^3, then the scalar product of X and Y is defined by

$$\langle X, Y\rangle_L = X_1Y_1 + X_2Y_2 - X_3Y_3,$$

which is called a Lorentz product. Furthermore, a Lorentz cross product $X \wedge_L Y$ is given by

$$X \wedge_L Y = (-X_2Y_3 + X_3Y_3, X_3Y_1 - X_1Y_3, X_1Y_2 - X_2Y_1)\,.$$

For $X \in L_1^3$, the norm of X is defined by $\|X\|_L = \sqrt{|\langle X,X\rangle_L|}$, and X is called a unit vector if $\|X\|_L = 1$.

2. Introduction

Take a curve which is drawn by the constant point X on K , moving Lorentz sphere and consider the parallel projecting the curve to any plane at any direction during $B^{'}=K/K^{'}$ closed Lorentz motion. In the paper, the projection area of a plane region made by parallel projecting of the curve is given.

Let the orthonormal system be $\{O,\overrightarrow{e_1},\overrightarrow{e_2},\overrightarrow{e_3}\}$ and E_1,E_2 and E_3 be the end points on Lorentz sphere of this orthonormal frame.The points E_1,E_2 , E_3 draw the closed Lorentz spherical curves c_1,c_2 and c_3, consequently, on the $K^{'}$ fixed lorentz sprehe during the one parameter motion $B^{'}=K/K^{'}$ of closed Lorentz sphere.The formula of area of region bounded by these closed Lorentz spherical curves on the Lorentz sphere with $K^{'}$ unit is

$$F_{E_i}=2\pi+\Lambda_{E_i},i=1,2,3$$

[3].

3. Some Theorems and Results

The position vector of the constant point $X\in K$ can be written in terms of orthonormal basis vectors $\overrightarrow{e_1},\overrightarrow{e_2}$ and $\overrightarrow{e_3}$ as

$$\overrightarrow{X(t)}=x_1\overrightarrow{e_1(t)}+x_2\overrightarrow{e_2(t)}+x_3\overrightarrow{e_3(t)}.$$

Then

$$\begin{bmatrix} d\overrightarrow{e_1} \\ d\overrightarrow{e_2} \\ d\overrightarrow{e_3} \end{bmatrix}=\begin{bmatrix} 0 & \lambda & 0 \\ -\lambda & 0 & \mu \\ 0 & \mu & 0 \end{bmatrix}\begin{bmatrix} \overrightarrow{e_1} \\ \overrightarrow{e_2} \\ \overrightarrow{e_3} \end{bmatrix}, \tag{3.1}$$

$$\overrightarrow{f}(c_i)=\oint\overrightarrow{e_i}\wedge d\overrightarrow{e_i}$$

$$\overrightarrow{f}(c_i,cj)=\frac{1}{2}\oint(\overrightarrow{e_i}\wedge d\overrightarrow{e_j}+\overrightarrow{e_j}\wedge d\overrightarrow{e_i}) \tag{3.2}$$

and

$$\oint d\overrightarrow{e_i}=0.$$

The area vector of the orbit of $c(X)$ during the one parameter closed Lorentz motion $B^{'}=K/K^{'}$ is

$$\overrightarrow{f}(c(X))=\oint\overrightarrow{x}\wedge d\overrightarrow{x}$$

$$\overrightarrow{f}(c(X))=\left(x_1^2+x_2^2\right)\overrightarrow{f}(c_1)+\left(x_3^2+x_2^2\right)\overrightarrow{f}(c_3)+2x_1x_3\overrightarrow{f}(c_1,c_3)$$

or

$$\overrightarrow{f}(c(X)) = \sum_{i=1}^{3} x_i^2 \overrightarrow{f}(c_i) + 2 \sum_{i,j=1,i<j}^{3} x_i x_j \overrightarrow{f}(c_i, c_j). \tag{3.3}$$

Here $\overrightarrow{f}(c_i)$ is called the area vector of the Lorentz spherical curve c_i and $\overrightarrow{f}(c_i, c_j)$ is called the mixed area vector of the Lorentz spherical curve c_j with the original Lorentz spherical curve c_i.

In addition to this, the area of the planar regions ontained by the projecting those curves on a plane at the direction of $\overrightarrow{n}$ ($\|\overrightarrow{n}\|_L = 1$) is

$$F(c_i^n) = \frac{1}{2} \left\langle \overrightarrow{n}, \overrightarrow{f}(c_i) \right\rangle_L, i = 1, 2, 3$$

$$F(c_i^n, c_j^n) = \frac{1}{2} \left\langle \overrightarrow{n}, \overrightarrow{f}(c_i, c_j) \right\rangle_L, (i < j) = 1, 2, 3.$$

If a we move from the normal projection to any parallel projection, the projection area will change by the amount of cosine hyperbole of the angle between the normals of a plane whose normals are timelike.

Theorem 1. *Let $c(X)$ be the orbit of a point X taken from the K moving Lorentz sphere during the one parameter closed Lorentzian spherical motion $B' = K/K'$. The orientated area $F\left(c(X)^P\right)$ of the planar region made by parallel projection of this orbit to any planar, in terms of the parallel projection area of the closed Lorentz c_1, c_2 and c_3 spherical curves is*

$$F\left(c(X)^P\right) = \sum_{i=1}^{3} x_i^2 F(c_i^P) + 2 \sum_{i,j=1,i>j}^{3} x_i x_j F(c_i^P, c_j^P). \tag{3.4}$$

Theorem 2. *Let M, N, X and Y be the four different points on the $\widehat{MN}$ segment of the arc of the K moving Lorentz sphere. While M and N draw the same (γ) curve during the one parameter closed Lorentz spherical motion $B' = K/K'$, let the points X and Y draw the $c(X)$ and $c(Y)$ curves. Let the projection area of these curves be $F(\gamma^P)$,$F(c(X)^P)$ and $F(c(Y)^P)$. Then,*

$$F = F(\gamma^P) - F(c(X)^P)$$

and

$$F' = F(\gamma^P) - F(c(Y)^P).$$

Proof. Let M and N be the two constant points on the $\widehat{MN}$ arc of the big Lorentz circle on the K moving Lorentz sphere.The position vectors of the constant points $\overrightarrow{M}$ and $\overrightarrow{N}$ can be written in terms of the orthonormal basis vectors $\overrightarrow{e_1}, \overrightarrow{e_2}$ and $\overrightarrow{e_3}$ of K moving Lorentz sphere

$$\overrightarrow{M} = \sum_{i=1}^{3} m_i \overrightarrow{e_i(t)}$$

and

$$\overrightarrow{N} = \sum_{i=1}^{3} n_i \overrightarrow{e_i(t)}.$$

The constant points M and N on K draw closed spherical curves $c(M)$ and $c(N)$ consequently on the constant sphere K' during the one parameter closed Lorentz spherical motion $B' = K/K'$.The area vectors of these curves are, from (3.3)

$$\overrightarrow{f}(c(M)) = \sum_{i=1}^{3} m_i^2 \overrightarrow{f}(c_i) + 2 \sum_{i,j=1, i<j}^{3} m_i m_j \overrightarrow{f}(c_i, c_j)$$

and

$$\overrightarrow{f}(c(N)) = \sum_{i=1}^{3} n_i^2 \overrightarrow{f}(c_i) + 2 \sum_{i,j=1, i<j}^{3} n_i n_j \overrightarrow{f}(c_i, c_j). \tag{3.5}$$

The areas of the plane region made by the parallel projection of the curves $c(M)$ and $c(N)$ on any plane are, from (3.4)

$$F\left(c(M)^P\right) = \sum_{i=1}^{3} m_i^2 F(c_i^P) + 2 \sum_{i,j=1, i<j}^{3} m_i m_j F(c_i^P, c_j^P)$$

and

$$F\left(c(N)^P\right) = \sum_{i=1}^{3} n_i^2 F(c_i^P) + 2 \sum_{i,j=1, i<j}^{3} n_i n_j F(c_i^P, c_j^P). \tag{3.6}$$

Take a point Y, $Y \neq X$, on $\widehat{MN}$ arc.The position vector $\overrightarrow{Y}$ of this point, can be written in terms of the base vectors $\overrightarrow{e_1}$, $\overrightarrow{e_2}$ and $\overrightarrow{e_3}$ of K,

$$\overrightarrow{Y(t)} = y_1 \overrightarrow{e_1(t)} + y_2 \overrightarrow{e_2(t)} + y_3 \overrightarrow{e_3(t)}.$$

The area vector of $c(Y)$ which is drown by this point during the same movement is;

$$\overrightarrow{f}(c(Y)) = \sum_{i=1}^{3} y_i^2 \overrightarrow{f}(c_i) + 2 \sum_{i,j=1, i<j}^{3} y_i y_j \overrightarrow{f}(c_i, c_j).$$

The oriantated projection area of the region made by the parallel projecting of $c(Y)$ on a plane is

$$F\left(c(Y)^P\right) = \sum_{i=1}^{3} y_i^2 F(c_i^P) + 2 \sum_{i,j=1, i<j}^{3} y_i y_j F(c_i^P, c_j^P). \tag{3.7}$$

After the selection of proper coordinates i.e. with an appropriate rotation from (3.4), (3.6) and (3.7), we get

$$\begin{aligned} F\left(c(X)^P\right) &= \sum_{i=1}^{3} {x'_i}^2 F(c_i^P) \\ F\left(c(M)^P\right) &= \sum_{i=1}^{3} {m'_i}^2 F(c_i^P) \\ F\left(c(N)^P\right) &= \sum_{i=1}^{3} {n'_i}^2 F(c_i^P) \\ F\left(c(Y)^P\right) &= \sum_{i=1}^{3} {y'_i}^2 F(c_i^P). \end{aligned}$$

While M and N on $\widehat{MN}$ arc are drawing the same orbit curve (γ) during $B' = K/K'$ motion. Let X and Y draw the different curves respectvely, $c(X)$ and $c(Y)$,during the same motion .If the area of the region between the curve which is the projection of (γ) and $c(X)$ on the plane, is F and the area of the region between the curves which is the projection of (γ) and $c(Y)$ on the plane is F' ,then

$$F = F(\gamma^P) - F(c(X)^P) = \sum_{i=1}^{3} \left({m'_i}^2 - {x'_i}^2\right) F(c_i^P) \tag{3.8}$$

and

$$F' = F(\gamma^P) - F(c(Y)^P) = \sum_{i=1}^{3} \left({m'_i}^2 - {y'_i}^2\right) F(c_i^P)$$

are obtained. ∎

Theorem 3. *Let M, N, X and Y be the four different points on $\widehat{MN}$ arc of K moving Lorentz sphere .While M and N draw the same curve (γ) during one parameter the closed Lorentz spherical motion $B' = K/K'$, let the points X and Y draw the curves $c(X)$ and $c(Y)$.Let the projection area of these curves are $F(\gamma^P)$, $F(c(X)^P)$ and $F(c(Y)^P)$. In this case,*

$$F = F(\gamma^P) - F(c(X)^P)$$

and

$$F' = F(\gamma^P) - F(c(Y)^P).$$

At it is seen, the ratio $\frac{F}{F'}$ is independent from the motion, actually it depends on the duoble ratio of the four points on $\widehat{MN}$ arc.

Proof. From (3.8) it can be easily seen that

$$\frac{F}{F'} = \frac{m_i'^2 - x_i'^2}{m_i'^2 - y_i'^2} = const.$$

Let's consider $\widehat{MN}$, which is a piece of arc on a big Lorentz circle on K moving Lorentz sphere and let X be another point on $\widehat{MN}$ arc .Let the position vectors of these points be $\overrightarrow{M}$, $\overrightarrow{N}$ and $\overrightarrow{X}$. θ_1,θ_2 and θ are the center angles of $\widehat{MX}$, $\widehat{XN}$ and $\widehat{MN}$ respectively. Then we can write

$$\overrightarrow{X} = \frac{\sinh\theta_2}{\sinh\theta}\overrightarrow{M} + \frac{\sinh\theta_1}{\sinh\theta}\overrightarrow{N}.$$

The area vector of $c(X)$ which is drawn by the point X during the Lorentz motion $B' = K/K'$, from (3.3) we obtain

$$\overrightarrow{f}\,(c(X)) = \frac{\sinh^2\theta_2}{\sinh^2\theta}\overrightarrow{f}\,(c(M))$$

$$+\frac{\sinh^2\theta_1}{\sinh^2\theta}\overrightarrow{f}\,(c(N)) + 2\frac{\sinh\theta_1\sinh\theta_2}{\sinh^2\theta}\overrightarrow{f}\,(c(M), c(N)) \tag{3.9}$$

If M and N points draw the same (γ) curve during the one parameter closed Lorentz spherical motion $B' = K/K'$, from (3.9) we have

$$\overrightarrow{f}\,(c(M)) = \overrightarrow{f}\,(c(N)) = \overrightarrow{f}\,(c(M), c(N)),$$

so it becomes

$$\overrightarrow{f}\,(c(X)) = \left(\frac{\sinh\theta_1 + \sinh\theta_2}{\sinh\theta}\right)^2\overrightarrow{f}\,(c(M)). \tag{3.10}$$

In the same way, let's consider a point Y $(Y \neq X)$, on the same $\widehat{MN}$ arc.Let θ_1, θ_2 and θ be the centre angles of $\widehat{MY}$, $\widehat{YN}$ and $\widehat{MN}$ respectively, and M and N draw the same curve (γ), therefore from (3.10) it can be written as

$$\overrightarrow{f}\,(c(Y)) = \left(\frac{\sinh\theta_1 + \sinh\theta_2}{\sinh\theta}\right)^2\overrightarrow{f}\,(c(M)).$$

If closed Lorentz curves (γ) , $c(X)$ and $c(Y)$ are projected on a plane, the oriantated projection areas of planar regions are

$$\overrightarrow{f}\left(c(X)^P\right) = \left(\frac{\sinh\theta_1 + \sinh\theta_2}{\sinh\theta}\right)^2\overrightarrow{f}\left(c(M)^P\right)$$

and then

$$\overrightarrow{f}\left(c(Y)^P\right) = \left(\frac{\sinh\theta_1 + \sinh\theta_2}{\sinh\theta}\right)^2\overrightarrow{f}\left(c(M)^P\right).$$

It the difference between the projected areas of (γ) and $c(X)$ is F, and the difference between projected areas of (γ) and $c(Y)$ is F', then we have

$$\frac{F}{F'} = \frac{\sinh^2\theta\,(\sinh\theta_1 + \sinh\theta_2)^2}{\sinh^2\theta\,(\sinh\theta_1 + \sinh\theta_2)^2} = const.$$

∎

Corollary 4. *The ratio* $\frac{F}{F'}$ *depends on the arc length of* $\overset{\frown}{MX}$, $\overset{\frown}{XN}$, $\overset{\frown}{MY}$, $\overset{\frown}{YN}$, $\overset{\frown}{MN}$ *of the points* M, X, Y *and* N *on* $\overset{\frown}{MN}$ *arc, in other words, it is independent from the motion.*

Theorem 5. *Let* M, N *be two different constant points on* K *moving Lorentz sphere and* X *be another constant point on* $\overset{\frown}{MN}$ *arc.Let* θ_1, θ_2 *and* θ *be the angles which are equal arc length of* $\overset{\frown}{MX}$, $\overset{\frown}{XN}$ *and* $\overset{\frown}{MN}$ *respectively.While the points* M *and* N *draw the same curve* (γ) *during motion* $B' = K/K'$ *, Let* X *draws* $c(X)$*.In this case,the mixed oriantated areas of the parallel projections of these curves are*

$$F(c(M)^P, c(X)^P) = \frac{\sinh\theta_1 + \sinh\theta_2}{\sinh\theta} F(c(M)^P).$$

Proof. The mixed area vectors of $c(M)$,$c(X)$ and $c(N)$ curves are

(3.11)

$$\left.\begin{aligned} \overrightarrow{f}(c(M), c(X)) &= \tfrac{\sinh\theta_2}{\sinh\theta}\overrightarrow{f}(c(M)) + \tfrac{\sinh\theta_1}{\sinh\theta}\overrightarrow{f}(c(M), c(N)) \\ \overrightarrow{f}(c(X), c(N)) &= \tfrac{\sinh\theta_2}{\sinh\theta}\overrightarrow{f}(c(M), c(N)) + \tfrac{\sinh\theta_1}{\sinh\theta}\overrightarrow{f}(c(N)) \\ \overrightarrow{f}(c(M), c(N)) &= \sum_{i=1}^{3} m_i n_i \overrightarrow{f}(c_i) + \sum_{i,j=1, i<j}^{3} (m_i n_j + m_j n_i)\overrightarrow{f}(c_i, c_j) \end{aligned}\right\}$$

If the points M and N draw the same (γ) curves during the one parameter closed Lorentz spherical motion $B' = K/K'$,then from (3.11)

$$\overrightarrow{f}(c(M), c(X)) = \overrightarrow{f}(c(X), c(N))$$

and

$$F(c(M)^P, c(X)^P) = \frac{\sinh\theta_1 + \sinh\theta_2}{\sinh\theta} F(c(M)^P).$$

∎

Corollary 6. *Let the points* M *and* N *on* $\overset{\frown}{MN}$ *arc of* K *moving Lorentz sphere draw the same curve* (γ) *and* X*, which is different from the points* M *and* N *taken from on* $\overset{\frown}{MN}$ *arc draw the curve* $c(X)$ *.Then, for* θ_1, θ_2 *and* θ *with* $\theta = \theta_1 + \theta_2$ *are being the angles equal to the arc lengths of* $\overset{\frown}{MX}$, $\overset{\frown}{XN}$ *and* $\overset{\frown}{MN}$ *;*

$$\frac{\sinh\theta_1 + \sinh\theta_2}{\sinh\theta} \neq 1.$$

Therefore M and X or N and X never draw the the same curve.

References

[1] Pottmann H.,"Uber Geschlossene Spharische Kurven",El. Math.,Vol:42,pp:62-68 Tu wien,1987

[2] Muller,H.R., "Spharische Kinematik",VEB D.,Vlg. d.,Wiss.,Berlin,1962

[3] Ozyilmaz E.,Yayli Y.,"O.Bonnet Integral Formula and Some Theorems in Minkowski Space", Hadronic Journal,Institute for Basic Research, USA 15,No:4,pp:397-414

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