

A Schur-type Theorem on Indefinite Quaternionic Kähler Manifolds

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Abstract

In this paper we give a Schur-type theorem on indefinite quaternionic Kähler manifolds.

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1 Introduction

The study of quaternionic Kähler manifolds with indefinite metric was initiated by J.D. Perez and F.G. Santos ([10]). Afterwards, this class of manifolds have been intensively studied by many authors ([5], [7], [9], [11], [12]). In section 2 we recall the definition and basic properties of indefinite quaternionic Kähler manifolds.

In section 3 we give some curvature properties of indefinite quaternionic Kähler manifolds. In section 4 we prove a Schur-type theorem on indefinite quaternionic Kähler manifolds.

2 Preliminaries

Let M be a differentiable manifold of dimension n and assume that there is a rank 3-subbundle σ of $End(TM)$ such that a local basis $\{J_1, J_2, J_3\}$ exists of sections of σ satisfying:

$$\begin{cases} J_\alpha^2 = -Id, \forall \alpha = \overline{1, 3} \\ J_1 J_2 = -J_2 J_1 = J_3 \end{cases} \quad (1)$$

Then the bundle σ is called a quaternionic structure on M and $\{J_1, J_2, J_3\}$ is called canonical local basis of σ . Moreover, (M, σ) is said to be an almost quaternionic manifold. It is easy to see that any almost quaternionic manifold is of dimension $n = 4m$.

A pseudo-Riemannian metric g is said to be adapted to the quaternionic structure σ if it satisfies:

$$g(J_\alpha X, J_\alpha Y) = g(X, Y), \forall \alpha = \overline{1, 3} \quad (2)$$

for all vector fields X, Y on M and any local basis $\{J_1, J_2, J_3\}$ of σ . Moreover, (M, σ, g) is said to be an indefinite almost quaternionic manifold. It is clear that the signature of g is $s = 4u, u \geq 0$.

If the bundle σ is parallel with respect to the Levi-Civita connection ∇ of g , then (M, σ, g) is said to be an indefinite quaternionic Kähler manifold. Equivalently, locally defined 1-forms $\omega_1, \omega_2, \omega_3$ exist such that:

$$\begin{cases} \nabla_X J_1 = \omega_3(X)J_2 - \omega_2(X)J_3 \\ \nabla_X J_2 = -\omega_3(X)J_1 + \omega_1(X)J_3 \\ \nabla_X J_3 = \omega_2(X)J_1 - \omega_1(X)J_2 \end{cases} \quad (3)$$

for any vector field X on M .

We remark that any indefinite quaternionic Kähler manifold is an Einstein manifold, provided that $\dim M > 4$ (see [5], [10]).

3 Curvature Properties of Indefinite Quaternionic Kähler Manifolds

Let (M, σ, g) be an indefinite quaternionic Kähler manifold. If R is the curvature tensor of M , then we have:

$$\begin{cases} [R(X, Y), J_1] = F_3(X, Y)J_2 - F_2(X, Y)J_3 \\ [R(X, Y), J_2] = -F_3(X, Y)J_1 + F_1(X, Y)J_3 \\ [R(X, Y), J_3] = F_2(X, Y)J_1 - F_1(X, Y)J_2 \end{cases} \quad (4)$$

for any vector fields X, Y on M , where:

$$[R(X, Y), J_\alpha] := R(X, Y)J_\alpha - J_\alpha R(X, Y), \forall \alpha = \overline{1, 3} \quad (5)$$

and

$$F_1 = d\omega_1 + \omega_2 \wedge \omega_3, F_2 = d\omega_2 + \omega_3 \wedge \omega_1, F_3 = d\omega_3 + \omega_1 \wedge \omega_2. \quad (6)$$

From (4) we obtain:

$$\begin{cases} R(X, Y, J_1 Z, J_1 W) = R(X, Y, Z, W) + F_2(X, Y)g(J_2 Z, W) + F_3(X, Y)g(J_3 Z, W) \\ R(X, Y, J_2 Z, J_2 W) = R(X, Y, Z, W) + F_1(X, Y)g(J_1 Z, W) + F_3(X, Y)g(J_3 Z, W) \\ R(X, Y, J_3 Z, J_3 W) = R(X, Y, Z, W) + F_1(X, Y)g(J_1 Z, W) + F_2(X, Y)g(J_2 Z, W) \end{cases} \quad (7)$$

for any vector field X, Y, Z, W on M .

Proposition 3.1 *Let (M^{4m}, σ, g) be an indefinite quaternionic Kähler manifold. If $\{E_i\}_{i=\overline{1,4m}}$ is an adapted local pseudo-orthonormal reper (i.e. $E_{m+i} = J_1E_i, E_{2m+i} = J_2E_i, E_{3m+i} = J_3E_i, \forall i = \overline{1, m}$), then the relation:*

$$F_\alpha(X, Y) = \frac{1}{2m} \sum_{i=1}^{4m} \epsilon_i R(X, Y, J_\alpha E_i, E_i) \tag{8}$$

holds for any $\alpha = 1, 2, 3$ and any local vector fields X, Y on M , where $\epsilon_i = g(E_i, E_i)$.

Proof. If we take $Z = J_1E_i$ and $W = E_i$ in (7), we obtain:

$$F_1(X, Y) = \epsilon_i [R(X, Y, J_3E_i, J_2E_i) + R(X, Y, J_1E_i, E_i)]. \tag{9}$$

Since $\{E_i\}_{i=\overline{1,4m}}$ is an adapted local pseudo-orthonormal reper, we have:

$$\sum_{i=1}^{4m} \epsilon_i R(X, Y, J_3E_i, J_2E_i) = \sum_{i=1}^{4m} \epsilon_i R(X, Y, J_1E_i, E_i). \tag{10}$$

From (9) and (10) we derive:

$$F_1(X, Y) = \frac{1}{2m} \sum_{i=1}^{4m} \epsilon_i R(X, Y, J_1E_i, E_i)$$

and thus we obtain the conclusion for $\alpha = 1$. Similarly we find the relation (8) for $\alpha = 2$ and 3.

Proposition 3.2 *Let (M^{4m}, σ, g) be an indefinite quaternionic Kähler manifold. If $\{E_i\}_{i=\overline{1,4m}}$ is an adapted local pseudo-orthonormal reper, then:*

$$F_\alpha(X, Y) = -\frac{1}{m} \sum_{i=1}^{4m} \epsilon_i R(X, E_i, Y, J_\alpha E_i) \tag{11}$$

for any $\alpha = 1, 2, 3$ and any local vector fields X, Y on M , where $\epsilon_i = g(E_i, E_i)$.

Proof. Since $\{E_i\}_{i=\overline{1,4m}}$ is an adapted local pseudo-orthonormal reper, we remark that:

$$\sum_{i=1}^{4m} \epsilon_i R(X, E_i, J_\alpha E_i, Y) = -\sum_{i=1}^{4m} \epsilon_i R(X, J_\alpha E_i, E_i, Y). \tag{12}$$

Using (12) and Bianchi identity, we have:

$$\begin{aligned} \sum_{i=1}^{4m} \epsilon_i R(X, E_i, J_\alpha E_i, Y) &= \frac{1}{2} \sum_{i=1}^{4m} \epsilon_i [R(X, E_i, J_\alpha E_i, Y) - R(X, J_\alpha E_i, E_i, Y)] \\ &= \frac{1}{2} \sum_{i=1}^{4m} \epsilon_i [R(X, E_i, J_\alpha E_i, Y) + R(J_\alpha E_i, X, E_i, Y)] \\ &= -\frac{1}{2} \sum_{i=1}^{4m} \epsilon_i R(E_i, J_\alpha E_i, X, Y) \end{aligned} \tag{13}$$

Finally, from (8) and (13) we obtain (11).

Corollary 3.3 *Let (M^{4m}, σ, g) be an indefinite quaternionic Kähler manifold with $m > 1$. Then:*

$$\text{Ric}(X, Y) = -(m + 2)F_\alpha(X, J_\alpha Y), \quad (14)$$

for any $\alpha = 1, 2, 3$ and any local vector fields X, Y on M .

Proof. Using Proposition 3.2 and the definition of Ricci tensor in the semi-Riemmanian case, we obtain:

$$\text{Ric}(X, Y) = -mF_1(X, J_1Y) - F_2(X, J_2Y) - F_3(X, J_3Y), \quad (15)$$

$$\text{Ric}(X, Y) = -F_1(X, J_1Y) - mF_2(X, J_2Y) - F_3(X, J_3Y) \quad (16)$$

and

$$\text{Ric}(X, Y) = -F_1(X, J_1Y) - F_2(X, J_2Y) - mF_3(X, J_3Y). \quad (17)$$

Since $m \neq 1$, from (15),(16) and (17) we deduce (14).

4 Schur's theorem for indefinite quaternionic Kähler manifolds

Let (M, σ, g) be an indefinite almost quaternionic manifold. If $X \in T_pM, p \in M$, then the 4-plane $Q(X)$ spanned by $\{X, J_1X, J_2X, J_3X\}$ is called a quaternionic 4-plane. A 2-plane in T_pM spanned by $\{X, Y\}$ is called half-quaternionic if $Q(X) = Q(Y)$.

Let $\pi = Sp\{X, Y\}$ be a plane tangent to M at a point $p \in M$. The sectional curvature $K(\pi)$ is defined by:

$$K(\pi) = \frac{R(X, Y, X, Y)}{g(X, X)g(Y, Y) - g(X, Y)^2}. \quad (18)$$

It is clear that this definition makes sense only for nondegenerate plane, i.e. those satisfying $g(X, X)g(Y, Y) - g(X, Y)^2 \neq 0$

The sectional curvature for a half-quaternionic 2-plane is called quaternionic sectional curvature. An indefinite quaternionic Kähler manifold is an indefinite quaternionic space form (denoted $M(c)$) if its quaternionic sectional curvatures are equal to a constant c .

Theorem 4.1 *Let (M^{4m}, σ, g) be an indefinite quaternionic Kähler manifold, with $m > 1$. If p is a point of M and the sectional curvatures of any half-quaternionic 2-plane $\pi \subset T_pM$ are given, then the Riemann-Christoffel curvature tensor is unique determined.*

Proof. We suppose that R and R' are two Riemann-Christoffel curvature tensors on M , such that:

$$R(X, Y, X, Y) = R'(X, Y, X, Y), \tag{19}$$

for any pseudo-orthonormal basis $\{X, Y\}$ of $\pi \subset T_pM$, with $Q(X) = Q(Y)$.

If we define $S := R - R'$, then we have:

$$S(X, Y, X, Y) = 0, \tag{20}$$

for any pseudo-orthonormal basis $\{X, Y\}$ of $\pi \subset T_pM$, with $Q(X) = Q(Y)$.

To complete the proof, it suffices to prove that the above identity holds true for any $X, Y \in T_pM$.

If F_1, F_2, F_3 and F'_1, F'_2, F'_3 are locally defined 2-forms corresponding to R and R' (cf. 6), then we consider:

$$F_\alpha^S := F_\alpha - F'_\alpha, \forall \alpha = \overline{1, 3}. \tag{21}$$

From (14) and (21) we obtain:

$$F_\alpha^S(X, Y) = \frac{Ric_R(X, J_\alpha Y)}{m+2} - \frac{Ric_{R'}(X, J_\alpha Y)}{m+2} \tag{22}$$

Since M is Einstein, we have $Ric_R = \lambda_R g$ and $Ric_{R'} = \lambda_{R'} g$, and from (22) we obtain:

$$F_\alpha^S(X, Y) = kg(X, J_\alpha Y), \tag{23}$$

where $k = \frac{\lambda_R - \lambda_{R'}}{m+2}$.

From (7), taking account of definition of S , we deduce:

$$S(J_1 X, J_3 X, J_2 X, X) = -S(X, J_2 X, X, J_2 X) - F_2^S(X, J_2 X)g(J_3 X, J_3 X) \tag{24}$$

for any vector fields X, Y, Z, W on M .

We have two possible situations.

Case 1. If X is a non-lightlike vector (i.e. $g(X, X) \neq 0$), then we can suppose that X is a unitary spacelike vector (i.e. $g(X, X) = 1$) or unitary timelike vector (i.e. $g(X, X) = -1$). We consider the first situation (the second situation is completely analogous). From (20), (23) and (24) we obtain:

$$S(X, J_2 X, J_3 X, J_1 X) = kg(X, X)^2. \tag{25}$$

Similarly, we find:

$$S(X, J_3 X, J_1 X, J_2 X) = kg(X, X)^2 \tag{26}$$

and

$$S(X, J_1 X, J_2 X, J_3 X) = kg(X, X)^2. \tag{27}$$

From (25), (26) and (27), using Bianchi identity, we derive:

$$3kg(X, X)^2 = 0 \quad (28)$$

and since X is a non-lightlike vector, we deduce $k = 0$ and thus we have also $F_\alpha^S = 0, \forall \alpha = \overline{1, 3}$. Now, we obtain easy from (7) that $S(X, Y, X, Y) = 0$ for any $X, Y \in T_pM$.

Case 2. If X is a lightlike vector (i.e. $g(X, X) = 0$), there exists a sequence $\{X_n\}_n$ of non-lightlike tangent vectors such that $X_n \rightarrow X$. Using above case, we obtain $S(X_n, Y, X_n, Y) = 0$, for any n and any $Y \in T_pM$. For $n \rightarrow \infty$, we obtain $S(X, Y, X, Y) = 0$ for any $X, Y \in T_pM$.

The proof is now complete.

Let (M^{4m}, σ, g) be an indefinite quaternionic Kähler manifold, with $m > 1$. Let R_0 be the tensor field on M given by:

$$R(X, Y, Z, W) = \frac{1}{4} \{g(X, Z)g(Y, W) - g(X, W)g(Y, Z) + \sum_{\alpha=1}^3 [g(X, J_\alpha Z)g(Y, J_\alpha W) - g(X, J_\alpha W)g(Y, J_\alpha Z) + 2g(X, J_\alpha Y)g(Z, J_\alpha W)]\} \quad (29)$$

for any $X, Y, Z, W \in T_pM$ and any local basis $\{J_1, J_2, J_3\}$ of σ at p . It is easy to see that above definition does not depend on the basis $\{J_1, J_2, J_3\}$ of σ chosen on a neighborhood of any point. Moreover, R_0 satisfies the properties of Riemann-Christoffel curvature tensor:

- i. $R_0(X, Y, Z, W) = -R_0(Y, X, Z, W) = -R_0(X, Y, W, Z)$;
- ii. $R_0(X, Y, Z, W) = R_0(Z, W, X, Y)$;
- iii. $R_0(X, Y, Z, W) + R_0(X, Z, W, Y) + R_0(X, W, Y, Z) = 0$

We remark that, if p is a point of M and the sectional curvatures of any half-quaternionic 2-plane $\pi \in T_pM$ are equal to a constant c , then, from Theorem 4.1, we deduce that in p we have: $R = cR_0$.

Theorem 4.2 *Let (M^{4m}, σ, g) be an indefinite quaternionic Kähler manifold, with $m > 1$. If M is connected and for each $p \in M$, the sectional curvature is constant on the half-quaternionic 2-planes in T_pM , then M is an indefinite quaternionic space form, having the quaternionic sectional curvature $c = \frac{\lambda}{m+2}$, where λ is the proportionality coefficient between Ric and g (since M is Einstein).*

Proof. From above considerations, we deduce that:

$$R' = R - cR_0 = 0.$$

Similarly as in Theorem 4.1, we obtain:

$$F_{\alpha}^{R'}(X, Y) = kg(X, J_{\alpha}Y), \quad (30)$$

where $k = \frac{\lambda}{m+2} - c$.

Since $R' = 0$, we have $F_{\alpha}^{R'} = 0$ and from (30) we derive $k = 0$ (see proof of Theorem 4.1).

Thus we conclude that $c = \frac{\lambda}{m+2}$ and the proof is now complete.

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