Int. J. Contemp. Math. Sci., Vol. 2, 2007, no. 10, 487 - 503

Analysis of Gyroscopic Motion with Graphical Operation Approach

I-Peng Chu and Tzong-Mou Wu

Department of Mechanical Engineering, Far East University
Tainan 744, Taiwan
ipchu@cc.feu.edu.tw, tm.wu@cc.feu.edu.tw

Abstract

In analyzing the gyroscopic motion problems, we usually need the "equations of motion" and some mathematical operations such as calculus, inner or cross product, etc. It will be a troublesome work for us to do the operation of algebra. For example, we must analyze carefully each component of the equations of motion about x, y, z axes in detail. This paper presents a new interesting mathematical method, graphical operation, to overcome this imperfection. By means of graphical operation, we could exactly simplify necessary algebraic operations to get the simplified equations and obtain the equations of motion in compact forms. Two examples are included to illustrate this method.

Mathematics Subject Classification: 53A17, 15A04, 15A24

Introduction

The analysis methods of dynamics have been carried out step by step far from the 17th century. The vector analysis, geometrical method, principle of work and energy and so on, are the well-known and popular dynamics analysis methods [1].

This paper develops a new analytical approach based on Denavit-Hartenberg kinematic notation [2] and Piogram symbolic representation [3-8] for the gyroscopic motion problems. Moreover, Wu et al. [9-13] made their effort to this operation method. The desired equations of motion can be obtained from this new technique.

With this method, it is found that a great advantage, in formulating and solving the problems of dynamics, is obtained; the necessary equations of motion will be obtained quickly and easily from the graphical operation. Two examples are given in this paper for illustrative purposes.

Graphical Operation

The basic graphical operation of rotational and translational matrix has been carried out by Gleen [3], Pio [4,5] and Itzhack [6]. The authors [7,8] derived some analog symbolic forms to decribe the D-H matrix [2] and its inversion. The closed form synthesis of planar and spatial linkages function generation with symbolic representation method had also been finished by Wu [9]. Owing to the method is operations process not only a representation, so we redefined the symbolic representation as graphical operation for more relevantly presenting this method. Hence Wu [10] applied the graphical operation method again to the dynamic and relative-motion problems. Furthermore, Wu [11] also completed the error analyses of robot manipulators by this graphical operation method. Wu [12] also provided some new general rules of graphical operation for vectors and matrices, which included major Square form, Division form and Simultaneous form, etc. Recently, Wu [13] defined the kinematics representations of the graphical operation method, which had displacement, velocity and acceleration graphical operation modules.

Firstly, the fundamental graphical operation rules of single path of rotational or translational graphical operation are shown in Fig. 1. In this figure, the *a* or *b* quantities are the translational components; the θ, α and ϕ are the rotational angles above *Z*, *X* and *Y* axis, respectively. The *K* value is the gain.

Secondly, we define some graphical operation to analogy with vector or matrices operation as following

Fundamental Graphical Operation Module

First, the +,– operation in vector is

$$(x_1, y_1, z_1) \pm (x_2, y_2, z_2) = (x_1 \pm x_2, y_1 \pm y_2, z_1 \pm z_2) \tag{1}$$

its graphical operation (denoted as GO for convenience) showed in Fig. 2.

The inner product of vector is

$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = x_1 x_2 + y_1 y_2 + z_1 z_2 \tag{2}$$

its GO is showed in Fig. 3.

The cross product of vector is

$$(x_1, y_1, z_1)\times(x_2, y_2, z_2) = (y_1z_2 - y_2z_1)\hat{\mathbf{i}} + (x_2z_1 - x_1z_2)\hat{\mathbf{j}} + (x_1y_2 - x_2y_1)\hat{\mathbf{k}} \tag{3}$$

its GO is showed in Fig. 4.

Then, we could apply the above basic GO module to finish the transformation matrices such as the rotation or translation matrices. In mathematics, we have the rotation matrices about X, Y, Z axis and translation matrix as following

$$\mathbf{Rot}(\mathrm{X},\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C\alpha & -S\alpha & 0 \\ 0 & S\alpha & C\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ {}^0\mathbf{r} = \mathbf{Rot}(\mathrm{X},\alpha)\,{}^1\mathbf{r} \tag{4}$$

$$\mathbf{Rot}(\mathrm{Y},\phi) = \begin{bmatrix} C\phi & 0 & S\phi & 0 \\ 0 & 1 & 0 & 0 \\ -S\phi & 0 & C\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ {}^0\mathbf{r} = \mathbf{Rot}(\mathrm{Y},\phi)\,{}^1\mathbf{r} \tag{5}$$

$$\mathbf{Rot}(\mathrm{Z},\theta) = \begin{bmatrix} C\theta & -S\theta & 0 & 0 \\ S\theta & C\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ {}^0\mathbf{r} = \mathbf{Rot}(\mathrm{Z},\theta)\,{}^1\mathbf{r} \tag{6}$$

$$\mathbf{Tran}(a,b,c) = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix},\ {}^0\mathbf{r} = \mathbf{Tran}(a,b,c)\,{}^1\mathbf{r} \tag{7}$$

The GO module of equations (4) – (7) are drawn in Fig. 5.

Finally, we introduce four basic GO calculus forms shown in Fig. 6 and Fig. 7. Their corresponding algebraic representations are as follows

$$\frac{\partial}{\partial\theta} \text{ corresponding to } \theta + 90^\circ,\ -\frac{\partial}{\partial\theta} \text{ corresponding to } \theta - 90^\circ \tag{8}$$

All the fundamental GO modules are showed as above. We could use these GO modules to synthesize any desired combination of vectors and matrices such as the well-known D-H matrix, Euler angle transformation, etc. Whatever, these matrices can be expressed with GO by serial connections. We will later introduce some practical examples and their simplified forms by this paper’s study.

Simplified Graphical Operation Module

If we firstly give an example, e.g., D-H matrix **Tran**$(0,0,b)$**Rot**(Z,θ) **Tran**$(a,0,0)$**Rot**(X,α), its GO representation is showed in Fig. 8, and the result of GO operation is

$$\begin{array}{lllll}
 & x\text{ col.} & y\text{ col.} & z\text{ col.} & \text{4th col.} \\
 & \downarrow & \downarrow & \downarrow & \downarrow \\
{}^0x = & (C\theta)^1x + & (-C\alpha S\theta)^1y + & (S\alpha S\theta)^1z + & (aC\theta) \\
{}^0y = & (S\theta)^1x + & (C\alpha C\theta)^1y + & (-S\alpha C\theta)^1z + & (aS\theta) \\
{}^0z = & & (S\alpha)^1y + & (C\alpha)^1z + & b
\end{array} \tag{9}$$

An advantage can be found from this operation, that is, if our interest is only the x column of 4x4 matrix, we need only enter $[{}^1x\ {}^1y\ {}^1z\ ]^{\mathbf{T}} = [x\ 0\ 0]^{\mathbf{T}}$ or $[{}^1x\ {}^1y\ {}^1z\ ]^{\mathbf{T}} = [1\ 0\ 0]^{\mathbf{T}}$ to get the exit $[{}^0x\ {}^0y\ {}^0z\ ]^{\mathbf{T}}$ as shown in Fig. 9.

If the rotation angle $\alpha = 0^{\circ}$ or $\alpha = 90^{\circ}$ or $\alpha = -90^{\circ}$ or $\alpha = 180^{\circ}$ or -180° in basic rotation matrix, the simplification of GO is shown in Fig. 10. In Fig. 10, the cross mark means passing through directly regardless of any other paths. From Fig. 10, we operate the GO and get the results of rotation easier than from multiplying matrix. Using GO, we can omit the multiplication of zero value, so that one will reduce the operation time in computer, especially when much multiplication of matrix occurs. For instance, when $(b,\ \theta,\ a,\ \alpha) = (0,\ 0^{\circ},\ a,\ 270^{\circ})$ in D-H matrix, we can separate the angle α into two parts of 90° and 180° shown in Fig. 11, and obtain the result very easily, as follows

$$\begin{aligned}
{}^0x &= {}^1x + a \\
{}^0y &= {}^1z \\
{}^0z &= -{}^1y
\end{aligned} \tag{10}$$

As we known, the nominal position and orientation of the *k*th frame, $(\mathrm{XYZ})_k$, with respect to the base frame $(\mathrm{XYZ})_1$ can be written as

$$^{1}\mathbf{T}_k = \prod_{i=1}^{k} \mathbf{A}_i \tag{11}$$

where the $\mathbf{A}_i$ is the basic rotational or translational matrix $^{i\text{-}1}\mathbf{A}_i$.

Using the graphical operation method, it is not necessary to operate matrix multiplication as in equation (11). For general *k* linkage mechanism, the displacement equation is

$$^{1}\mathbf{r} = {}^{1}\mathbf{T}_k\, {}^{k}\mathbf{r} \quad \text{or} \quad \mathbf{r}_1 = {}^{1}\mathbf{T}_k \mathbf{r}_k \tag{12}$$

where the $^{k}\mathbf{r}$ or $\mathbf{r}_k$ is the displacement vector of analyzed point with respect to frame $(\mathrm{XYZ})_k$.

Examples

In this section we introduce two examples to show how to use the new graphical operation technique to gyroscopic motion problems.

Two Disks Motion

In Fig. 12, the disk *A* is spinning about its horizontal axis *y* with a constant angular velocity ω_s = 3 rad/s, the disk *O* is spinning about its vertical axis *Z* with a constant angular velocity ω_p = 1 rad/s. The frame (*xyz*) is attached to disk *O* and rotates about fixed frame (*XYZ*). At the instant $\theta = 0^0$, the point *A* is at the position shown in Fig. 12. Its position, velocity and acceleration relative to frame (*xyz*) are respectively

$$^{1}\mathbf{r} = [0\ 1\ 0.25]^{\mathrm{T}},\ {}^{1}\dot{\mathbf{r}} = [0.75\ 0\ 0]^{\mathrm{T}},\ {}^{1}\ddot{\mathbf{r}} = [0\ 0\ -2.25]^{\mathrm{T}} \tag{13}$$

where the *x* component of velocity is $\omega_s \times \mathrm{r} = 3 \times 0.25 = 0.75$, the *z* component of acceleration is $-\omega_s \times \omega_s \times \mathrm{r} = -3 \times 3 \times 0.25 = -2.25$.

The angular velocity and angular acceleration of frame (*xyz*) about (*XYZ*) are $\dot{\theta} = \omega_p = 1\ \mathrm{rad/s}$, $\ddot{\theta} = \dot{\omega}_{p,XYZ} = 0\ \mathrm{rad/s^2}$. Determine the angular velocity and angular acceleration of disk *A* and the linear velocity and linear acceleration of point *A* on the disk in the position shown, i.e., the absolute velocity $^{0}\dot{\mathbf{r}}$ and the absolute acceleration ${}^{0}\ddot{\mathbf{r}}$ of point A with respect to fixed frame (XYZ) at the instant $\theta = 0^{0}$.

Firstly, from equations (11) and (12) we can get easily the displacement vector of point A from Fig. 12 is

$$ {}^{0}\mathbf{r} = {}^{0}\mathbf{A}_{1}{}^{1}\mathbf{r} = \mathbf{Rot}(Z,\theta)^{1}\mathbf{r} \qquad (14) $$

The figure of equation (14) is drawn in Fig. 13 (a).

Secondly, we differentiate equation (14) to yield the velocity and acceleration vector as following

$$ \begin{aligned} {}^{0}\dot{\mathbf{r}} &= {}^{0}\mathbf{A}_{1}{}^{1}\dot{\mathbf{r}} + {}^{0}\dot{\mathbf{A}}_{1}{}^{1}\mathbf{r} \\ {}^{0}\ddot{\mathbf{r}} &= {}^{0}\mathbf{A}_{1}{}^{1}\ddot{\mathbf{r}} + {}^{0}\ddot{\mathbf{A}}_{1}{}^{1}\mathbf{r} + 2{}^{0}\dot{\mathbf{A}}_{1}{}^{1}\dot{\mathbf{r}} \end{aligned} \qquad (15) $$

The graphical operation of above two equations are shown in Fig. 13 (b) and (c). From this two figures, we will determine quickly our desired as the following results (with the method of vector analysis [1], we must determine firstly the angular velocity ω and angular acceleration α of disk A, i.e. equations (17) and (20), to find the linear velocity and acceleration)

$$ {}^{0}\dot{\mathbf{r}} = [-0.25\ 0\ 0]^{\mathrm{T}},\ {}^{0}\ddot{\mathbf{r}} = [0\ \ 0.5\ \ -2.25]^{\mathrm{T}} \qquad (16) $$

On the other hand, from Fig. 12, the angular velocity of disk A is

$$ \omega = \omega_{s} + \omega_{p} = [0\ 3\ 1]^{\mathrm{T}} \qquad (17) $$

Because of the magnitude of ω_{s} is constant, the time rate of change of ω_{s} as seen from x-y-z coordinate is zero, i.e., $\dot{\omega}_{s,xyz} = 0$. The time derivative of ω_{p} relative to x-y-z coordinate is also zero, i.e., $\dot{\omega}_{p,xyz} = 0$. By basic relationship of relative-motion, $\dot{\mathbf{r}}_{XYZ} = \dot{\mathbf{r}}_{xyz} + \Omega \times \mathbf{r}$, we have

$$ \begin{aligned} &\text{For } \omega_{s},\ \Omega = \omega_{p} \Rightarrow \dot{\omega}_{s,XYZ} = \dot{\omega}_{s,xyz} + \omega_{p} \times \omega_{s} = 0 + [0\ \ 0\ \ 1]^{\mathrm{T}} \times [0\ \ 3\ \ 0]^{\mathrm{T}} \\ &\text{For } \omega_{p},\ \Omega = 0 \Rightarrow \dot{\omega}_{p,XYZ} = \dot{\omega}_{p,xyz} + 0 \times \omega_{p} = 0 \end{aligned} \qquad (18) $$

The angular acceleration of disk A is therefore

$$ \alpha = \dot{\omega}_{s,XYZ} + \dot{\omega}_{p,XYZ} = \dot{\omega}_{s,XYZ} \qquad (19) $$

The GO representation of equation (19) are shown in Fig. 14. The result of equation (19) easily yields from this figure

$$\alpha = [-3\ 0\ 0]^{\mathrm{T}} \tag{20}$$

The Gyroscopic Motion

In general gyroscopic motion, the gyro has three components of angular motion shown in Fig. 15. We will determine the angular velocity and the angular acceleration of the gyro at the instant $\theta = 0^0$. From Fig. 15, we obtain the angular velocity of the gyro

$$\omega = [-\omega_n\ 0\ \omega_s + \omega_p]^{\mathrm{T}} = [-3\ \ 0\ \ 15]^{\mathrm{T}} \tag{21}$$

Similarly, the basic three relationships of relative-motion between frame (*XYZ*) and (*xyz*) are

$$\begin{aligned} &\text{For } \omega_s,\ \Omega = \omega_n + \omega_p \\ &\Rightarrow \dot{\omega}_{s,XYZ} = \dot{\omega}_{s,xyz} + (\omega_n + \omega_p) \times \omega_s = [0\ 0\ 6]^{\mathrm{T}} + [-3\ 0\ 5]^{\mathrm{T}} \times [0\ 0\ 10]^{\mathrm{T}} \\ &\text{For } \omega_n,\ \Omega = \omega_p \\ &\Rightarrow \dot{\omega}_{n,XYZ} = \dot{\omega}_{n,xyz} + \omega_p \times \omega_n = [-2\ 0\ 0]^{\mathrm{T}} + [0\ 0\ 5]^{\mathrm{T}} \times [-3\ 0\ 0]^{\mathrm{T}} \\ &\text{For } \omega_p,\ \Omega = 0 \\ &\Rightarrow \dot{\omega}_{p,XYZ} = \dot{\omega}_{p,xyz} + 0 \times \omega_p = [0\ 0\ 4]^{\mathrm{T}} \end{aligned} \tag{22}$$

We plot the GO of above equation in Fig. 16. Thus, the angular acceleration of the gyro is

$$\begin{aligned} \alpha &= \dot{\omega}_{s,XYZ} + \dot{\omega}_{n,XYZ} + \dot{\omega}_{p,XYZ} \\ &= [0\ 30\ 6]^{\mathrm{T}} + [-2\ \ -15\ 0]^{\mathrm{T}} + [0\ 0\ 4]^{\mathrm{T}} \\ &= [-2\ 15\ 10]^{\mathrm{T}} \end{aligned} \tag{23}$$

Conclusion

Typically, it is usually a big trouble and disadvantage for us to do the algebraic operation, for example, the inner or outer product of vectors and calculus. In analyzing the gyroscopic motion problems, we usually need the "equations of motion" and some mathematical operations such as calculus, inner or cross product, etc. So that we must more carefully consider the required mathematical operations.

In general, we could not use well recent computer software to finish our desired, such as numerical software, even the symbolic programs, e.g., Maple, Macsyma, and Mathematica, because of we can not entirely simplify our desired equations. It always needs some additional derive by ourselves before using the above computer software.

With the method of vector analysis, we must determine firstly the angular velocity and angular acceleration to find the linear velocity and acceleration. It will not be convenient for us to analyze the gyroscopic motion problem. This paper presents some simple graphical operation rules to overcome this imperfection. By means of graphical operation, we could exactly simplify desired algebraic operations to get the simplified equations and obtain the equations of motion in compact forms. The interesting idea also provides another analysis approach. It is hoped that the work presented here will contribute towards progress in the dynamics analysis technique and machine design.

References

[1] R.C. Hibbeler, Engineering Mechanics – Dynamics, Prentice-Hall, Inc., New Jersey, 2001.

[2] J. Denavit, R.S. Hartenberg, A kinematic notation for lower-pair mechanisms based on matrices, Journal of Applied Mechanics, Transactions of ASME, 22 (1955), 215-221.

[3] Q. Gleen, A topological approach to the analysis of star tracking system, IEEE Transactions on Aerospace and Navigational Electronics, (1963), 206-212.

[4] R.L. Pio, Symbolic representation of coordinate transformation, IEEE Transactions on Aerospace and Navigational Electronics, ANE-11 (1964), 128-134.

[5] R.L. Pio, Euler angle transformations, IEEE Transactions on Automatic Control, AC-11 (4) (1966), 707-715.

[6] Itzhack Y. Bar-Itzhack, Symbolic representation of translatory motion in multivarying-link mechanisms, Journal of Guidance, 6 (3) (1983), 193-199.

[7] P.Y. Cheng, C.I. Weng and C.K. Chen, Symbolic derivation of dynamic equations of motion for robot manipulators using piogram symbolic method, IEEE Journal of Robtics and Automation, 4 (6) (1988), 599-609.

[8] P.Y. Cheng, C.K. Chen and C.I. Weng, Application of symbolic representation method to the inverse kinematic problem of robot manipulators, Mechanism and Machine Theory, 24 (1) (1989), 61-72.

[9] T.M. Wu, C.K. Chen, Closed form synthesis of planar and spatial linkage function generation with symbolic representation, Mathematics and Computers in Simulation, 44 (1997), 155-162.

[10] T.M. Wu, C.K. Chen, Application of graphical operation method to the dynamics and relative-motion problems, Applied Mathematics and Computation, 132 (2002), 205-218.

[11] T.M. Wu, Extension of graphical operation method to the sensitivity and error response of robot manipulators, Journal of the Chinese Society of Mechanical Engineers, 23 (3) (2002), 261-270.

[12] T.M. Wu, C.K. Chen, Graphical operation of vectors and matrices, Applied Mathematics and Computation, 149 (2004), 259-276.

[13] T.M. Wu, C.K. Chen, Kinematics representations of graphical operation methods, Applied Mathematics and Computation, 155 (2004), 573-590.

Received: July 29, 2006

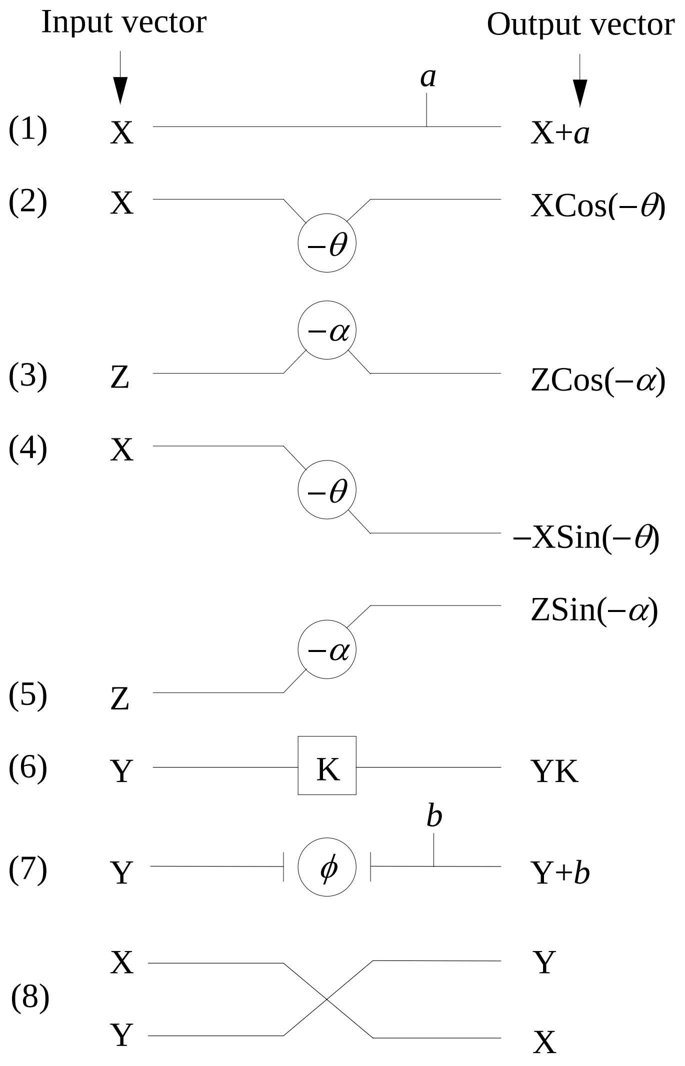


Fig. 1. The fundamental graphical operation rule of single path of rotational or translational graphical operation

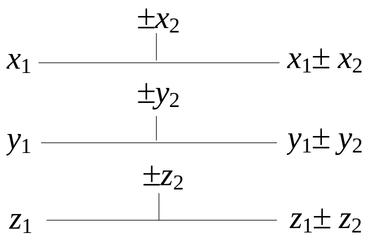


Fig. 2. The GO of vector $\mathbf{X}_1 \pm \mathbf{X}_2$ operation

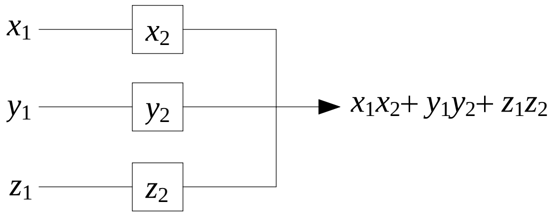


Fig. 3. The GO of inner product $\mathbf{X}_1 \cdot \mathbf{X}_2$ operation

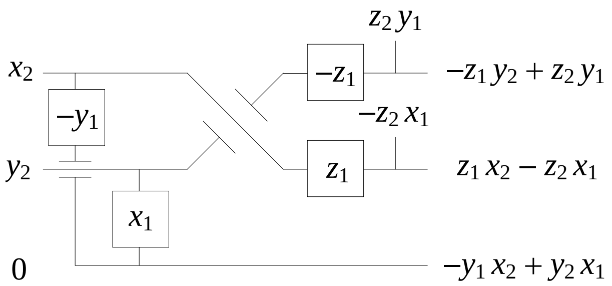
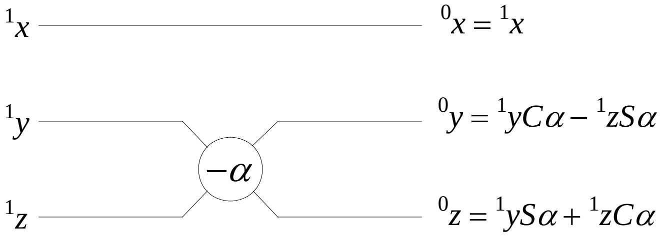
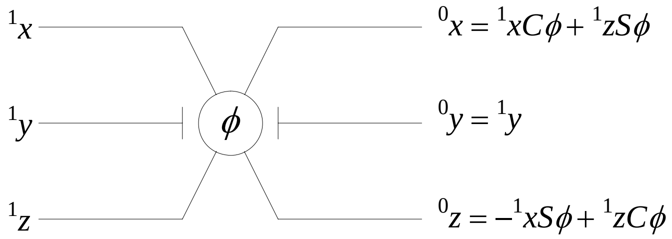


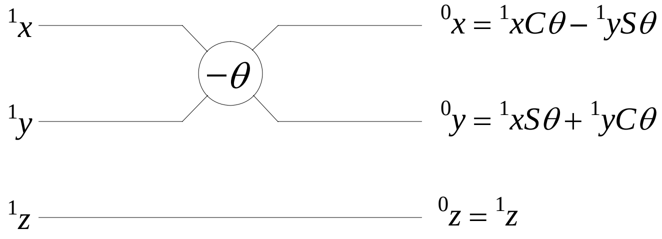
Fig. 4. The GO of vector cross $\mathbf{X}_1 \times \mathbf{X}_2$ operation



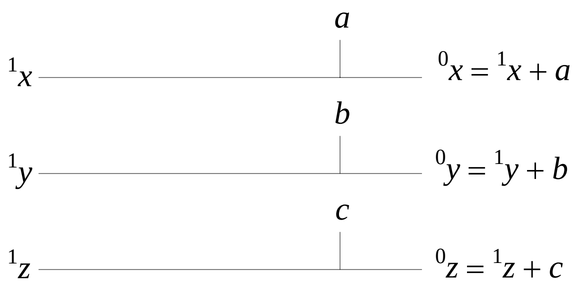
(a) GO of $^0\mathbf{r} = \mathbf{Rot}(X, \alpha)\, {}^1\mathbf{r}$



(b) GO of $^0\mathbf{r} = \mathbf{Rot}(Y, \phi)\, {}^1\mathbf{r}$



(c) GO of $^0\mathbf{r} = \mathbf{Rot}(Z, \theta)\, {}^1\mathbf{r}$



(d) GO of $^0\mathbf{r} = \mathbf{Tran}(a, b, c)\, {}^1\mathbf{r}$

Fig. 5. GO of basic rotation and translation matrices

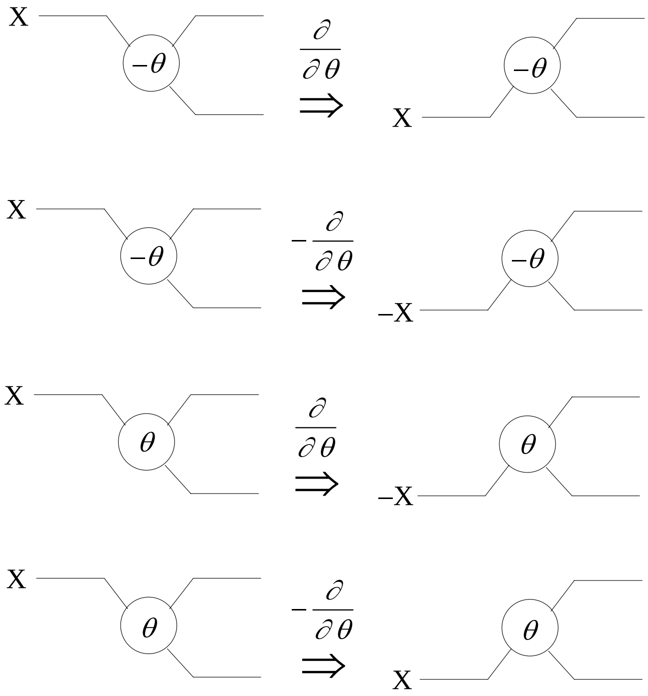


Fig. 6. The four basic GO calculus form with θ

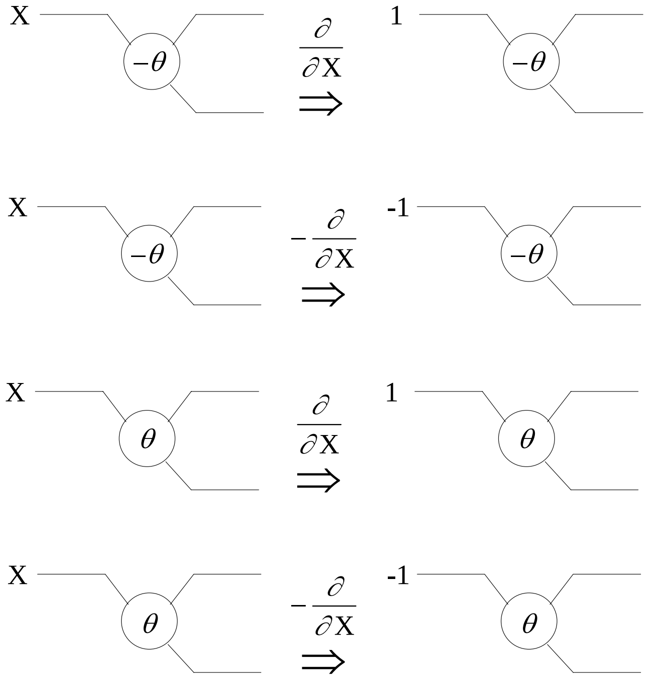


Fig. 7. The four basic GO calculus form with X

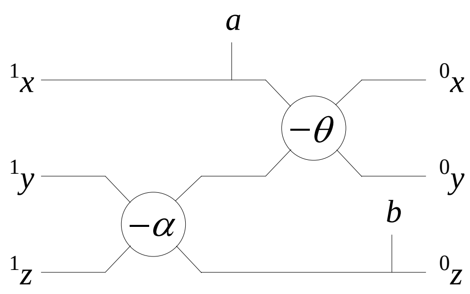


Fig. 8. GO of D-H matrix

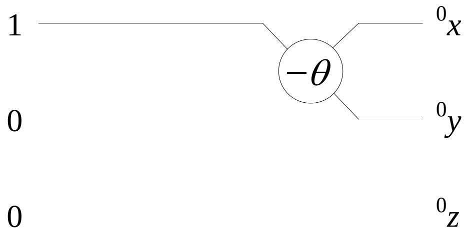
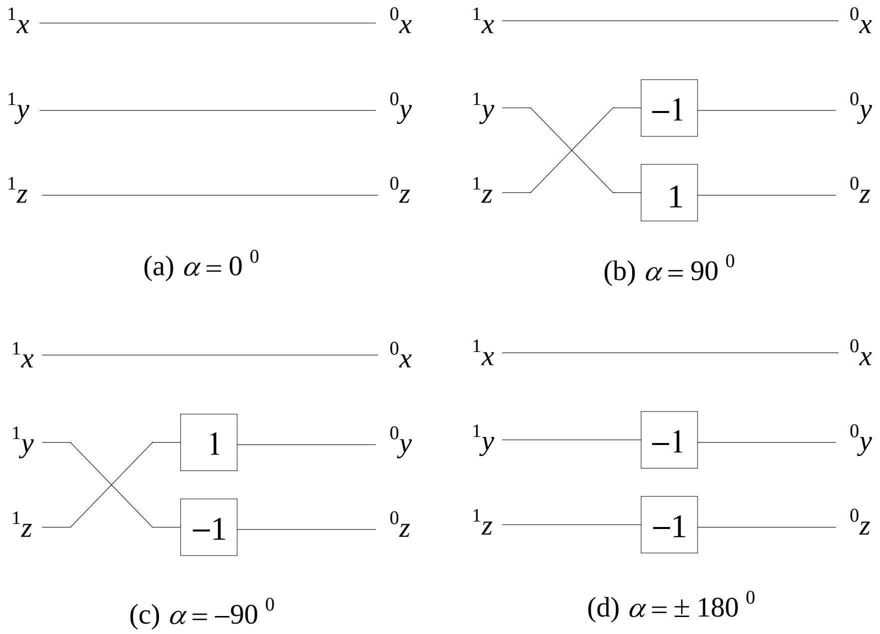


Fig. 9. GO of D-H matrix with *x* column



(a) $\alpha = 0^0$

(b) $\alpha = 90^0$

(c) $\alpha = -90^0$

(d) $\alpha = \pm 180^0$

Fig. 10. Simplification form of GO

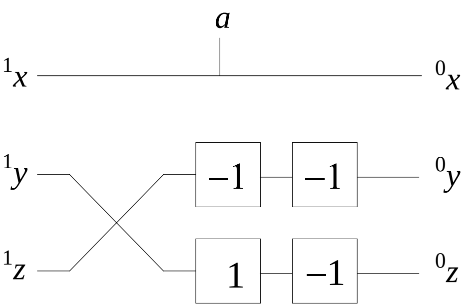


Fig. 11. Special combination of (b, θ, a, α) = (0, 0°, a, 270°) in D-H matrix

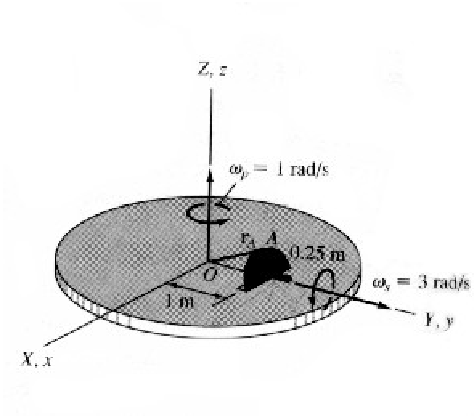
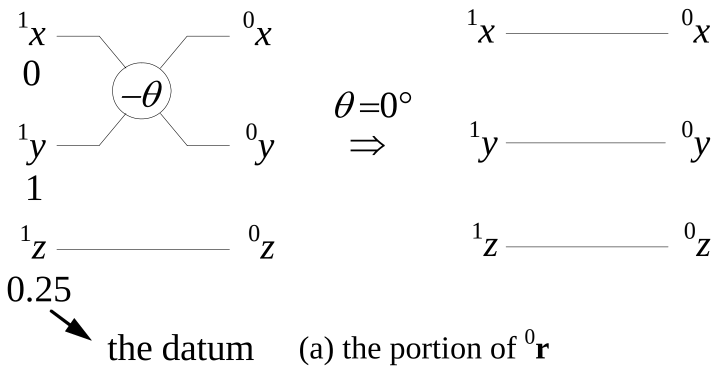
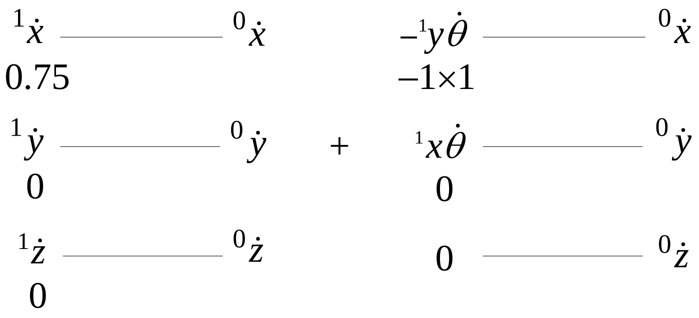


Fig. 12. The coordinate system of two disks motion



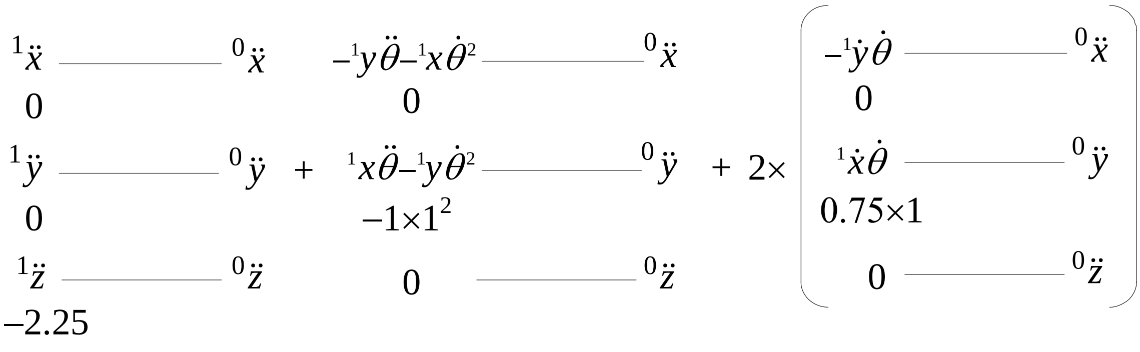
(a) the portion of $^0\mathbf{r}$

⇓ differentiate



(b) the portion of $^0\dot{\mathbf{r}}$

⇓ differentiate



(c) the portion of $^0\ddot{\mathbf{r}}$

Fig. 13. The GO of equations of motion of point *A*

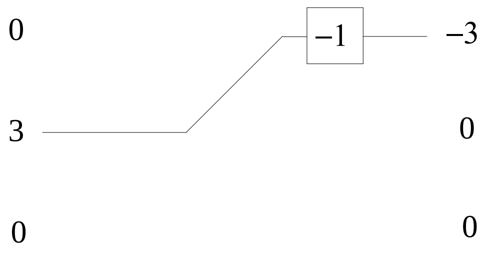


Fig. 14. The GO of vector cross $[0\ 0\ 1]^T \times [0\ 3\ 0]^T$

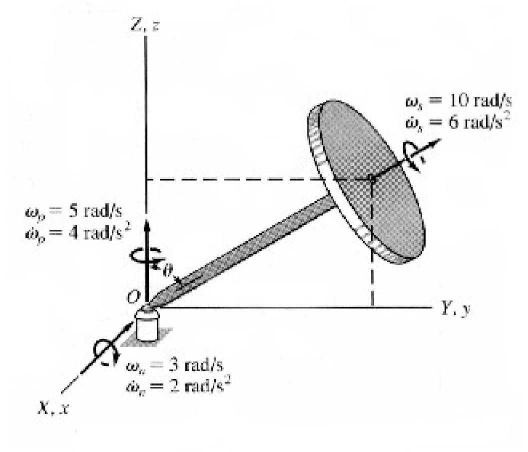
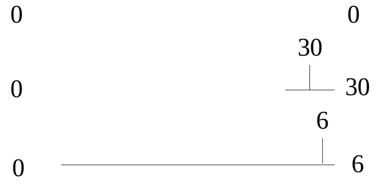
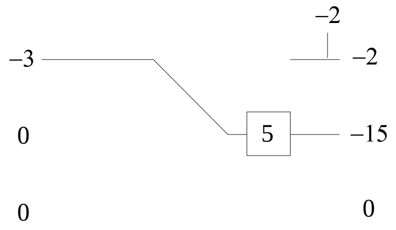


Fig. 15. The coordinate system of gyroscopic motion



(a) the portion of $\dot{\omega}_{s,XYZ}$



(b) the portion of $\dot{\omega}_{n,XYZ}$

Fig. 16. The GO of angular acceleration of the gyro