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von Neumann Universe: A Perspective

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Abstract

In this commemorating note we attempt to show that the von Neumann universe V has a profound mathematical structure.

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1 Introduction

To commemorate the 49*th* anniversery of his death, falling on 25*th* July, John von Neumann(1903 − 1957), popularly called Johnny, may be complimented by remembering him as the last complete mathematician after Carl Friedrich Gauss (1777 − 1855). Based on various sources, von Neumann's work can be seen to include: mathematical logic; pure mathematics; quantum physics; computing (his invention); cybernetics and automata theory; the Bomb; turbulence game theory (another invention) and economics, evolutionary biology and theory of war and conflict; artificial life; cellular automata (a third invention); the theory of self reproduction and artificial evolution; computer modeling and simulation; complexity theory.

In this note we attempt to show that the von Neumann Universe V has a profound mathematical structure

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2 Structure and Some Properties of von Neumann Universe

Historically the origin of mathematics can be seen to consists in construction of numbers, that of natural numbers, in particular. In pre-set theoretic days, an attempt was made to define natural numbers in the language of properties: 1 being the property common to all singlets, 2 being the property common to all doublets, etc. The proposal couldnot carried out far enough because of the cumbersomeness of the theory of property especially eschewed in philosophy and linguistics. To circumvent the difficulty, another proposal in fregean spirit, was put forward: 1 as the set of all sets equipollent with a given singlet $\{\phi\}$, say; 2 as the set of all sets equipollent with given doublet $\{\phi, \{\phi\}\}$, etc. This approach again got rejected in the light of the fact that 'there does not exist a set of all sets equipollent with a given nonempty set'.

It was first proposed by Zermelo [7] and latter bettered by von Neumann [6] that all numbers could be bootstrapped out of the empy set by applying set theoretic operations:

Zermelo : $0 = \phi$, $1 = \{\phi\}$, $2 = \{\{\phi\}\}$, etc.

von Neumann : $0 = \phi$, $1 = \{\phi\}$, $2 = \{\phi, \{\phi\}\}$, etc.

Hence, with von Neumann, a natural number n is conveniently represented as $\{0, 1, 2, \cdots, n-1\}$. All sets in V are constructed from nothing. The von Neumann universe is constructed inductively, starting from ϕ and successively applying the Powerset opertaion P. That is, $V_0 = \phi$, $V_1 = P(\phi) = \{\phi\}, \cdots, V_{n+1} = P(V_n)$ and $V_n \subset V_{n+1}$. The only restrictions required to be observed are the following: ' all elements of a set must themselves be sets' and 'the collection of all sets is not a set'. The strategy is to give a sufficiently complete description of which operations do not take us outside V.

But this alone is not going to provide adequate means to do sufficiently rich mathematics which, by its nature, is infinitistic. The level V_n consists of only finite sets, whose elements again are finite sets, and so on. Thus, we cannot go beyound finite sets unless we consider all the V_n' s as already constructed and apply the Powerset operation P to the union of V_n' s. That is, we need to resort to transfinite iteration giving rise to:

$V\omega_0 = \bigcup_{n=1}^{\infty} V_n =$ union of the first ω_0 levels of the von Neumann universe,

$V\omega_{0+1} = P(V\omega_0)$, $\cdots$, where $\omega_0, \omega_{0+1}, \cdots$ are infinite ordinals. In turn, the notion of limiting the size of a set so as to be in V could be captured as follows:

$(\forall x)\,(\exists \text{ an ordinal } \alpha)\,[x \in V_\alpha]\,.$

It is to be noted that in all these constructions for legitimizing the existence of an infinite set, the axiom of infinity was introduced.

von Neumann's form of axiom of infinity:

$(\exists\, y)\ \ [\varphi \in y \wedge (\forall x)\ [x \in y \longrightarrow x \cup \{x\} \in y]\,]$

Again, in order to introduce ordinals as an extension of natural numbers, von Neumann's form of axiom of infinity has an edge over that of zermelo's form:

$(\exists\, y)\ \ [\varphi \in y \wedge (\forall x)\ [x \in y \longrightarrow \{x\} \in y]\,]$.

At this point, in consonance with strong forms of axiom of infinity([3], [4]) and also for providing adequate linguistic framework as demanded in category theory, etc., where it needs to deal with a set of "all categories", etc., one would wish to "sufficiently" stimulate V so as to accommodate 'set of all sets in a given universe', avoiding to introduce 'class' and 'set' terms separately as it appeared in von Neumann's system for set theory, for example.

In order to stimulate V so as to accommodate arbitrarily large sets such as "a set of all groups", etc., besides gauranteeing the consequences of von Neumann's form of the axiom of infinity alnog with, we adopt the following definition:

A non empty set U will be said to be a "universe", a form of stimulated V, symbolized by Univ(U) , if

1. $(\forall\ x)\ \ [x \in U \longrightarrow x \subset U]\,;$
2. $(\forall\ x)\ \ [x \in U \longrightarrow P(x) \in U]\,;$
3. $(\forall\ x)\ (\forall y)\ [x \in U\ \wedge\ y \in U\,]\supset \{x, y\}\ \in U]\,;$ and
4. $(\forall\ x)\ \ (\forall y)\ \ \left[x \in U\ \ \wedge\ \ y \in U\right] \supset \bigcup_{i \in x} y_i\ \in U\Big]$. Hence, follows a strong form of axiom of infinity:

$(\forall x)\,(\exists y)\,[x \in y\ \ \wedge\ \ Univ\ (y)]\,.$

It is instructive to point out that the stimulated V is closed with respect to all mathematical constructions needed for modeling large (if not all) part of intutive mathematics. We also point out that the multiset theory can be conveniently dealt with in stimulated V (see [2] for basics of multiset theory and related matters).

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References

[1] Beller A., Jensen R. B., and Welch P. (1982), Coding the Universe, volume 47 of London Mathematical Society Lecture Notes, Cambridge University Press.

[2] Blizard Wayne D. (1989), Multiset Theory, Norte Dame Journal of Formal Logic, Volume 30(1), pp. 36 - 65

[3] Fraenkel A. A. (1953), Abstract Set Theory, North-Holland.

[4] Levy A. (1960), Axiom Schemata of strong infinity in axiomatic set theory, Pacific Journal of Mathematics: 10: 223 - 238.

[5] von Heijenoort J (ed.) (1967), from Frege to Godel: A source book in mathematical logic 1879 - 1931, Harvard University Press, second edition 1971.

[6] von Neumann J. (1923), Zur Einfuhrung der transfiniten Zahlen, Act Univ Szeged, Sect Mat 1:199-208.Reprinted in [5]

[7] Zermelo E. (1904), Beweis, dass Jede Menge wohgeo rdnet Werden lcann, Math. Ann. 50:514-516.Reprinted in [5]

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