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Non-Existence of Umbilic Invariant Hypersurfaces of Sasakian Space Forms

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Abstract

In the present paper, we consider invariant hypersurfaces of a Sasakian space form. The main purpose of this paper is to investigate non-existence of umbilic invariant hypersurfaces of a Sasakian space forms.

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1 Introduction

Let $\widetilde{M}$ be a $(2n+1)-$dimensional differentiable manifold equipped with a triple (ϕ, ξ_1, η^1), where ϕ is a $(1,1)$-tensor field, ξ_1 is a vector field, η^1 is a $1-$form on $\widetilde{M}$ such that

$$\eta^1(\xi_1) = 1, \quad \phi^2 = -I + \xi_1 \otimes \eta^1 \tag{1}$$

which implies

$$\phi\xi_1 = 0 \quad \eta^1 \circ \phi = 0 \quad rank(\phi) = 2n. \tag{2}$$

If $\widetilde{M}$ admits a Riemannian metric $\widetilde{g}$, such that

$$\widetilde{g}(\phi X, \phi Y) = \widetilde{g}(X, Y) - \eta^1(X)\,\eta^1(Y), \tag{3}$$
$$g(\xi_1, X) = \eta^1(X), \tag{4}$$

then $\widetilde{M}$ is said to admit a $(\phi, \xi_1, \eta^1, \widetilde{g})-$structure. If moreover

$$\left(\widetilde{\nabla}_X \phi\right) Y = \widetilde{g}(X, Y)\xi_1 - \eta^1(Y)\,X. \tag{5}$$

and

$$\widetilde{\nabla}_X \xi_1 = -\phi X \tag{6}$$

where $\widetilde{\nabla}$ denotes the Riemannian connection of $\widetilde{g}$ hold, then $(\widetilde{M}, \phi, \xi_1, \eta^1, \widetilde{g})$ is called a Sasakian manifold (see, [1]).

A plane section Π in $T_p\widetilde{M}$ of an almost contact metric manifold $\widetilde{M}$ is called a $\phi-$section if $\Pi \perp \xi_1$ and $\phi(\Pi) = \Pi$. $\widetilde{M}$ is of constant $\phi-$sectional curvature if at each point $p \in \widetilde{M}$, the sectional curvature $\widetilde{K}(\Pi)$ does depend on the choice of the $\phi-$section Π of $T_p\widetilde{M}$. If $\widetilde{K}(X)$ is constant for all non-null vectors in Π, we call $\widetilde{M}$ to be of constant $\phi-$sectional curvature at point p. The function of c defined by $c(p) = \widetilde{K}(\Pi)$ is called the $\phi-$sectional curvature of $\widetilde{M}$. A Sasakian manifold $\widetilde{M}$ with constant $\phi-$sectional curvature c is said to be a Sasakian space form and is denoted by $\widetilde{M}(c)$.

The curvature tensor $\widetilde{R}$ of a Sasakian space form $\widetilde{M}(c)$ is given by (see, [1])

$$\begin{aligned}\widetilde{R}(X,Y)Z = {} & \frac{c+3}{4}\left[\widetilde{g}(Y,Z)X - \widetilde{g}(X,Z)Y\right] \\ & \frac{c-1}{4}[\eta^1(X)\eta^1(Z)Y - \eta^1(Y)\eta^1(Z)X && (7) \\ & +\widetilde{g}(X,Z)\eta^1(Y)\xi_1 - \widetilde{g}(Y,Z)\eta^1(X)\xi_1 && (8) \\ & +\widetilde{g}(\phi Y,Z)\phi X + \widetilde{g}(X,\phi Z)\phi Y - 2\widetilde{g}(\phi X,Y)\phi Z].\end{aligned}$$

Throughout the paper, all manifolds and maps are differentiable of class C^∞. We denote $\mathcal{F}(\widetilde{M})$ the algebra of the differentiable functions on $\widetilde{M}$ and by $\Gamma(E)$ the $\mathcal{F}(\widetilde{M})-$module of the sections of a vector bundle E over $\widetilde{M}$.

2 Main Results

Let $\widetilde{M}$ be a $(2n+1)-$dimensional Sasakian manifold with structure $(\phi, \xi_1, \eta^1, \widetilde{g})$ and (M, g) a hypersurface tangent to ξ_1, called invariant, where g is induced metric on M. We denote by N the unit normal vector field to M and put

$$-\xi_2 = \phi N. \tag{9}$$

Then, since $\eta^1(N) = 0$, we have

$$\overline{g}(\xi_2, \xi_2) = \overline{g}(\phi N, \phi N) = \overline{g}(N, N) - \eta^1(N)\eta^1(N) = 1 \tag{10}$$

and

$$\overline{g}(\xi_2, N) = -\overline{g}(\phi N, N) = 0,\ \overline{g}(\xi_2, \xi_1) = -\eta^1(\phi N) = 0. \tag{11}$$

Hence, $\xi_2 \in \Gamma(TM)$.

We denote by $D^{\perp}$ =span$\{\xi_2\}$ the $1-$dimensional distribution generated by ξ_2, and D the orthogonal complement of $D^{\perp} \oplus \{\xi_1\} \oplus \{N\}$. Thus, we have following decomposition:

$$\begin{aligned} T\widetilde{M} &= D \oplus D^{\perp} \oplus \{\xi_1\} \oplus \{N\} \\ &= \widetilde{D} \oplus \{N\}\,;\ \widetilde{D} = D \oplus D^{\perp} \oplus \{\xi_1\} \end{aligned} \tag{12}$$

$$\begin{aligned} TM &= D \oplus D^{\perp} \oplus \{\xi_1\} \\ &= \widehat{D} \oplus D^{\perp};\ \ \widehat{D} = D \oplus \{\xi_1\} \end{aligned} \tag{13}$$

Based on the decomposition (12) We set

$$\phi X = fX + w(X)N,\ \forall X \in \Gamma(TM) \tag{14}$$

where w and f are tensor fields on M of type $(0,1)$ and $(1,1)$ respectively, also fX represents the tangent part of ϕX. In the sequel, we set $w \neq 0$. Clearly, from (14)

$$w(X) = g(X, \xi_2) = \eta^2(X),$$

where η^2 is the $1-$form dual to ξ_2 on M. Moreover it is easy to verify that

$$\phi\xi_2 = N;\ f\xi_i = 0,\ \eta^i \circ f = 0, \text{ for all } i \in \{1,2\}\,. \tag{15}$$

Let M be a invariant hypersurface of Sasakian manifold $\widetilde{M}$ Denote by $\widetilde{\nabla}$ and ∇ the Levi-Civita on $\widetilde{M}$ and induced Levi-Civita connection M respectively. By using (12) and (13), the Gauss and Weingarten formulae are

$$\widetilde{\nabla}_X Y = \nabla_X Y + B(X,Y)N \tag{16}$$

$$\widetilde{\nabla}_X N = -AX,\ \forall X, Y \in \Gamma(TM), \tag{17}$$

where A is the shape operator. It is known that

$$B(X,Y) = g(AX, Y),\ \forall X, Y \in \Gamma(TM), \tag{18}$$

In the usual way, we derive the Gauss and Codazzi equations:

$$R(X,Y)Z = \frac{c+3}{4}\left[\widetilde{g}(Y,Z)X - \widetilde{g}(X,Z)Y\right]$$

$$\frac{c-1}{4}[\eta^1(X)\eta^1(Z)Y-\eta^1(Y)\eta^1(Z)X$$
$$+\widetilde{g}(X,Z)\eta^1(Y)\xi_1-\widetilde{g}(Y,Z)\eta^1(X)\xi_1 \qquad (19)$$
$$+\widetilde{g}(\phi Y,Z)\phi X+\widetilde{g}(X,\phi Z)\phi Y-2\widetilde{g}(\phi X,Y)\phi Z]-g(A_N Z,Y)A_N X+g(A_N Z,X)A_N Y.$$
$$(\nabla_X A)\,Y-(\nabla_Y A)\,X=\frac{c-1}{4}[\eta^2(Y)\phi X-\eta^2(X)\phi Y+2\widetilde{g}(\phi X,Y)\xi_2]$$

Lemma 1 *Let M be a invariant hypersurface of a Sasakian space form $\widetilde{M}(c)$. We have*

$$\triangledown_X\xi_2=fAX \qquad (20)$$

and

$$(\triangledown_X f)\,Y=\left(\widetilde{\nabla}_X\phi\right)Y+g(A_N X,Y)\xi_2-\eta^2(Y)A_N X \qquad (21)$$

on M.

Proof. By using (9) and (16), we obtain (20). From (5) and (16), we have (21). ■

Lemma 2 *Let M be a invariant hypersurface of a Sasakian space form $\widetilde{M}(c)$.* If $c\neq 1$ then $\triangledown\xi_2\neq 0$.

Proof. By Lemma 1, $\triangledown_X\xi_2=0$ if and only if $fA_N X=0$. Suppose that this condition holds for all X. Then $AX=\sum_{i=1}^{2}\eta^i(A_N X)\xi_i$. Thus, for all X and Y,

$$(\triangledown_X A)\,Y=\triangledown_X(AY)-A\triangledown_X Y\in\operatorname{Span}\{\xi_1,\xi_2\}.$$

Applying Codazzi equation, we have

$$\frac{c-1}{4}[\eta^2(Y)\phi X-\eta^2(X)\phi Y+2\widetilde{g}(\phi X,Y)\xi_2]\in\operatorname{Span}\{\xi_1,\xi_2\}. \qquad (22)$$

In particular, by putting $Y=\xi_2$ in (22) we get

$$\frac{c-1}{4}[fX]\in\operatorname{Span}\{\xi_1,\xi_2\}. \qquad (23)$$

This is contradiction since $c\neq 1$, i.e, $\triangledown\xi_2\neq 0$. ■

Theorem 3 *Let M be a invariant hypersurface of a Sasakian space form $\widetilde{M}(c)$. Then the shape operator A cannot be parallel.*

Proof. Suppose that $\nabla A=0$. If we take $X\neq 0$ orthogonal to ξ_2 and $Y=\xi_2$ in the Codazzi equation, we have $\frac{c-1}{4}fX=0$. This is clearly imposible since $c\neq 1$. ■

Theorem 4 *Let M be a invariant hypersurface of a Sasakian space form $\widetilde{M}(c)$. No identity of the form $A = \lambda I$ can hold, even with λ nonconstant . In particular, Umbilic hypersurfaces cannot occur.*

Proof. Let us assume that $A_N = \lambda I$. By using Codazzi equation, we have

$$(X\lambda)\,Y - (Y\lambda)\,X = \frac{c-1}{4}\left[\eta^2(X)\phi Y - \eta^2(Y)\phi X + 2\overline{g}(X,\phi Y)\xi_2\right]. \tag{24}$$

If we put $Y = \xi_2$ in (24), we obtain

$$(X\lambda)\,\xi_2 - (\xi_2\lambda)\,X = \frac{c-1}{4} fX. \tag{25}$$

For $X \neq 0$ orthogonal to ξ_2, the set $\{X, fX, \xi_2\}$ is linearly independent, and so $c = 1$ which contradicts the hypothesis. Thus, the proof is compeletd. ■

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