Non-Existence of Umbilic Invariant Hypersurfaces of Sasakian Space Forms

Nesip Aktan
Afyon Kocatepe University
Art and Sciences Faculty
Department of Mathematics, Afyon

Erdal Özüasağlam
Eskişehir Osmangazi University
Art and Sciences Faculty
Department of Mathematics, Eskişehir

Abstract
In the present paper, we consider invariant hypersurfaces of a Sasakian space form. The main purpose of this paper is to investigate non-existence of umbilic invariant hypersurfaces of a Sasakian space forms.

Mathematics Subject Classification: 53D10, 53C25, 53C21

Keywords: Sasakian manifold, Space form, Hypersurface

1 Introduction

Let $\tilde{M}$ be a $(2n+1)$–dimensional differentiable manifold equipped with a triple $(\phi, \xi_1, \eta^1)$, where $\phi$ is a $(1,1)$-tensor field, $\xi_1$ is a vector field, $\eta^1$ is a 1–form on $\tilde{M}$ such that

$$\eta^1(\xi_1) = 1, \quad \phi^2 = -I + \xi_1 \otimes \eta^1 \quad (1)$$

which implies

$$\phi \xi_1 = 0 \quad \eta^1 \circ \phi = 0 \quad rank(\phi) = 2n. \quad (2)$$
If $\tilde{M}$ admits a Riemannian metric $\tilde{g}$, such that
\[
\tilde{g}(\phi X, \phi Y) = \tilde{g}(X, Y) - \eta^1(X) \eta^1(Y),
\]
\[
g(\xi_1, X) = \eta^1(X),
\]
then $\tilde{M}$ is said to admit a $(\phi, \xi_1, \eta^1, \tilde{g})$-structure. If moreover
\[
(\tilde{\nabla}_X \phi)Y = \tilde{g}(X, Y)\xi_1 - \eta^1(Y)X.
\]
and
\[
\tilde{\nabla}_X \xi_1 = -\phi X
\]
where $\tilde{\nabla}$ denotes the Riemannian connection of $\tilde{g}$ hold, then $(\tilde{M}, \phi, \xi_1, \eta^1, \tilde{g})$ is called a Sasakian manifold (see, [1]).

A plane section $\Pi$ in $T_p\tilde{M}$ of an almost contact metric manifold $\tilde{M}$ is called a $\phi$-section if $\Pi \perp \xi_1$ and $\phi(\Pi) = \Pi$. $\tilde{M}$ is of constant $\phi$-sectional curvature if at each point $p \in \tilde{M}$, the sectional curvature $\tilde{K}(\Pi)$ does depend on the choice of the $\phi$-section $\Pi$ of $T_p\tilde{M}$. If $\tilde{K}(X)$ is constant for all non-null vectors in $\Pi$, we call $\tilde{M}$ to be of constant $\phi$-sectional curvature at point $p$. The function of $c$ defined by $c(p) = \tilde{K}(\Pi)$ is called the $\phi$-sectional curvature of $\tilde{M}$. A Sasakian manifold $\tilde{M}$ with constant $\phi$-sectional curvature $c$ is said to be a Sasakian space form and is denoted by $\tilde{M}(c)$.

The curvature tensor $\tilde{R}$ of a Sasakian space form $\tilde{M}(c)$ is given by (see, [1])
\[
\tilde{R}(X, Y)Z = \frac{c+3}{4} [\tilde{g}(Y, Z)X - \tilde{g}(X, Z)Y] + \frac{c-1}{4} [\eta^1(X)\eta^1(Z)Y - \eta^1(Y)\eta^1(Z)X + \tilde{g}(X, Z)\eta^1(Y)\xi_1 - \tilde{g}(Y, Z)\eta^1(X)\xi_1 + \tilde{g}(\phi Y, Z)\phi X + \tilde{g}(X, \phi Z)\phi Y - 2\tilde{g}(\phi X, Y)\phi Z].
\]

Throughout the paper, all manifolds and maps are differentiable of class $C^\infty$. We denote $\mathcal{F}(\tilde{M})$ the algebra of the differentiable functions on $\tilde{M}$ and by $\Gamma(E)$ the $\mathcal{F}(\tilde{M})$-module of the sections of a vector bundle $E$ over $\tilde{M}$.

## 2 Main Results

Let $\tilde{M}$ be a $(2n+1)$-dimensional Sasakian manifold with structure $(\phi, \xi_1, \eta^1, \tilde{g})$ and $(M, g)$ a hypersurface tangent to $\xi_1$, called invariant, where $g$ is induced metric on $M$. We denote by $N$ the unit normal vector field to $M$ and put
\[
-\xi_2 = \phi N.
\]
Then, since $\eta^1(N) = 0$, we have
\[
\mathcal{J} (\xi_2, \xi_2) = \mathcal{J} (\phi N, \phi N) = \mathcal{J} (N, N) - \eta^1(N)\eta^1(N) = 1
\] (10)
and
\[
\mathcal{J} (\xi_2, N) = -\mathcal{J} (\phi N, N) = 0, \quad \mathcal{J} (\xi_2, \xi_1) = -\eta^1(\phi N) = 0.
\] (11)
Hence, $\xi_2 \in \Gamma(TM)$.

We denote by $D^\perp = \text{span}\{\xi_2\}$ the 1-dimensional distribution generated by $\xi_2$, and $D$ the orthogonal complement of $D^\perp \oplus \{\xi_1\} \oplus \{N\}$. Thus, we have the following decomposition:
\[
TM = D \oplus D^\perp \oplus \{\xi_1\} \oplus \{N\}
\] (12)
\[
= D^\perp \oplus \{N\}; \quad D = D \oplus D^\perp \oplus \{\xi_1\}
\] (13)

Based on the decomposition (12) we set
\[
\phi X = fX + w(X)N, \quad \forall X \in \Gamma(TM)
\] (14)
where $w$ and $f$ are tensor fields on $M$ of type $(0, 1)$ and $(1, 1)$ respectively, also $fX$ represents the tangent part of $\phi X$. In the sequel, we set $w \neq 0$. Clearly, from (14)
\[
w(X) = g(X, \xi_2) = \eta^2(X),
\]
where $\eta^2$ is the 1-form dual to $\xi_2$ on $M$. Moreover it is easy to verify that
\[
\phi \xi_2 = N; \quad f \xi_i = 0, \quad \eta^i \circ f = 0, \text{ for all } i \in \{1, 2\}.
\] (15)

Let $M$ be an invariant hypersurface of Sasakian manifold $\tilde{M}$. Denote by $\tilde{\nabla}$ and $\nabla$ the Levi-Civita on $\tilde{M}$ and induced Levi-Civita connection $M$ respectively. By using (12) and (13), the Gauss and Weingarten formulae are
\[
\tilde{\nabla}_X Y = \nabla_X Y + B(X, Y)N
\] (16)
\[
\tilde{\nabla}_X N = -AX, \quad \forall X, Y \in \Gamma(TM),
\] (17)
where $A$ is the shape operator. It is known that
\[
B(X, Y) = g(AX, Y), \quad \forall X, Y \in \Gamma(TM),
\] (18)
In the usual way, we derive the Gauss and Codazzi equations:
\[
R(X, Y)Z = \frac{c+3}{4} [\tilde{g}(Y, Z)X - \tilde{g}(X, Z)Y]
\]
\[
\frac{c-1}{4} \left[ \eta^1(X) \eta^1(Z) Y - \eta^1(Y) \eta^1(Z) X \right] \\
+ \tilde{g}(X, Z) \eta^1(Y) \xi_1 - \tilde{g}(Y, Z) \eta^1(X) \xi_1 \\
+ \tilde{g}(\phi Y, Z) \phi X + \tilde{g}(X, \phi Z) \phi Y - 2 \tilde{g}(\phi X, Y) \phi Z - g(A_N Z, Y) A_N X + g(A_N Z, X) A_N Y.
\]

\[
(\nabla_X A) Y - (\nabla_Y A) X = \frac{c-1}{4}[\eta^2(Y) \phi X - \eta^2(X) \phi Y + 2 \tilde{g}(\phi X, Y) \xi_2]
\]

\textbf{Lemma 1} Let \( M \) be a invariant hypersurface of a Sasakian space form \( \tilde{M}(c) \). We have

\[
\nabla_X \xi_2 = f A X
\]

and

\[
(\nabla X f) Y = (\nabla_X \phi) Y + g(A_N X, Y) \xi_2 - \eta^2(Y) A_N X
\]

on \( M \).

\textbf{Proof.} By using (9) and (16), we obtain (20). From (5) and (16), we have (21). \( \blacksquare \)

\textbf{Lemma 2} Let \( M \) be a invariant hypersurface of a Sasakian space form \( \tilde{M}(c) \). If \( c \neq 1 \) then \( \nabla \xi_2 \neq 0 \).

\textbf{Proof.} By Lemma 1, \( \nabla_X \xi_2 = 0 \) if and only if \( f A_N X = 0 \). Suppose that this condition holds for all \( X \). Then \( A X = \sum_{i=1}^{2} \eta^i(A_N X) \xi_i \). Thus, for all \( X \) and \( Y \),

\[
(\nabla_X A) Y = \nabla_X (A Y) - A \nabla_X Y \in \text{Span} \{ \xi_1, \xi_2 \}.
\]

Applying Codazzi equation, we have

\[
\frac{c-1}{4}[\eta^2(Y) \phi X - \eta^2(X) \phi Y + 2 \tilde{g}(\phi X, Y) \xi_2] \in \text{Span} \{ \xi_1, \xi_2 \}.
\]

In particular, by putting \( Y = \xi_2 \) in (22) we get

\[
\frac{c-1}{4}[f X] \in \text{Span} \{ \xi_1, \xi_2 \}.
\]

This is contradiction since \( c \neq 1 \), i.e, \( \nabla \xi_2 \neq 0 \). \( \blacksquare \)

\textbf{Theorem 3} Let \( M \) be a invariant hypersurface of a Sasakian space form \( \tilde{M}(c) \). Then the shape operator \( A \) cannot be parallel.

\textbf{Proof.} Suppose that \( \nabla A = 0 \). If we take \( X \neq 0 \) orthogonal to \( \xi_2 \) and \( Y = \xi_2 \) in the Codazzi equation, we have

\[
\frac{c-1}{4} f X = 0.
\]

This is clearly impossible since \( c \neq 1 \). \( \blacksquare \)
Theorem 4 Let $M$ be an invariant hypersurface of a Sasakian space form $\tilde{M}(c)$. No identity of the form $A = \lambda I$ can hold, even with $\lambda$ nonconstant. In particular, Umbilic hypersurfaces cannot occur.

Proof. Let us assume that $A_N = \lambda I$. By using Codazzi equation, we have

$$
(X\lambda) Y - (Y\lambda) X = \frac{c-1}{4} \left[ \eta^2(X)\phi Y - \eta^2(Y)\phi X + 2\eta(X,\phi Y)\xi_2 \right].
$$

(24)

If we put $Y = \xi_2$ in (24), we obtain

$$
(X\lambda) \xi_2 - (\xi_2\lambda) X = \frac{c-1}{4} fX.
$$

(25)

For $X \neq 0$ orthogonal to $\xi_2$, the set $\{X, fX, \xi_2\}$ is linearly independent, and so $c = 1$ which contradicts the hypothesis. Thus, the proof is completed.

References


Received: October 18, 2006