

# On the Order of Schur Multipliers of Finite Abelian $p$ -Groups

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## Abstract

Let  $G$  be a finite  $p$ -group of order  $p^n$  with  $|M(G)| = p^{\frac{n(n-1)}{2}-t}$ , where  $M(G)$  is the Schur multiplier of  $G$ . Ya.G. Berkovich, X. Zhou, and G. Ellis have determined the structure of  $G$  when  $t = 0, 1, 2, 3$ . In this paper, we are going to find some structures for an abelian  $p$ -group  $G$  with conditions on the exponents of  $G$ ,  $M(G)$ , and  $S_2M(G)$ , where  $S_2M(G)$  is the metabelian multiplier of  $G$ .

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## 1 Introduction and Preliminaries

Let  $G$  be any group with a presentation  $G \cong F/R$ , where  $F$  is a free group. Then the Baer invariant of  $G$  with respect to the variety of groups  $\mathcal{V}$ , denoted by  $\mathcal{V}M(G)$ , is defined to be

$$\mathcal{V}M(G) = \frac{R \cap V(F)}{[RV^*F]},$$

where  $V$  is the set of words of the variety  $\mathcal{V}$ ,  $V(F)$  is the verbal subgroup of  $F$  and

$$[RV^*F] = \langle v(f_1, \dots, f_{i-1}, f_i r, f_{i+1}, \dots, f_n)v(f_1, \dots, f_i, \dots, f_n)^{-1} | r \in R, f_i \in F, v \in V, 1 \leq i \leq n, n \in N \rangle.$$

In particular, if  $\mathcal{V}$  is the variety of abelian groups,  $\mathcal{A}$ , then the Baer invariant of the group  $G$  will be  $(R \cap F')/[R, F]$  which is isomorphic to the well-known notion the Schur multiplier of  $G$ , denoted by  $M(G)$  (see [5,6] for further details).

If  $\mathcal{V}$  is the variety of polynilpotent groups of class row  $(c_1, \dots, c_t)$ ,  $\mathcal{N}_{c_1, c_2, \dots, c_t}$ , then the Baer invariant of a group  $G$  with respect to this variety is as follows:

$$\mathcal{N}_{c_1, c_2, \dots, c_t}M(G) = \frac{R \cap \gamma_{c_t+1} \circ \dots \circ \gamma_{c_1+1}(F)}{[R, c_1 F, c_2 \gamma_{c_1+1}(F), \dots, c_t \gamma_{c_{t-1}+1} \circ \dots \circ \gamma_{c_1+1}(F)]}, \quad (1)$$

where  $\gamma_{c_t+1} \circ \dots \circ \gamma_{c_1+1}(F) = \gamma_{c_t+1}(\gamma_{c_{t-1}+1}(\dots(\gamma_{c_1+1}(F))\dots))$  are the term of iterated lower central series of  $F$ . See [4] for the equality

$$[RN_{c_1, \dots, c_t}^*F] = [R, c_1 F, c_2 \gamma_{c_1+1}(F), \dots, c_t \gamma_{c_{t-1}+1} \circ \dots \circ \gamma_{c_1+1}(F)].$$

In particular, if  $c_i = 1$  for  $1 \leq i \leq t$ , then  $\mathcal{N}_{c_1, c_2, \dots, c_t}$  is the variety of solvable groups of length at most  $t \geq 1$ ,  $\mathcal{S}_t$ .

In 1956, J.A. Green [3] showed that the order of the Schur multiplier of a finite  $p$ -group of order  $p^n$  is bounded by  $p^{\frac{n(n-1)}{2}}$ , and hence equals to  $p^{\frac{n(n-1)}{2}-t}$ , for some nonnegative integer  $t$ . In 1991, Ya.G. Berkovich [1] has determined all finite  $p$ -groups  $G$  for which  $t = 0, 1$ . The groups for which  $t = 0$  are exactly elementary abelian  $p$ -groups, and the groups for which  $t = 1$  are cyclic groups of order  $p^2$  or the nonabelian group of order  $p^3$  with exponent  $p > 2$ . In 1994, X. Zhou [7] found all finite  $p$ -groups for  $t = 2$ . He showed that these groups are the direct product of two cyclic groups of order  $p^2$  and  $p$  or the direct product of a cyclic group of order  $p$  and the nonabelian group of order  $p^3$  and exponent  $p > 2$  or the dihedral group of order 8. G. Ellis [2] determined all finite  $p$ -groups  $G$  with  $t = 0, 1, 2, 3$  in a quite different method to that of [1] and [7] as follows:

**Theorem 1.1** ([2]). *Let  $G$  be a group of prime-power order  $p^n$ . Suppose that  $M(G)$  has order  $p^{\frac{n(n-1)}{2}-t}$ . Then  $t \geq 0$  and*

- (i)  $t = 0$  if and only if  $G$  is elementary abelian;
- (ii)  $t = 1$  if and only if  $G \cong \mathbf{Z}_{p^2}$  or  $G \cong E_1$ ;
- (iii)  $t = 2$  if and only if  $G \cong \mathbf{Z}_p \times \mathbf{Z}_{p^2}$ ,  $G \cong D$  or  $G \cong \mathbf{Z}_p \times E_1$ ;
- (iv)  $t = 3$  if and only if  $G \cong \mathbf{Z}_{p^3}$ ,  $G \cong \mathbf{Z}_p \times \mathbf{Z}_p \times \mathbf{Z}_{p^2}$ ,  $G \cong D \times \mathbf{Z}_{p^2}$ ,  $G \cong E_2$ ,  $G \cong Q$  or  $G \cong \mathbf{Z}_p \times \mathbf{Z}_p \times E_1$ .

Here  $\mathbf{Z}_{p^m}$  denotes the cyclic group of order  $p^m$ ,  $D$  denotes the dihedral group of order 8,  $Q$  denotes the quaternion group of order 8,  $E_1$  denotes the extra special group of order  $p^3$  with odd exponent  $p$ , and  $E_2$  denotes the extra special

group of order  $p^3$  with odd exponent  $p^2$ .

Now, in this paper, we are going to find some structures for the  $p$ -group  $G$  when  $G$  is abelian and  $|\Phi(G)| = p^a$  with conditions on the exponents of  $G, M(G)$ , and  $S_2M(G)$ . The following useful theorem of I. Schur is frequently used in our method.

**Theorem 1.2** (I. Schur [5]). *Let  $G \cong \mathbf{Z}_{n_1} \oplus \mathbf{Z}_{n_2} \oplus \dots \oplus \mathbf{Z}_{n_k}$ , where  $n_{i+1} | n_i$  for all  $i \in 1, 2, \dots, k - 1$  and  $k \geq 2$ , and let  $\mathbf{Z}_n^{(m)}$  denote the direct product of  $m$  copies of  $\mathbf{Z}_n$ . Then*

$$M(G) \cong \mathbf{Z}_{n_2} \oplus \mathbf{Z}_{n_3}^{(2)} \oplus \dots \oplus \mathbf{Z}_{n_k}^{(k-1)}$$

**Remark 1.3.** Let  $G$  be an abelian group with a free presentation  $F/R$ . Since  $F' \leq R$ ,  $\mathcal{N}_{c_1}M(G) = \gamma_{c_1+1}(F)/[R, {}_{c_1}F]$ . Now, we can consider  $\gamma_{c_1+1}(F)/[R, {}_{c_1}F]$  as a free presentation for  $\mathcal{N}_{c_1}M(G)$  and hence

$$\mathcal{N}_{c_2}M(\mathcal{N}_{c_1}M(G)) = \frac{\gamma_{c_2+1}(\gamma_{c_1+1}(F))}{[R, {}_{c_1}F, {}_{c_2}\gamma_{c_1+1}F]}.$$

Therefore by (1) we have

$$\mathcal{N}_{c_1, c_2}M(G) = \mathcal{N}_{c_2}M(\mathcal{N}_{c_1}M(G)).$$

By continuing the above process we can show that

$$\mathcal{N}_{c_1, c_2, \dots, c_t}M(G) = \mathcal{N}_{c_t}M(\dots \mathcal{N}_{c_2}M(\mathcal{N}_{c_1}M(G)) \dots).$$

In particular, if  $c_1 = c_2 = 1$ , then we have  $S_2M(G) = M(M(G))$ .

## 2 Main Results

Through out the paper we assume that  $G$  is an abelian  $p$ -group of order  $p^n$  with  $|M(G)| = p^{\frac{n(n-1)}{2}-t}$ .

**Lemma 2.1** . *Let  $\Phi(G)$ , the Frattini subgroup of  $G$ , be of order  $p^a$ . Then  $n = (a(a + 1) + 2t + 2m)/2a$ , for some  $m \in \mathbf{N}_0$ .*

*Proof.* Let  $G = \mathbf{Z}_{p^{\alpha_1}} \oplus \mathbf{Z}_{p^{\alpha_2}} \oplus \dots \oplus \mathbf{Z}_{p^{\alpha_{n-a}}}$ , where  $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_{n-a}$ . By Theorem 1.2 ,  $M(G) \cong \mathbf{Z}_{p^{\alpha_2}} \oplus \mathbf{Z}_{p^{\alpha_3}}^{(2)} \oplus \dots \oplus \mathbf{Z}_{p^{\alpha_{n-a}}}^{(n-a-1)}$  and so  $|M(G)| =$

$p^{\alpha_2+2\alpha_3+\dots+(n-a-1)\alpha_{n-a}}$ . But  $M(G)$  has order  $p^{\frac{n(n-1)}{2}-t}$ . Therefore

$$\begin{aligned} \frac{n(n-1)}{2} - t &= \alpha_2 + 2\alpha_3 + \dots + (n-a-1)\alpha_{n-a} \\ &\geq 1 + 2 + \dots + (n-a-1) \\ &= \frac{(n-a)(n-a-1)}{2} \\ &= \frac{n^2 - (2a+1)n + a(a+1)}{2}. \end{aligned}$$

Hence  $2an \geq 2t + a(a+1)$ , and the result holds.

**Lemma 2.2** . *With the assumption and notation of the previous lemma we have the following inequalities for the exponent of  $G$ ,*

$$p^{a-m+1} \leq \exp(G) \leq p^{a+1}.$$

*Proof.* Clearly  $G/\Phi(G)$  is an elementary abelian  $p$ -group of order  $p^{n-a}$  and so  $\exp(G) \leq p^{a+1}$ .

For the other inequality let  $G \cong \mathbf{Z}_{p^{\alpha_1}} \oplus \mathbf{Z}_{p^{\alpha_2}} \oplus \dots \oplus \mathbf{Z}_{p^{\alpha_{n-a}}}$ , where  $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_{n-a} \geq 1$ . Then similar to the proof of previous lemma we have

$$\begin{aligned} \frac{n(n-1)}{2} - t &= \alpha_2 + 2\alpha_3 + \dots + (n-a-1)\alpha_{n-a} \\ &\geq \alpha_2 + (2+3+\dots+n-a-1) \\ &\geq \frac{2\alpha_2 - 2 + n^2 - (2a+1)n + a(a+1)}{2} \end{aligned}$$

Therefore  $n \geq \frac{a(a+1)+2\alpha_2-2+2t}{2a}$  and hence by Lemma 2.1 we have  $\alpha_2 \leq m+1$ . Now, suppose by contrary  $\exp(G) = p^{a-k+1}$ , where  $k > m$ , then  $\alpha_3 \geq 2$ . Thus by Theorem 1.2 we have

$$\begin{aligned} \frac{n(n-1)}{2} - t &= \alpha_2 + 2\alpha_3 + \dots + (n-a-1)\alpha_{n-a} \\ &= (\alpha_2 + \alpha_3 + \dots + \alpha_{n-a}) + \alpha_3 + 2\alpha_4 + \dots + (n-a-2)\alpha_{n-a} \\ &\geq (n-a+k-1) + (2+2+3+\dots+n-a-2) \\ &= n-a+k + \frac{(n-a-2)(n-a-1)}{2}. \end{aligned}$$

Hence  $n \geq \frac{2k+a(a+1)+2+2t}{2a} > \frac{2m+a(a+1)+2+2t}{2a}$  which is a contradiction by Lemma 2.1.

**Theorem 2.3** . With the above notation and assumptions, let  $G$  be of exponent  $p^{a-m+1}$ . Then  $G \cong \mathbf{Z}_{p^{a-m+1}} \oplus \mathbf{Z}_{p^{m+1}} \oplus \underbrace{\mathbf{Z}_p \oplus \dots \oplus \mathbf{Z}_p}_{n-a-2\text{-copies}}$ .

*Proof.* let  $G \cong \mathbf{Z}_{p^{\alpha_1}} \oplus \mathbf{Z}_{p^{\alpha_2}} \oplus \dots \oplus \mathbf{Z}_{p^{\alpha_{n-a}}}$ , where  $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_{n-a} \geq 1$ . By the proof of previous lemma we have  $\alpha_2 \leq m + 1$ . If  $\alpha_2 \leq m$ , then  $\alpha_3 \geq 2$  and we have

$$\begin{aligned} \frac{n(n-1)}{2} - t &= \alpha_2 + 2\alpha_3 + \dots + (n-a-1)\alpha_{n-a} \\ &= (\alpha_2 + \alpha_3 + \dots + \alpha_{n-a}) + \alpha_3 + 2\alpha_4 + \dots + (n-a-2)\alpha_{n-a} \\ &\geq (n-a+m-1) + (2+2+3+\dots+n-a-2) \\ &= n-a+m + \frac{(n-a-2)(n-a-1)}{2}. \end{aligned}$$

Therefore  $n \geq \frac{2m+a(a+1)+2+2t}{2a}$  which is a contradiction by Lemma 2.1. Hence the result holds.

**Theorem 2.4** . Further to the previous notation and assumptions, let  $m = k + s$  ( $k, s \in \mathbf{N}_0$ ),  $\exp(G) = p^{a-k+1}$ , and  $\exp(M(G)) + \exp(S_2M(G)) = p^{k+r}$ . Then

$$G \cong \mathbf{Z}_{p^{a-k+1}} \oplus \mathbf{Z}_{p^x} \oplus \mathbf{Z}_{p^{k+r-x}} \oplus \mathbf{Z}_{p^{h_1}} \oplus \dots \oplus \mathbf{Z}_{p^{h_f}} \oplus \underbrace{\mathbf{Z}_p \oplus \dots \oplus \mathbf{Z}_p}_{n-a-(f+3)\text{-copies}},$$

where

$$x = k - s + 2r - 3 + 3(h_1 - 1) + \dots + (f + 2)(h_f - 1), \quad h_1 \geq h_2 \geq \dots \geq h_f \geq 2, \text{ and } f \leq -r + 2.$$

*Proof.* Let  $G \cong \mathbf{Z}_{p^{\alpha_1}} \oplus \mathbf{Z}_{p^{\alpha_2}} \oplus \dots \oplus \mathbf{Z}_{p^{\alpha_{n-a}}}$ , where  $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_{n-a}$ . By Theorem 1.2 and Remark 1.3 it is easy to see that  $\exp(M(G)) = p^{\alpha_2}$ ,  $\exp(S_2M(G)) = p^{\alpha_3}$ , and so by hypothesis  $\alpha_2 + \alpha_3 = k + r$ , and  $\alpha_1 = a - k + 1$ . If  $\alpha_3 = 1$ , then it is easy to see that  $s = 0$  and so  $\exp(G) = a - m + 1$ . Hence by Theorem 2.3  $G \cong \mathbf{Z}_{p^{a-m+1}} \oplus \mathbf{Z}_{p^{m+1}} \oplus \underbrace{\mathbf{Z}_p \oplus \dots \oplus \mathbf{Z}_p}_{n-a-2\text{-copies}}$ .

Now, we can assume that  $\alpha_3 \geq 2$  and hence  $G$  may have the following structure:

$$G \cong \mathbf{Z}_{p^{a-k+1}} \oplus \mathbf{Z}_{p^x} \oplus \mathbf{Z}_{p^{k+r-x}} \oplus \mathbf{Z}_{p^{h_1}} \oplus \dots \oplus \mathbf{Z}_{p^{h_f}} \oplus \underbrace{\mathbf{Z}_p \oplus \dots \oplus \mathbf{Z}_p}_{n-a-(f+3)\text{-copies}},$$

where  $f \geq 0$  and  $h_1 \geq h_2 \geq \dots \geq h_f \geq 2$ . If  $f \geq 1$  since  $G$  is a group of order  $p^n$ , we have

$(a-k+1)+(k+r)+(h_1+\dots+h_f)+(n-a-f-3) = n$  so  $h_1+\dots+h_f = -r+f+2$  and hence  $h_f = -r+f+2-h_1-\dots-h_{f-1}$ .

But  $h_1 \geq h_2 \geq \dots \geq h_f \geq 2$ , so that  $-r+f+2 \geq 2f$  and so  $0 \leq f \leq -r+2$ .

Now, by Theorem 1.2 we have

$$\frac{n^2 - n - 2t}{2} = z + (1 + 2 + \dots + n - a - 1),$$

where

$$z = x + 2k + 2r - 2x + 3h_1 + 4h_2 + \dots + (f + 2)h_f - (1 + 2 + \dots + f + 2).$$

Thus

$$n = \frac{2z + a(a + 1) + 2t}{2a}.$$

On the other hand by the hypothesis  $n = \frac{2(k+s)+a(a+1)+2t}{2a}$ , hence we have  $z = k + s$ , and the result follows.

**Corollary 2.5** . *With the notation and assumptions of previous theorem we have*

(i)  $G \cong \mathbf{Z}_{p^{a-k+1}} \oplus \mathbf{Z}_{p^{k-s+1}} \oplus \mathbf{Z}_{p^{s+1}} \oplus \underbrace{\mathbf{Z}_p \oplus \dots \oplus \mathbf{Z}_p}_{n-a-3\text{-copies}}$ , if  $r = 2$ ;

(ii)  $G \cong \mathbf{Z}_{p^{a-k+1}} \oplus \mathbf{Z}_{p^{k-s+2}} \oplus \mathbf{Z}_{p^{s-1}} \oplus \mathbf{Z}_{p^2} \oplus \underbrace{\mathbf{Z}_p \oplus \dots \oplus \mathbf{Z}_p}_{n-a-4\text{-copies}}$ , if  $r = 1$ ;

(iii) if  $r = 0$ , then

$$G \cong \begin{cases} \mathbf{Z}_{p^{a-k+1}} \oplus \mathbf{Z}_{p^{k-s+4}} \oplus \mathbf{Z}_{p^{s-4}} \oplus \mathbf{Z}_{p^2} \oplus \mathbf{Z}_{p^2} \oplus \underbrace{\mathbf{Z}_p \oplus \dots \oplus \mathbf{Z}_p}_{n-a-5\text{-copies}} \\ \text{or} \\ \mathbf{Z}_{p^{a-k+1}} \oplus \mathbf{Z}_{p^{k-s+3}} \oplus \mathbf{Z}_{p^{s-3}} \oplus \mathbf{Z}_{p^3} \oplus \underbrace{\mathbf{Z}_p \oplus \dots \oplus \mathbf{Z}_p}_{n-a-4\text{-copies}} \end{cases} ;$$

(iv) if  $r = -1$ , then

$$G \cong \begin{cases} \mathbf{Z}_{p^{a-k+1}} \oplus \mathbf{Z}_{p^{k-s+8}} \oplus \mathbf{Z}_{p^{s-9}} \oplus \mathbf{Z}_{p^4} \oplus \underbrace{\mathbf{Z}_p \oplus \dots \oplus \mathbf{Z}_p}_{n-a-4\text{-copies}} \\ \mathbf{Z}_{p^{a-k+1}} \oplus \mathbf{Z}_{p^{k-s+5}} \oplus \mathbf{Z}_{p^{s-6}} \oplus \mathbf{Z}_{p^3} \oplus \mathbf{Z}_{p^2} \oplus \underbrace{\mathbf{Z}_p \oplus \dots \oplus \mathbf{Z}_p}_{n-a-5\text{-copies}} \\ \mathbf{Z}_{p^{a-k+1}} \oplus \mathbf{Z}_{p^{k-s+7}} \oplus \mathbf{Z}_{p^{s-8}} \oplus \mathbf{Z}_{p^2} \oplus \mathbf{Z}_{p^2} \oplus \mathbf{Z}_{p^2} \oplus \underbrace{\mathbf{Z}_p \oplus \dots \oplus \mathbf{Z}_p}_{n-a-6\text{-copies}} \end{cases} .$$

*Proof.* i) If  $r = 2$ , then  $f = 0$ . Therefore

$$G \cong \mathbf{Z}_{p^{a-k+1}} \oplus \mathbf{Z}_{p^x} \oplus \mathbf{Z}_{p^{k+r-x}} \oplus \underbrace{\mathbf{Z}_p \oplus \dots \oplus \mathbf{Z}_p}_{n-a-3\text{-copies}},$$

where  $x = k - s + 1$ . Hence the result follows.

ii) If  $r = 1$ , then  $f = 0, 1$ . If  $f = 0$ , then  $n = a - k + 1 + k + 1 + n - a - 3$  which is a contradiction. Then  $f = 1$  and

$$G \cong \mathbf{Z}_{p^{a-k+1}} \oplus \mathbf{Z}_{p^x} \oplus \mathbf{Z}_{p^{k+r-x}} \oplus \mathbf{Z}_{p^{h_1}} \oplus \underbrace{\mathbf{Z}_p \oplus \dots \oplus \mathbf{Z}_p}_{n-a-4\text{-copies}},$$

where  $h_1 = -r + f + 2 = 2$  and  $x = k - s + 2 + 6 - (1 + 2 + 3) = k - s + 2$ . Hence the result follows.

iii) If  $r = 0$ , then  $f = 0, 1, 2$ . If  $f = 0$ , then  $h_1 + h_2 + \dots + h_f = 0$  but we have  $h_1 + h_2 + \dots + h_f = -r + f + 2 = 2$  which is a contradiction. If  $f = 1$ , then  $x = k - s + 4$  and if  $f = 2$ , then  $x = k - s + 3$ .

iv) By a routine calculation similar to (ii) the result holds.

Note that we can continue the above corollary for other integers  $r < -1$ , but with a boring calculations.

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