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New Insights into Cost Malmquist Productivity Measure

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Abstract

DEA-based Malmquist productivity index measures the productivity change over time. Cost Malmquist index is defined in terms of input cost rather than input quantity distance functions. Hence, productivity change is decomposed into overall efficiency and cost technical change. In this paper, we provide an extension to the cost Malmquist approach by further analyzing these two cost Malmquist components. Our proposed approach not only reveals patterns of productivity change and presents a new interpretation along with the managerial implication of each Malmquist component, but also identifies the strategy shifts of individual DMUs in particular time period.

Mathematics Subject Classification: Operation Research, No. 90

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1 Introduction

One of the most popular approaches to measuring productivity changes is based on using Malmquist Productivity Indexes. The Malmquist index was first suggested by Malmquist (1953)as a quantity index for use in the analysis of consumption of inputs, Fare et al. (1992) combined ideas on the measurement of efficiency from Farrell (1957) and the measurement of productivity from Caves et al. (1982) to construct a Malmquist productivity index directly from input and output data using DEA. They decomposed productivity change into two component, one measuring change in efficiency and the other measuring technical change or equivalently change in the frontier technology. Yao

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Chen et al. (2004) provided an extension to the Malmquist approach by further analyzing these two Malmquist components. Thanassoulis and Manidakis (2004) provided cost Malmquist productivity index (CM) when producers are cost minimisers and input prices are known. They decomposed productivity index into overall efficiency and cost technical change. Following Yao Chen et al. (2004), we further examine the two components to reveal sources and patterns of productivity change that are obscured by the aggregated nature of the cost Malmquist index. Our proposed approach not only reveals patterns of productivity change and presents a new interpretation along with the managerial implication of each cost Malmquist component, but also identifies the strategy shifts of individual DMUs in a particular time period. We can make judgments on whether or not such strategy shifts are favorable and promising. The current paper is organized as follows. The next section is background. The new approach is presented in section 3. In section 4 we compute the index and its component. An example is provided in section 5. Concluding remarks are given in the last section.

2 Background

Consider that in time period t, DMUs are using inputs $x^t \in R_+^n$, to produce outputs $y^t \in R_+^m$. Define now the production technology of period t in terms of the *input requerment set* which is
$L^t(y^t) = \{x^t : x^t can\ produce\ \ y^t\}$ (1)
$L^t(y^t)$ contains all input vectors that can produce output y^t. Assume that $L^t(y^t)$ is non-empty, closed, convex, bounded and satisfies strong disposability of inputs and outputs. $L^t(y^t)$ is bounded from below by the *input isoquant*, that is:
$\text{Isoq}L^t(y^t) = \{x^t : x^t \in L^t(y^t), \lambda x^t L^t(y^t) for \lambda < 1\}$ (2)
$\text{Isoq}L^t(y^t)$ defines a boundary (frontier) to the input requirement set in the sense that any radial contraction of input vectors that lie on the frontier is not possible within $L^t(y^t)$. The *input distance function* (Shephard (1953), (1970)) is defined as follows:
$D_i^t(y^t, x^t) = \sup\{\theta : \frac{x^t}{\theta} \in L^t(y^t), \theta > 0\}$ (3)
where the subscript i denotes input orientation. Note that $D_i^t(y^t, x^t) \geq 1$ and if $D_i^t(y^t, x^t) = 1 \Longleftrightarrow x^t \in \text{Isoq}L^t(y^t)$.
The *cost function* define as follows:
$C^t(y^t, w^t) = \min\{\ w^t x^t : x^t \in L^t(y^t), w > 0\}$ (4)
in which $w^t \in R_+^n$ is input prices. $C^t(y^t, w^t)$ defines the minimum cost of producing a given output vector y^t given the input prices w^t and the technology of period t. Cost boundary is defined as follows:
$\text{Isoq}\overline{C}^t(y^t, w^t) = \{x^t : w^t x^t = C^t(y^t, w^t)\}.$ (5)

This set is contain all vector inputs that can produce output y^t at the minimum cost (i.e. $C^t(y^t, w^t)$). Farrell (1957) defined the measure input oriented overall efficiency for (y^t, x^t) under input prices w^t as
$OE_i^t = \frac{C^t(y^t,w^t)}{w^t x^t}$. (6)
The Malmquist index was introduced by Fare et al. (1989) as follows: Assume two time period t and $t+1$ and (y^t, x^t) be DMU_t in period t and (y^{t+1}, x^{t+1}) be this in period $t+1$. The input oriented Malmquist index (IM) is
$IM^t = \frac{D_i^t(y^{t+1},x^{t+1})}{D_i^t(y^t,x^t)}$ (7)

In fact IM^t compares (y^{t+1}, x^{t+1}) and (y^t, x^t) with respect to constant returns to scale (CRS) technology frontier in period t. In similar fashion one my define the following index:

$$IM^{t+1} = \frac{D_i^{t+1}(y^{t+1},x^{t+1})}{D_i^{t+1}(y^t,x^t)} \qquad (8)$$

In fact IM^{t+1} compares (y^{t+1}, x^{t+1}) and (y^t, x^t) with respect to CRS technology frontier in period $t+1$. To avoid an arbitrary choice of a reference period Fare and et al. (1989) use the geometric mean of IM^t and IM^{t+1} so that the IM is

$$IM = [\frac{D_i^t(y^{t+1},x^{t+1})}{D_i^t(y^t,x^t)} \frac{D_i^{t+1}(y^{t+1},x^{t+1})}{D_i^{t+1}(y^t,x^t)}]^{\frac{1}{2}} \qquad (9)$$

When $IM > 1$ on average the input levels in x^{t+1} are further from the efficient boundary than are the inputs in x^t for securing the corresponding outputs and so we have a deterioration in productivity between t and $t+1$. Thanassoulis and Maniadakis (2004) defined the cost Malmquist (CM) productivity index of period t, $t+1$ and their geometric mean as follows:

$$CM^t = [\frac{w^t x^{t+1}/C^t(y^{t+1},w^t)}{w^t x^t/C^t(y^t,w^t)}], \qquad (10)$$

$$CM^{t+1} = [\frac{w^{t+1}x^{t+1}/C^{t+1}(y^{t+1},w^{t+1})}{w^{t+1}x^t/C^{t+1}(y^t,w^{t+1})}], \qquad (11)$$

$$CM = [\frac{w^t x^{t+1}/C^t(y^{t+1},w^t)}{w^t x^t/C^t(y^t,w^t)} \frac{w^{t+1}x^{t+1}/C^{t+1}(y^{t+1},w^{t+1})}{w^{t+1}x^t/C^{t+1}(y^t,w^{t+1})}]^{\frac{1}{2}} \qquad (12)$$

where $w^t x^t = \Sigma_{n=1}^N w_n^t x_n^t$, n denotes the nth input and $C^t(y^t, w^t)$ is as defined in (4) with reference to the CRS technology. The cost ratio $w^t x^t / C^t(y^t, w^t)$ measures the extent to which the aggregate production cost in period t can be reduced while still securing the output vector y^t under the input price vector w^t. This ratio measures the distance between the observed cost $w^t x^t$ and the cost boundary $C^t(y^t, w^t)$. Minimum value of this distance will be 1, (when the two costs coincide). This (cost) distance is the reciprocal of the input oriented measure of overall efficiency defined in (6). The rest of the cost ratios in (10)-(12) are defined in an analogous manner. We use the CRS cost boundaries as benchmarks for productivity measurement. A CM index value less than

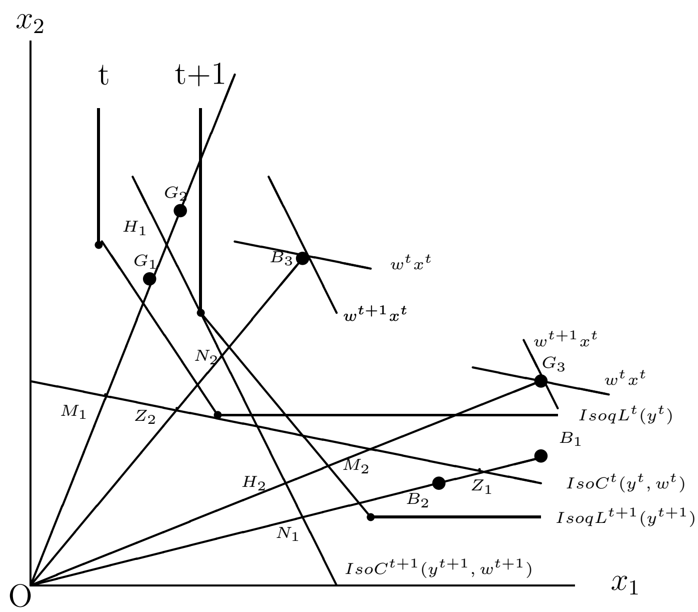

Fig. 1.

1 implies productivity progress, a value greater than 1 implies regress and a value of 1 indicates constant productivity. The CM and its components have been illustrated in Fig. 1. Obvious that

$$CM = [((OB/OZ)/(OG/OM)) \times ((OB/ON)/(OG/OH))]^{\frac{1}{2}}$$

Thanassoulis and Maniadakis (2004) also demonstrated that the CM can be decomposed into two components: *overall efficiency change* (OEC) and *cost-technical change* (CTC) as follows:

$$CM = \frac{w^{t+1}x^{t+1}/C^{t+1}(y^{t+1},w^{t+1})}{w^t x^t/C^t(y^t,w^t)} \times [\frac{w^t x^{t+1}/C^t(y^{t+1},w^t)}{w^{t+1}x^{t+1}/C^{t+1}(y^{t+1},w^{t+1})} \times \frac{w^t x^t/C^t(y^t,w^t)}{w^{t+1}x^t/C^{t+1}(y^t,w^{t+1})}]^{\frac{1}{2}} \quad (13)$$

The term outside brackets in (13) captures the OEC between period t and $t+1$. This measure indicate that whether production unit "catches up" cost boundary when going from period t to $t+1$. The term inside brackets, referred to CTC, measures the shift of cost boundary evaluated at the input mixes x^t and x^{t+1}. If the value of CTC less than one indicates a positive shift or technical progress, a value of CT greater than one indicates a negative shift or technical regress and value of CTC equal to one indicates no shift in cost frontier. OEC and CTC have been illustrated in terms of the distances in Fig. 1. Obvious that

$$OEC = \frac{OB/ON}{OG/OM}$$

$$CTC = [\frac{OB/OZ}{OB/ON} \times \frac{OG/OM}{OG/OH}]^{\frac{1}{2}} = [\frac{ON/OZ}{OH/OM}]^{\frac{1}{2}}$$

Note that here measures of CTC compare cost and consequently technologies at standard input prices. The technical change components compares minimum cost securing certain output in one period relative to that in another period.

3 New insights from cost Malmquist productivity approach

In this section we further analyze the two ratio components defined in

$[\frac{w^t x^{t+1}/C^t(y^{t+1},w^t)}{w^{t+1}x^{t+1}/C^{t+1}(y^{t+1},w^{t+1})} \times \frac{w^t x^t/C^t(y^t,w^t)}{w^{t+1}x^t/C^{t+1}(y^t,w^{t+1})}]^{\frac{1}{2}}$,

since these two components provide a behavior of DMU in time t and $t+1$, with respect to the frontier in both time periods. Suppose we have three DMUs, denoted as G_t^1, G_t^2 and G_t^3 in period t, the possible locations of these three DMUs are denoted as B_{t+1}^1, B_{t+1}^2, B_{t+1}^3, as illustrated in Fig. 1 For example, if G_t^1 represents DMU_0 in period t then DMU_0 can be lie in location of B_{t+1}^2 (or B_{t+1}^1 or B_{t+1}^3) in period $t+1$, i.e. DMU_0 is moved from G_t^1 to B_{t+1}^2 from period t to period $t+1$. Associated with these six points, nine possible movements from period t to period $t+1$ can occur denoted in Table 1, where the nine movements are characterized by

$\frac{w^t x^{t+1}/C^t(y^{t+1},w^t)}{w^{t+1}x^{t+1}/C^{t+1}(y^{t+1},w^{t+1})}$ and $\frac{w^t x^t/C^t(y^t,w^t)}{w^{t+1}x^t/C^{t+1}(y^t,w^{t+1})}$

In Fig. 1, a downward cost frontier shift (towards the origin) from period t to period $t+1$ represents a positive shift, indicating a cost technology progress, and an upward frontier shift (away from the origin) represents a negative shift, indicating a cost technology decline. On the basis of the nine movements in Fig. 1, four possible cases can be occur. They are

(a) If

$\frac{w^t x^{t+1}/C^t(y^{t+1},w^t)}{w^{t+1}x^{t+1}/C^{t+1}(y^{t+1},w^{t+1})} > 1$ and $\frac{w^t x^t/C^t(y^t,w^t)}{w^{t+1}x^t/C^{t+1}(y^t,w^{t+1})} > 1$

then the CTC must be greater than one, indicates that the cost frontier has a negative shift and the cost technology of DMU_0 regresses (Cases 3 and 9 in Table 1)

(b) If

$\frac{w^t x^{t+1}/C^t(y^{t+1},w^t)}{w^{t+1}x^{t+1}/C^{t+1}(y^{t+1},w^{t+1})} < 1$ and $\frac{w^t x^t/C^t(y^t,w^t)}{w^{t+1}x^t/C^{t+1}(y^t,w^{t+1})} < 1$

then the CTC must be less than one, indicates that the cost frontier has a positive shift and the cost technology of DMU_0 progresses (Cases 4 and 5 in Table 1).

(c) If

$\frac{w^t x^{t+1}/C^t(y^{t+1},w^t)}{w^{t+1}x^{t+1}/C^{t+1}(y^{t+1},w^{t+1})} > 1$ and $\frac{w^t x^t/C^t(y^t,w^t)}{w^{t+1}x^t/C^{t+1}(y^t,w^{t+1})} < 1$

Case	Movement $t \longrightarrow t+1$	$\frac{w^t x^{t+1}/C^t(y^{t+1},x^t)}{w^{t+1}x^{t+1}/C^{t+1}(y^{t+1},x^{t+1})}$	$\frac{w^t x^t/C^t(y^t,x^t)}{w^{t+1}x^t/C^{t+1}(y^t,x^{t+1})}$	CTC_0
1	$G_t^2 \longrightarrow B_{t+1}^1$	<1	>1	?
2	$G_t^2 \longrightarrow B_{t+1}^2$	<1	>1	?
3	$G_t^2 \longrightarrow B_{t+1}^3$	>1	> 1	>1
4	$G_t^3 \longrightarrow B_{t+1}^1$	<1	< 1	< 1
5	$G_t^3 \longrightarrow B_{t+1}^2$	<1	<1	<1
6	$G_t^3 \longrightarrow B_{t+1}^3$	>1	<1	?
7	$G_t^1 \longrightarrow B_{t+1}^1$	<1	>1	?
8	$G_t^1 \longrightarrow B_{t+1}^2$	<1	>1	?
9	$G_t^1 \longrightarrow B_{t+1}^3$	>1	>1	> 1

Table 1.

then CTC could be larger or less than one. We can conclude that DMU_0 move from a positive shift toward a negative shift. Furthermore, CTC<1 indicates that the change resulted from the negative shift facet is larger than that of the positive shift facet, and on average the cost technology of DMU_0 progresses. CTC>1 indicates that the change resulted from the positive shift facet is larger than that of the negative shift facet, and on average the cost technology of DMU_0 regresses. CTC=1 indicates that on average the cost technology of DMU_0 remains the same (Case 6).

(d) If

$$\frac{w^t x^{t+1}/C^t(y^{t+1},w^t)}{w^{t+1}x^{t+1}/C^{t+1}(y^{t+1},w^{t+1})} < 1 \text{ and } \frac{w^t x^t/C^t(y^t,w^t)}{w^{t+1}x^t/C^{t+1}(y^t,w^{t+1})} > 1$$

then CTC could be larger or less than one. We can conclude that DMU_0 move from a negative shift toward a positive shift. Furthermore, Similarly as in case (c) CTC>1 indicates that the change resulted from the negative shift facet is larger than that of the positive shift facet, and on average the cost technology of DMU_0 regresses. CTC<1 indicates that the change resulted from the negative shift facet is less than that of the positive shift facet, and on average the cost technology of DMU_0 progresses. CTC=1 indicates that on average the cost technology of DMU_0 remains the same (Cases 1, 2, 7 and 8 in Table 1). From productivity viewpoint, (d) is more favorable situation for DMU_0 than (c). Note that $CM_0 = OEC_0 * CTC_0$ and (i) $OEC_0 > 1$ implies that DMU_0 in period t is closer to the cost frontier in period $t+1$; (ii) $OEC_0 < 1$ implies that DMU_0 in period t is further away from the cost frontier in period $t+1$; and (iii) $OEC_0 = 1$ implies that distance DMU_0 from cost frontier in period t is same to that in period $t+1$; which is very rare condition except when DMU_0 is overall efficient in both period t and $t+1$. Table 2 reports the detail information about components of the cost Malmquist index. In Table 2, only in cases 1 and 2 the value of cost Malmquist index productivity is certain. For all other cases, we do not have certain value for cost Malmquist index productivity for DMU_0. Case II is the best case for cost productivity index because it is associated with positive cost frontier shift and an improvement in overall efficiency change. Case I is the worse case for cost Malmquist index productivity because it is associated with negative cost frontier shift and an decline in overall efficiency change. Case III is associated with a positive cost frontier shift but DMU_0 have regress in overall efficiency change, indicating that DMU_0 in period t is closer to the cost frontier in period $t+1$. Case IV is associated with negative cost frontier shift but DMU_0 have an improvement in overall efficiency change, indicating that DMU_0 in period t is not closer to the cost frontier in period $t+1$. Assume $CM_0 < 1$ Case V is most favorable in last four cases in Table 2. Because the cost Malmquist productivity gain is from not only an overall efficiency change improvement but also a cost frontier movement from negative shift to positive shift, indicate that DMU_0 has favorable strategy shift. Assume that $CM_0 > 1$ case VIII is the least favorable situation of the last four cases Because the cost Malmquist productivity loss is from not only an cost overall efficiency decline but also a cost frontier movement from positive shift to negative shift, indicate that DMU_0 has unfavorable strategy shift. Also, for case VI $CM_0 < 1$(or $CM_0 > 1$) indicates that cost Malmquist productivity gain is from the combined effects of average cost technology progress (or regress) and overall efficiency change decline. The fact that the cost technology moves from a negative shift facet to positive shift facet may indicate that DMU_0 has a favorable strategy change. For case VII $CM_0 < 1$ (or $CM_0 > 1$) indicate that cost Malmquist productivity gain (or loss) is due to the combined effects of average cost technology progress (or regress) and overall efficiency change improvement. The fact that the cost technology moves from a positive shift facet to negative shift facet may indicate that DMU_0 has a unfavorable strategy change.

4 Computation of index and its component

We can use DEA for compute CM index as follows: Let us have in each period $j = 1, ..., J$ production units. In period t, DMU_0 uses input x_0^t at price w_0^t for produce output y_0^t. For DMU_0 the cost denoted $w^t x^t$ in (12) is $w^t x^t = \sum_{n=1}^{N} w_{0n}^t x_{0n}^t$. Similarly $w^{t+1}x^{t+1}$, $w^{t+1}x^t$ and $w^t x^{t+1}$ are respectively $\sum_{n=1}^{N} w_{0n}^{t+1} x_{0n}^{t+1}$, $\sum_{n=1}^{N} w_{0n}^{t+1} x_{0n}^t$ and $\sum_{n=1}^{N} w_{0n}^t x_{0n}^{t+1}$.

Case	$\frac{w^t x^{t+1}/C^t(y^{t+1},x^t)}{w^{t+1}x^{t+1}/C^{t+1}(y^{t+1},x^{t+1})}$	$\frac{w^t x^t/C^t(y^t,x^t)}{w^{t+1}x^t/C^{t+1}(y^t,x^{t+1})}$	CTC_0	$\frac{w^{t+1}x^{t+1}/C^{t+1}(y^{t+1},x^{t+1})}{w^t x^t/C^t(y^t,x^t)}$	CM_0
I	>1	>1	>1	≥ 1	>1
II	<1	<1	<1	≤ 1	<1
III	<1	<1	<1	≥ 1	?
IV	>1	>1	<1	≥ 1	?
V	<1	>1	d	≤ 1	?
VI	<1	>1	d	≥ 1	?
VII	>1	<1	c	≤ 1	?
VIII	>1	<1	c	≥ 1	?

Table 2.

For DMU_0 the term$C^t(y^t, w^t)$ can be computed by following model:

$$\begin{aligned} C^t(y^t, w^t) = \min \quad & w_0^t x \\ s.t. \quad & \sum_{j=1}^{J} \lambda_j x_j^t \leq x, \\ & \sum_{j=1}^{J} \lambda_j y_j^t \geq y_0^t, \quad (14) \\ & \lambda_j \geq 0, \qquad j = 1, ..., J \\ & x \geq 0 \end{aligned}$$

The term $C^t(y^{t+1}, w^t)$ can be computed using following model:

$$\begin{aligned} C^t(y^{t+1}, w^t) = \min \quad & w_0^t x \\ s.t. \quad & \sum_{j=1}^{J} \lambda_j x_j^t \leq x, \\ & \sum_{j=1}^{J} \lambda_j y_j^t \geq y_0^{t+1}, \quad (15) \\ & \lambda_j \geq 0, \qquad j = 1, ..., J \\ & x \geq 0 \end{aligned}$$

The terms $C^{t+1}(y^{t+1}, w^{t+1})$ and $C^{t+1}(y^t, w^{t+1})$ can be computed using models (14) and (15) respectively, after changing round the time periods t and $t + 1$.

5 An example

In this section we illustrate the proposed approach using simple numerical example (see Thanassoulis and Manidakis (2004)). Suppose DMUs A, B, C, D and E use two inputs to produce a single output in level one. For simplicity it is assumed that input price in each period is the same. Assume that in period 1 each unit except E reduce their input levels by 30 percent. Unit E moves from E^0 in period 0 to B^1 in period 1 and the price of input 1 reduces by 33.33 percent. The data is graphed in Fig. 2. $A^0B^0C^0D^0$ is the boundary of production technology in period 0 and $A^1B^1C^1D^1$ is that in period 1. The cost boundary is represented by MC^0 in period 0 and by MC^1 in period 1. Unit E moves from E^0 in period 0 to E^1 in period 1 and its production cost shift from OC^0 to OC^1 in period 1. The data is given in Table 3 and equivalently in Fig. 2. Now we can compute the components of CM with respect to unit E as follows:

$\frac{w^0x^0}{C^0(y^0,w^0)} = 31.5/18 = 1.75,$

$\frac{w^1x^1}{C^1(y^1,w^1)} = 1,\ \frac{w^0x^1}{C^0(y^1,w^0)} = \frac{12.6}{18} = 0.7,\ \frac{w^1x^0}{C^1(y^0,w^1)} = \frac{24.00075}{10.50021} = 2.286$

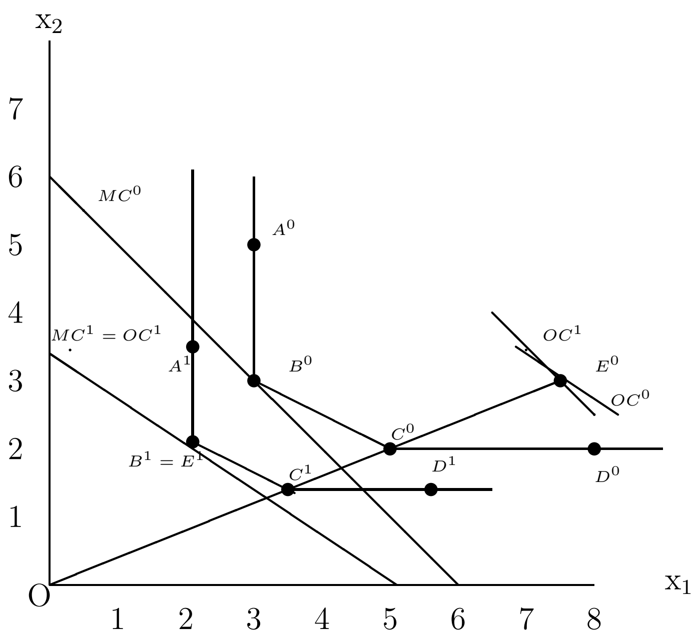

Fig. 2.

Unit	Input 1	Input 2	Output	Input 1 price	Input 2 price
A	3	5	1	3	3
B	3	3	1	3	3
C	5	2	1	3	3
D	8	2	1	3	3
E	7.5	3	1	3	3

Table 3. (Numerical example data for period 0)

We can see that unit E is in the case II (best case). Also unit E is cost efficient in period 1 and cost frontier has positive shift in period 0 and 1 thus CTC<1 indicate that improvement cost boundary shift is due to improvement cost boundary shift in each period. Also OEC<1 indicate that unit E has an improvement in overall efficiency. Thus CM(=0.418) is less than 1 indicate that unit E has cost Malmquist productivity gain and it is due to improvement in cost boundary shift and overall efficiency.

6 Conclusion

This paper analysis each individual component of cost Malmquist productivity index. The new Malmquist approach can identify the strategic shift of individual companies in a particular time period. Furthermore, we are able to make judgments on whether or not the strategic shift is favorable and promising.

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