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Some Criteria for Subsemigroups of a Symmetric Numerical Semigroup

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Abstract

In this paper, we give some criteria for subsemigroups of symmetric numerical semigroups which are generated by two elements.

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1. Introduction

Let $\mathbb{N} = \{0, 1, 2, \cdots n, \cdots\}$ and $S \subseteq \mathbb{N}$. S is called a numerical semigroup if S is subsemigroup of $(\mathbb{N}, +)$ with $0 \in S$. It is known that every numerical semigroup is finitely generated, i.e. there exist elements of S, say $n_0, n_1, \cdots, n_p$ such that $n_0 < n_1 < \cdots < n_p$ and

$$S = \langle n_0, n_1, \cdots, n_p \rangle = \{\sum_{i=0}^{p} k_i n_i \quad : k_i \in \mathbb{N}\}$$

and

$$G.C.D.(n_0, n_1, \cdots, n_p) = 1 \Leftrightarrow Card(\mathbb{N} \backslash S) < \infty$$

by [1]. For S numerical semigroup, we define the following:

$g(S) = max\{x \in \mathbb{Z} : x \notin S\}$ is called the Frobenius number of S, where $\mathbb{Z}$ is the integer set. Thus, S numerical semigroup is $S = \{0, n_0, n_1, \cdots, g(S) + 1 \rightarrow \cdots\}$ (The arrow "$\rightarrow$" means that every integer which is greater then $g(S) + 1$ belongs to S).

We say that a numerical semigroup is symmetric if for every $x \in \mathbb{Z}\backslash S$ we have $g(S) - x \in S$. If numerical semigroup S generated by a, b elements then, we know that S is symmetric and $g(S) = ab - a - b$ by [4].
The elements of $\mathbb{N}\backslash S$, denote by $H(S)$, are called gaps of S by [2]. Let S be a symmetric numerical semigroup. We say $S_1 \subset S$, is a symmetric numerical subsemigroup if S_1 is a symmetric numerical semigroup. If S is a symmetric numerical semigroup generated by a, b elements then, we know that $\sharp(H(S)) = \frac{(a-1)(b-1)}{2}$ by [3].

2. Main Results

In this section, we will give some criteria for subsemigroups of symmetric numerical semigroups which are generated by two elements.

Theorem 1. Let S be a symmetric numerical semigroup such that $S = \langle a, b\rangle$, if $S_1 = \langle b, c\rangle$ (or $S_1 = \langle c, b\rangle$) such that $a|b$ then, $S_1 \subset S$ where a, b, c are nonnegative integer numbers.
Proof. Let $x \in S_1$. Then, there exist $r, m \in \mathbb{N}$ such that $x = br + cm$. On the other hand,there exists $q \in \mathbb{N}$ such that $c = aq$ since $a|c$. Thus, we write $x = br + aqm \in \langle a, b\rangle = S$.

Example 2. If we get $S = \langle 2, 5\rangle = \{0, 2, 4, 5, 6, 7, \to \cdots\}$ and
$S_1 = \langle 5, 6\rangle = \{0, 5, 6, 10, 11, 12, 15, 16, 17, 18, 20, \to \cdots\}$ or

$S_1 = \langle 4, 5\rangle = \{0, 4, 5, 8, 9, 10, 12, 13, \to \cdots\}$ then, we write $S_1 \subset S$ from Theorem 1.

Theorem 3. Let S be a symmetric numerical semigroup such that $S = \langle a, b\rangle$, if $S_1 = \langle a, c\rangle$ such that $(b, c) = d \geq 1$ and $c = b + ad$, then, $S_1 \subset S$ where a, b, c are nonnegative integer numbers.
Proof. If $x \in S_1 = \langle a, c\rangle$ then, $x = ap + cr, p, r \in \mathbb{N}$. Thus, we obtain that $x = ap + bk$ since $c = b + ad$ and $(b, c) = d \geq 1$. Therefore, we find that $x \in \langle a, b\rangle = S$.

Example 4. If we get $S = \langle 2, 9\rangle = \{0, 2, 4, 6, 8, 9, 10, \to \cdots\}$ and
$S_1 = \langle 2, 15\rangle = \{0, 2, 4, 6, 8, 10, 12, 14, 15, 16, \to \cdots\}$ then, we write $S_1 \subset S$ from Theorem 3.

Theorem 5. Let S be a symmetric numerical semigroup such that $S = \langle a, b\rangle$, if $S_1 = \langle c, d\rangle$ such that $a|c$ and $b|d$, then, $S_1 \subset S$ where a, b, c, d are nonnegative integer numbers.
Proof. If $x \in S_1 = \langle c, d\rangle$ then, $x = cp + dm, p, m \in \mathbb{N}$. Thus, we have $x = at + bk$ since $a|c$ and $b|d$, where $k, t \in \mathbb{N}$. Finally,we obtain $x \in \langle a, b\rangle = S$.

Example 6. If we get $S = \langle 4, 5\rangle = \{0, 4, 5, 8, 9, 10, 12, \rightarrow \cdots\}$ and $S_1 = \langle 8, 15\rangle = \{0, 8, 15, 16, 23, 24, 29, 30, 31, 32, \rightarrow \cdots\}$, then, we write $S_1 \subset S$ from Theorem 5.

Theorem 7. Let S be a symmetric numerical semigroup such that $S = \langle a, b\rangle$, if $S_1 = \langle c, d\rangle$ such that $c = a+b$ and $b|d$ (or $d = a+b$ and $a|c$), Then, $S_1 \subset S$ where a, b, c, d are nonnegative integer numbers.

Proof. If $x \in S_1 = \langle c, d\rangle$ then, $x = cp + dm, p, m \in \mathbb{N}$. Thus, we have $x = (a + b)p + (br)m$ since $c = a + b$ and $b|d$. Hence, $x = at + bk$ where $k, t \in \mathbb{N}$. Finally,we obtain $x \in \langle a, b\rangle = S$. The same operations can be also made for $d = a + b$ and $a|c$.

Example 8. If we get $S = \langle 3, 4\rangle = \{0, 3, 4, 6, 7, 8, 9, \rightarrow \cdots\}$ and $S_1 = \langle 7, 8\rangle = \{0, 7, 8, 14, 15, 16, 21, 22, 23, 24, 28, 29, \rightarrow \cdots\}$, then we write $S_1 \subset S$ from Theorem 7.

Theorem 9. Let S_1 and S_2 be two subsemigroups of S symmetric numerical semigroup. If $S_1 \subset S_2$, then, $H(S_2) \subset H(S_1)$.

Proof. If $x \in H(S_2)$ then, $x \in \mathbb{N}/S_2$, namely $x \in {S_2}^c$. Thus, we find that $x \in H(S_1)$.

Theorem 10. Let S_1 and S_2 be two subsemigroups of S symmetric numerical semigroup. If $S_1 \subset S_2$, then, $g(S_1) \geq g(S_2)$.

Proof. It is trivial, by definition of Frobenius number and Theorem 9.

Example 11. Let S be symmetric numerical semigroup such that $S = \langle 2, 3\rangle = \{0, 2, 3, 4, 5, 6, \rightarrow \cdots\}$. If $S_1 = \langle 2, 5\rangle = \{0, 2, 4, 5, 6, 7, \rightarrow \cdots\}$ and $S_2 = \langle 4, 5\rangle = \{0, 4, 5, 8, 9, 10, 12, 13, \rightarrow \cdots\}$ are subsemigroups of S, then we find that $H(S_1) = \{1, 3\}$, $H(S_2) = \{1, 3, 6, 7, 11\}$, $g(S_1) = 3$ and $g(S_2) = 11$.

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