Neighborhoods of a Certain Class of Analytic Functions with Negative Coefficients

H. Özlem Güney and S. Sümer Eker

University of Dicle, Faculty of Science and Arts
Department of Mathematics, 21280, Diyarbakır, Turkey
ozlemg@dicle.edu.tr & sevtaps@dicle.edu.tr

Abstract

By making use of the familiar concept of neighborhoods of analytic functions, we prove several inclusion relations associated with the \((n, \delta)\)-neighborhoods of various subclass of univalent functions with negative coefficients.

Mathematics Subject Classification: 30C45

Keywords: analytic function, negative coefficient, \((n, \delta)\)-neighborhood

1. Introduction

Let \(A(n)\) denote the class of functions \(f(z)\) of the form

\[
f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \quad (a_k \geq 0 \, , \, n \in \mathbb{N} := \{1, 2, 3, \ldots \})
\]  

which are analytic in the open unit disk \(U = \{z \in \mathbb{C} : |z| < 1\} \).

For any \(f(z) \in A(n)\) and \(\delta \geq 0\) we define

\[
N_{n,\delta}(f) = \{g \in A(n) : g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k, \sum_{k=n+1}^{\infty} k|a_k - b_k| \leq \delta\}
\]  

which was called \((n, \delta)\)-neighborhoods of \(f(z)\). So, for \(e(z) = z\), we see that

\[
N_{n,\delta}(e) = \{g \in A(n) : g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k, \sum_{k=n+1}^{\infty} k|b_k| \leq \delta\}.
\]
The concept of neighborhoods was firstly by A.W. Goodman [1] and then generalized by ST. Ruscheweyh [2].

The main object of the present paper is to investigate the neighborhoods of the following subclass of class $A(n)$ of univalent functions with negative coefficients.

A function $f(z) \in A(n)$ is said to be in the class $P_{\gamma}(n, \lambda, \alpha, r)$ if it satisfies

$$
Re \left\{ \frac{2^r \lambda r f''(z) z^{2^r - 1} + \lambda f''(z) z^{2^r} + (\lambda - \gamma) r f'(z) z^{2^r - 1} + (\lambda - \gamma) f''(z) z^r + (1 - \lambda + \gamma) f'(z) z^{r - 1} + (1 - \lambda + \gamma) f(z)}{\gamma z^{2^r} f''(z) + (\lambda - \gamma) f'(z) z^r + (1 - \lambda + \gamma) f(z)} \right\} > \alpha
$$

for some $r = 1, 2, \ldots$, $\alpha (0 \leq \alpha < 1), \lambda, \gamma (0 \leq \gamma \leq \lambda \leq 1)$ and for all $z \in U$ [7]. We note that $P_0(n, \lambda, \alpha, r) \equiv P(n, \lambda, \alpha r)$. The classes $P(n, \lambda, \alpha r)$ and $P(n, \lambda, \alpha r, r)$ were studied by O. Altintas [3] and M. Kamali et al. [4], respectively. $P_{\gamma}(n, \lambda, \alpha, r)$ was introduced by M. Kamali et al. [5]. The class $P_{\gamma}(n, \lambda, \alpha, r)$ is generalization of $P(n, \lambda, \alpha r)$ by O. Altintas [3].

2. A Set of Inclusion Relations Involving $N_{n,\delta}(e)$

In our investigation of the inclusion relations involving $N_{n,\delta}(e)$, we shall require the following Lemma which was proved in [7].

**Lemma** Let $f(z) \in A(n)$ is in the class $P_{\gamma}(n, \lambda, \alpha, r)$ if and only if

$$
\sum_{k=n+1}^{\infty} \{(k - \alpha)[(k - 1)(\lambda \gamma k + \lambda - \gamma) + 1]k(r - 1)[2 \lambda \gamma (k - 1) + \lambda - \gamma] \} a_k \leq (1 - \alpha) + (\lambda - \gamma)(r - 1).
$$

(2.1)

Our first inclusion relation involving $N_{n,\delta}(e)$ is given by the following:

**Theorem 1** Let

$$
\delta_1 = \frac{(n+1)(1-\alpha)}{(n+1-\alpha)[n(\lambda \gamma (n+1) + \lambda - \gamma) + 1]}
$$

and

$$
\delta_2 = \frac{\{(n+1)[n(\lambda \gamma (n+1) + \lambda - \gamma) + 1] + (n+1)(r-1)[2 \lambda \gamma n + \lambda - \gamma]\}}{\{(n+1-\alpha)[n(\lambda \gamma (n+1) + \lambda - \gamma) + 1] + (n+1)(r-1)[2 \lambda \gamma n + \lambda - \gamma]\}} \times
$$

$$
\frac{[(1 - \alpha) + (\lambda - \gamma)(r - 1)]}{\{(r-1)[2 \lambda \gamma n + \lambda - \gamma] + [n(\lambda \gamma (n+1) + \lambda - \gamma) + 1]\}}.
$$

then

$$
P_{\gamma}(n, \lambda, \alpha, 1) \subset N_{n,\delta_1}(e).
$$
and
\[ P_\gamma(n, \lambda, \alpha, r) \subset N_{n, \delta_2}(e) \text{ for } r = 2, 3, \ldots, \]

respectively.

**Proof** For \( f \in P_\gamma(n, \lambda, \alpha, 1) \), Lemma is,
\[
\sum_{k=n+1}^{\infty} (k-\alpha)((k-1)(\lambda \gamma k + \lambda - \gamma) + 1)a_k \leq (1 - \alpha). \tag{2.2}
\]

Therefore, Lemma immediately yields,
\[
(n + 1 - \alpha)[n(\lambda \gamma(n + 1) + \lambda - \gamma) + 1] \sum_{k=n+1}^{\infty} a_k \leq (1 - \alpha).
\]

so that
\[
\sum_{k=n+1}^{\infty} a_k \leq \frac{1 - \alpha}{(n + 1 - \alpha)[n(\lambda \gamma(n + 1) + \lambda - \gamma) + 1]}. \tag{2.3}
\]

On the other hand, we also find (2.2) and (2.3) that
\[
\sum_{k=n+1}^{\infty} k[(k-1)(\lambda \gamma k + \lambda - \gamma) + 1]a_k + \sum_{k=n+1}^{\infty} (-\alpha)[(k-1)(\lambda \gamma k + \lambda - \gamma) + 1]a_k
\]
\[
\leq (1 - \alpha)[n(\lambda \gamma(n + 1) + \lambda - \gamma) + 1] \sum_{k=n+1}^{\infty} ka_k
\]
\[
\leq (1 - \alpha) + \alpha[n(\lambda \gamma(n + 1) + \lambda - \gamma) + 1] \sum_{k=n+1}^{\infty} a_k
\]
\[
\sum_{k=n+1}^{\infty} ka_k \leq \frac{(n + 1)(1 - \alpha)}{(n + 1 - \alpha)[n(\lambda \gamma(n + 1) + \lambda - \gamma) + 1]} = \delta_1
\]

and for \( f \in P_\gamma(n, \lambda, \alpha, r) \) Lemma immediately yields
\[
\{(n + 1 - \alpha)[n(\lambda \gamma(n + 1) + \lambda - \gamma) + 1] + (n + 1)(r - 1)(2\lambda \gamma n + \lambda - \gamma)\} \sum_{k=n+1}^{\infty} a_k
\]
\[
\leq (1 - \alpha) + (\lambda - \gamma)(r - 1)
\]
so that
\[
\sum_{k=n+1}^{\infty} a_k \leq \frac{(1 - \alpha) + (\lambda - \gamma)(r - 1)}{\{(n + 1 - \alpha)[n(\lambda \gamma(n + 1) + \lambda - \gamma) + 1] + (n + 1)(r - 1)[2\lambda \gamma n + \lambda - \gamma]\}}.
\]

On the other hand, we also find (2.1) and (2.4) that
\[
(r - 1)[2\lambda \gamma n + \lambda - \gamma] + [n(\lambda \gamma(n + 1) + \lambda - \gamma) + 1] \sum_{k=n+1}^{\infty} ka_k
\]
\[
\leq (1 - \alpha) + (\lambda - \gamma)(r - 1) + \alpha[n(\lambda \gamma(n + 1) + \lambda - \gamma) + 1] \sum_{k=n+1}^{\infty} a_k
\]
\[
\leq (1 - \alpha) + (\lambda - \gamma)(r - 1)
\]
\[
+ \frac{\alpha[n(\lambda \gamma(n + 1) + \lambda - \gamma) + 1][(1 - \alpha) + (\lambda - \gamma)(r - 1)]}{\{(n + 1 - \alpha)[n(\lambda \gamma(n + 1) + \lambda - \gamma) + 1] + (n + 1)(r - 1)[2\lambda \gamma n + \lambda - \gamma]\}}
\]
that is
\[
\sum_{k=n+1}^{\infty} ka_k \leq \frac{\{(n + 1)[n(\lambda \gamma(n + 1) + \lambda - \gamma) + 1] + (n + 1)(r - 1)[2\lambda \gamma n + \lambda - \gamma]\}}{\{(n + 1 - \alpha)[n(\lambda \gamma(n + 1) + \lambda - \gamma) + 1] + (n + 1)(r - 1)[2\lambda \gamma n + \lambda - \gamma]\}} \times
\]
\[
\frac{[(1 - \alpha) + (\lambda - \gamma)(r - 1)]}{\{(r - 1)[2\lambda \gamma n + \lambda - \gamma] + [n(\lambda \gamma(n + 1) + \lambda - \gamma) + 1]\}}.
\]

3. Neighborhoods for the class \(P^{(\beta)}(n, \lambda, \alpha, r)\)

In this section, we determine the neighborhoods for the class \(P^{(\beta)}(n, \lambda, \alpha, r)\) which we define as follows. A function \(f(z) \in A(n)\) is said to be in the class \(P^{(\beta)}(n, \lambda, \alpha, r)\) if there exists a function \(g \in P_{\gamma}(n, \lambda, \alpha, r)\) such that
\[
\left| \frac{f(z)}{g(z)} - 1 \right| < 1 - \beta
\]
for $\beta(0 \leq \beta < 1)$ and $z \in U$.

**Theorem 2** If $g \in P_\gamma(n, \lambda, \alpha, r)$ and

$$
\beta = 1 - \frac{n(n + 2r - 1 - \alpha)[\lambda \gamma(n + 1) + (\lambda - \gamma)] + (\lambda - \gamma)(r - 1)(1 - \alpha)}{n(n + 1)[\lambda \gamma(n + 1)(n + 2r - 1 - \alpha) + (n + r - \alpha)(\lambda - \gamma) + 1]},
$$

then

$$
N_{n, \delta}(g) \subset P_\gamma^{(\beta)}(n, \lambda, \alpha, r).
$$

**Proof** Suppose that $f \in N_{n, \delta}(g)$. Then we find from (1.2) that

$$
\sum_{k=n+1}^{\infty} k|a_k - b_k| \leq \delta
$$

which readily implies the coefficients inequality

$$
\sum_{k=n+1}^{\infty} |a_k - b_k| \leq \frac{\delta}{n + 1}, \quad n \in \mathbb{N}.
$$

Next, since $g \in P_\gamma(n, \lambda, \alpha, r)$ we have from (2.2)

$$
\sum_{k=n+1}^{\infty} b_k
$$

$$
\leq \frac{(1 - \alpha) + (\lambda - \gamma)(r - 1)}{(n + 1 - \alpha)[n(\lambda \gamma(n + 1) + (\lambda - \gamma)) + (n + 1)(r - 1)(2\lambda \gamma n + \lambda - \gamma)]},
$$

so that

$$
\left| \frac{f(z)}{g(z)} - 1 \right| < \frac{\sum_{k=n+1}^{\infty} |a_k - b_k|}{1 - \sum_{k=n+1}^{\infty} b_k}
$$

$$
\leq \frac{\delta}{n + 1} \frac{n(n + 2r - 1 - \alpha)[\lambda \gamma(n + 1) + (\lambda - \gamma)] + (\lambda - \gamma)(r - 1)(1 - \alpha)}{n[\lambda \gamma(n + 1)(n + 2r - 1 - \alpha) + (n + r - \alpha)(\lambda - \gamma) + 1]} = 1 - \beta
$$

provided that $\beta$ is given precisely by (3.2). Thus, by definition of $P_\gamma^{(\beta)}(n, \lambda, \alpha, r)$, $f \in P_\gamma^{(\beta)}(n, \lambda, \alpha, r)$ for $\beta$ given by (3.2), which evidently completes our proof of Theorem 2.
References


Received: June 29, 2006