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Neighborhoods of a Certain Class of Analytic Functions with Negative Coefficients

H. Özlem Güney and S. Sümer Eker

University of Dicle, Faculty of Science and Arts
Department of Mathematics, 21280, Diyarbakır, Turkey
ozlemg@dicle.edu.tr & sevtaps@dicle.edu.tr

Abstract

By making use of the familiar concept of neighborhoods of analytic functions, we prove several inclusion relations associated with the (n, δ)-neighborhoods of various subclass of univalent functions with negative coefficients.

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1. Introduction

Let $\mathcal{A}(n)$ denote the class of functions $f(z)$ of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \quad (a_k \geq 0 \ , n \in \mathbb{N} := \{1, 2, 3, \dots\}) \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

For any $f(z) \in \mathcal{A}(n)$ and $\delta \geq 0$ we define

$$\mathcal{N}_{n,\delta}(f) = \{g \in \mathcal{A}(n) : g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k, \sum_{k=n+1}^{\infty} k|a_k - b_k| \leq \delta\} \tag{1.2}$$

which was called (n, δ)-neighborhoods of $f(z)$. So, for $e(z) = z$,we see that

$$\mathcal{N}_{n,\delta}(e) = \{g \in \mathcal{A}(n) : g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k, \sum_{k=n+1}^{\infty} k|b_k| \leq \delta\}. \tag{1.3}$$

The concept of neighborhoods was firstly by A.W.Goodman[1] and then generalized by ST.Ruscheweyh [2].

The main object of the present paper is to investigate the neighborhoods of the following subclass of class $\mathcal{A}(n)$ of univalent functions with negative coefficients.

A function $f(z) \in \mathcal{A}(n)$ is said to be in the class $\mathcal{P}_\gamma(n, \lambda, \alpha, r)$ if it satisfies

$$Re\left\{ z\, \frac{2\gamma\lambda r f''(z) z^{2r-1} + \lambda\gamma f'''(z) z^{2r} + (\lambda-\gamma) r f'(z) z^{r-1} + (\lambda-\gamma) f''(z) z^r + (1-\lambda+\gamma) f'(z)}{\gamma\lambda z^{2r} f''(z) + (\lambda-\gamma) f'(z) z^r + (1-\lambda+\gamma) f(z)} \right\} > \alpha$$

for some $r = 1, 2, ...,$ $\alpha (0 \leq \alpha < 1), \lambda, \gamma (0 \leq \gamma \leq \lambda \leq 1)$ and for all $z \in U$ [7]. We note that $\mathcal{P}_0(n, \lambda, \alpha, r) \equiv \mathcal{P}(n, \lambda, \alpha r)$. The classes $\mathcal{P}(n, \lambda, \alpha r)$ and $\mathcal{P}(n, \lambda, \alpha r, r)$ were studied by O.Altintas [3] and M.Kamali et al. [4], respectively. $\mathcal{P}_\gamma(n, \lambda, \alpha, r)$ was introduced by M.Kamali et al.[5]. The class $\mathcal{P}_\gamma(n, \lambda, \alpha, r)$ is generalization of $\mathcal{P}(n, \lambda, \alpha r)$ by O.Altintas [3].

2. A Set of Inclusion Relations Involving $\mathcal{N}_{n,\delta}(e)$

In our investigation of the inclusion relations involving $\mathcal{N}_{n,\delta}(e)$, we shall require the following Lemma which was proved in [7].

Lemma Let$f(z) \in \mathcal{A}(n)$ is in the class $\mathcal{P}_\gamma(n, \lambda, \alpha, r)$ if and only if

$$\sum_{k=n+1}^{\infty} \{(k-\alpha)[(k-1)(\lambda\gamma k + \lambda - \gamma) + 1] k (r-1)[2\lambda\gamma(k-1) + \lambda - \gamma]\}\, a_k$$
$$\leqq (1-\alpha) + (\lambda-\gamma)(r-1). \tag{2.1}$$

Our first inclusion relation involving $\mathcal{N}_{n,\delta}(e)$ is given by the following:

Theorem 1 Let

$$\delta_1 = \frac{(n+1)(1-\alpha)}{(n+1-\alpha)[n(\lambda\gamma(n+1) + \lambda - \gamma) + 1]}$$

and

$$\delta_2 = \frac{\{(n+1)[n(\lambda\gamma(n+1) + \lambda - \gamma) + 1] + (n+1)(r-1)[2\lambda\gamma n + \lambda - \gamma]\}}{\{(n+1-\alpha)[n(\lambda\gamma(n+1) + \lambda - \gamma) + 1] + (n+1)(r-1)[2\lambda\gamma n + \lambda - \gamma]\}} \times$$

$$\frac{[(1-\alpha) + (\lambda-\gamma)(r-1)]}{\{(r-1)[2\lambda\gamma n + \lambda - \gamma] + [n(\lambda\gamma(n+1) + \lambda - \gamma) + 1]\}}.$$

then

$$\mathcal{P}_\gamma(n, \lambda, \alpha, 1) \subset \mathcal{N}_{n,\delta_1}(e).$$

and

$$\mathcal{P}_{\gamma}(n,\lambda,\alpha,r) \subset \mathcal{N}_{n,\delta_2}(e) \ \ for\ r = 2, 3, ...,$$

respectively.

Proof For $f \in \mathcal{P}_{\gamma}(n,\lambda,\alpha,1)$, Lemma is,

$$\sum_{k=n+1}^{\infty} (k-\alpha)[(k-1)(\lambda\gamma k + \lambda - \gamma) + 1]a_k \leqq (1-\alpha). \tag{2.2}$$

Therefore, Lemma immediately yields,

$$(n+1-\alpha)[n(\lambda\gamma(n+1) + \lambda - \gamma) + 1] \sum_{k=n+1}^{\infty} a_k \leqq (1-\alpha).$$

so that

$$\sum_{k=n+1}^{\infty} a_k \leqq \frac{1-\alpha}{(n+1-\alpha)[n(\lambda\gamma(n+1) + \lambda - \gamma) + 1]}. \tag{2.3}$$

On the other hand, we also find (2.2) and (2.3) that

$$\sum_{k=n+1}^{\infty} k[(k-1)(\lambda\gamma k + \lambda - \gamma) + 1]a_k + \sum_{k=n+1}^{\infty} (-\alpha)[(k-1)(\lambda\gamma k + \lambda - \gamma) + 1]a_k$$

$$\leqq (1-\alpha)[n(\lambda\gamma(n+1) + \lambda - \gamma) + 1] \sum_{k=n+1}^{\infty} ka_k$$

$$\leqq (1-\alpha) + \alpha[n(\lambda\gamma(n+1) + \lambda - \gamma) + 1] \sum_{k=n+1}^{\infty} a_k$$

$$\sum_{k=n+1}^{\infty} ka_k \leqq \frac{(n+1)(1-\alpha)}{(n+1-\alpha)[n(\lambda\gamma(n+1) + \lambda - \gamma) + 1]} = \delta_1$$

and for $f \in \mathcal{P}_{\gamma}(n,\lambda,\alpha,r)$ Lemma immediately yields

$$\{(n+1-\alpha)[n(\lambda\gamma(n+1) + \lambda - \gamma) + 1] + (n+1)(r-1)[2\lambda\gamma n + \lambda - \gamma]\} \sum_{k=n+1}^{\infty} a_k$$

$$\leqq (1-\alpha)+(\lambda-\gamma)(r-1)$$

so that

$$\sum_{k=n+1}^{\infty} a_k \leqq$$

$$\frac{(1-\alpha)+(\lambda-\gamma)(r-1)}{\{(n+1-\alpha)[n(\lambda\gamma(n+1)+\lambda-\gamma)+1]+(n+1)(r-1)[2\lambda\gamma n+\lambda-\gamma]\}}. \tag{2.4}$$

On the other hand, we also find (2.1) and (2.4) that

$$(r-1)[2\lambda\gamma n+\lambda-\gamma]+[n(\lambda\gamma(n+1)+\lambda-\gamma)+1]\sum_{k=n+1}^{\infty} k a_k$$

$$\leqq (1-\alpha)+(\lambda-\gamma)(r-1)+\alpha[n(\lambda\gamma(n+1)+\lambda-\gamma)+1]\sum_{k=n+1}^{\infty} a_k$$

$$\leqq (1-\alpha)+(\lambda-\gamma)(r-1)$$

$$+\frac{\alpha[n(\lambda\gamma(n+1)+\lambda-\gamma)+1][(1-\alpha)+(\lambda-\gamma)(r-1)]}{\{(n+1-\alpha)[n(\lambda\gamma(n+1)+\lambda-\gamma)+1]+(n+1)(r-1)[2\lambda\gamma n+\lambda-\gamma]\}}$$

that is

$$\sum_{k=n+1}^{\infty} k a_k \leq \frac{\{(n+1)[n(\lambda\gamma(n+1)+\lambda-\gamma)+1]+(n+1)(r-1)[2\lambda\gamma n+\lambda-\gamma]\}}{\{(n+1-\alpha)[n(\lambda\gamma(n+1)+\lambda-\gamma)+1]+(n+1)(r-1)[2\lambda\gamma n+\lambda-\gamma]\}}\times$$

$$\frac{[(1-\alpha)+(\lambda-\gamma)(r-1)]}{\{(r-1)[2\lambda\gamma n+\lambda-\gamma]+[n(\lambda\gamma(n+1)+\lambda-\gamma)+1]\}}.$$

3. Neighborhoods for the class $\mathcal{P}_{\gamma}^{(\beta)}(n,\lambda,\alpha,r)$

In this section , we determine the neighborhoods for the class $\mathcal{P}_{\gamma}^{(\beta)}(n,\lambda,\alpha,r)$ which we define as follows. A function $f(z)\in\mathcal{A}(n)$ is said to be in the class $\mathcal{P}_{\gamma}^{(\beta)}(n,\lambda,\alpha,r)$ if there exists a function $g\in\mathcal{P}_{\gamma}(n,\lambda,\alpha,r)$ such that

$$\left|\frac{f(z)}{g(z)}-1\right|<1-\beta \tag{3.1}$$

for $\beta(0 \leq \beta < 1)$ and $z \epsilon U$.

Theorem 2 If $g \in \mathcal{P}_\gamma(n, \lambda, \alpha, r)$ and

$$\beta = 1 - \frac{\{n(n+2r-1-\alpha)[\lambda\gamma(n+1) + (\lambda-\gamma)] + (\lambda-\gamma)(r-1)(1-\alpha)\}\delta}{n(n+1)\{\lambda\gamma(n+1)(n+2r-1-\alpha) + (n+r-\alpha)(\lambda-\gamma) + 1\}}, \tag{3.2}$$

then

$$\mathcal{N}_{n,\delta}(g) \subset \mathcal{P}_\gamma^{(\beta)}(n, \lambda, \alpha, r).$$

Proof Suppose that $f \in \mathcal{N}_{n,\delta}(g)$. Then we find from (1.2) that

$$\sum_{k=n+1}^{\infty} k|a_k - b_k| \leq \delta$$

which readily implies the coefficients inequality

$$\sum_{k=n+1}^{\infty} |a_k - b_k| \leq \frac{\delta}{n+1},\ n \in \mathbb{N}.$$

Next, since $g \in \mathcal{P}_\gamma(n, \lambda, \alpha, r)$ we have from (2.2)

$$\sum_{k=n+1}^{\infty} b_k$$

$$\leqq \frac{(1-\alpha) + (\lambda-\gamma)(r-1)}{\{(n+1-\alpha)[n(\lambda\gamma(n+1) + \lambda - \gamma) + 1] + (n+1)(r-1)[2\lambda\gamma n + \lambda - \gamma]\}}.$$

so that

$$\left|\frac{f(z)}{g(z)} - 1\right| < \frac{\sum_{k=n+1}^{\infty} |a_k - b_k|}{1 - \sum_{k=n+1}^{\infty} b_k}$$

$$\leq \frac{\delta}{n+1} \frac{n(n+2r-1-\alpha)[\lambda\gamma(n+1) + (\lambda-\gamma)] + (\lambda-\gamma)(r-1)(1-\alpha)}{n\{\lambda\gamma(n+1)(n+2r-1-\alpha) + (n+r-\alpha)(\lambda-\gamma) + 1\}} = 1 - \beta$$

provided that β is given precisely by (3.2). Thus, by definition of $\mathcal{P}_\gamma^{(\beta)}(n, \lambda, \alpha, r)$, $f \in \mathcal{P}_\gamma^{(\beta)}(n, \lambda, \alpha, r)$ for β given by (3.2),which evidently completes our proof of Theorem 2.

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