Sakaguchi - Type Harmonic Univalent Functions with Negative Coefficients

H. Özlem Güney

University of Dicle, Faculty of Science and Art
Department of Mathematics, 21280 Diyarbakır, Turkey
ozlemg@dicle.edu.tr

Abstract

In [1], a sufficient condition had been given for the class $SH(\alpha)$. We show that these coefficient conditions are also necessary when $h$ has negative and $g$ has positive coefficients. These lead to distortion bounds.

Mathematics Subject Classification: 30C45, 30C50, 30C55

Keywords: harmonic univalent function, Sakaguchi function, negative coefficient

1. Introduction

Let $\mathbb{U} = \{ z : |z| < 1 \}$ denote the open unit disc and $S_H$ denote the class of all complex-valued, harmonic and sense-preserving univalent functions $f$ in $\mathbb{U}$ normalized by $f(0) = f_z(0) - 1 = 0$. Each $f \in S_H$ can be expressed as $f = h + \overline{g}$, where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n$$

(1)

are analytic in $\mathbb{U}$. A necessary and sufficient condition for $f$ to be locally univalent and sense-preserving in $\mathbb{U}$ is that $|h'(z)| > |g'(z)|$ in $\mathbb{U}$. Clunie and Sheil-Small [2] studied $S_H$ together with some geometric subclasses of $S_H$. Observe that $S_H$ reduces to $S$, the class of normalized univalent analytic functions, if the co-analytic part of $f$ is zero.

On the other hand, Sakaguchi [3] introduced the class $S_S$ of analytic univalent functions in $\mathbb{U}$ which are starlike with respect to symmetrical points. An
analytic function $f$ is said to be starlike with respect to symmetrical points if and only if

$$\Re \frac{2zf'(z)}{f(z) - f(-z)} > 0. \quad (2)$$

Extending the definition (2) to include the harmonic functions in [1], Ahuja and Jahangiri denoted the class $\text{SH}(\alpha)$ of complex-valued, sense-preserving, harmonic starlike functions $f$ of the form (1) which satisfy the condition

$$\text{Im} \left\{ \frac{2\frac{\partial}{\partial \theta} f(re^{i\theta})}{f(re^{i\theta}) - f(-re^{i\theta})} \right\} \ge \alpha, \quad (3)$$

where $z = re^{i\theta}$, $0 \le r < 1$ and $0 \le \theta < 2\pi$.

We further denote by $\mathcal{FH}(\alpha)$ the subclasses of $\text{SH}(\alpha)$ such that the functions $h$ and $g$ in $f = h + \bar{g}$ are of the form

$$h(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n \quad a_n, b_n \ge 0 \quad (4)$$

In this paper, we provide that coefficient condition which had proved sufficient in [1] is necessary when $h$ has negative and $g$ has positive coefficients. Furthermore we give distortion bounds for functions in this class.

2. Main Results

Firstly, we shall need the following lemma which include a sufficient condition for harmonic functions in $\text{SH}(\alpha)$ in [1].

**Lemma** For $h$ and $g$ as in (1), let the harmonic function $f = h + \bar{g}$ satisfy

$$\sum_{n=1}^{\infty} \left\{ \frac{2(n-1)}{1-\alpha}(|a_{2n-2}| + |b_{2n-2}|) + \frac{2n-1-\alpha}{1-\alpha}|a_{2n-1}| + \frac{2n-1+\alpha}{1-\alpha}|b_{2n-1}| \right\} \le 2 \quad (5)$$

where $a_1 = 1$ and $0 \le \alpha < 1$. Then $f$ is sense-preserving harmonic univalent in $U$, and $f \in \text{SH}(\alpha)$.

**Theorem 1** Let $f = h + \bar{g}$ be given by (4). Then $f \in \mathcal{FH}(\alpha)$ if and only if

$$\sum_{n=1}^{\infty} \left\{ \frac{2(n-1)}{1-\alpha}(|a_{2n-2}| + |b_{2n-2}|) + \frac{2n-1-\alpha}{1-\alpha}|a_{2n-1}| + \frac{2n-1+\alpha}{1-\alpha}|b_{2n-1}| \right\} \le 2 \quad (6)$$
where \( a_1 = 1 \) and \( 0 \leq \alpha < 1 \).

Proof. The if part follows from Lemma upon noting that if the analytic and co-analytic parts of \( f = h + \bar{g} \in SH(\alpha) \) are of the form (4) then \( f = h + \bar{g} \in \mathcal{FH}(\alpha) \). For the only part, we show that \( f \notin \mathcal{FH}(\alpha) \) if the condition (6) does not hold. Note that a necessary and sufficient condition for \( f = h + \bar{g} \) given by (4) to be starlike of order \( \alpha, 0 \leq \alpha < 1 \), is that \( \Re \left\{ \frac{2(zh'(z) - zg'(z))}{(h(z) - h(-z)) + (g(z) - g(-z))} \right\} - \alpha \geq 0, \ 0 \leq \alpha < 1 \). This is equivalent to

\[
0 \leq \Re \left\{ \frac{2(1 - \alpha)z - \sum_{n=2}^{\infty} (2n - \alpha - (-1)^n \alpha) a_n z^n - \sum_{n=1}^{\infty} (2n + \alpha - (-1)^n \alpha) b_n \bar{z}^n}{2z - \sum_{n=2}^{\infty} 2a_{2n-1} z^{2n-1} + \sum_{n=1}^{\infty} 2b_{2n-1} \bar{z}^{2n-1}} \right\}
\]

\[
= \Re \left\{ \frac{2(1 - \alpha)z - \sum_{n=2}^{\infty} 2(2n - 2) a_{2n-2} z^{2n-2} + (2(2n - 1) - 2\alpha) a_{2n-1} z^{2n-1}}{2z - \sum_{n=2}^{\infty} 2a_{2n-1} z^{2n-1} + \sum_{n=1}^{\infty} 2b_{2n-1} \bar{z}^{2n-1}} \right\}
\]

\[
\leq \left| \frac{(1 - \alpha)z - \sum_{n=2}^{\infty} 2(n - 1) a_{2n-2} z^{2n-2} + (2n - 1 - \alpha) a_{2n-1} z^{2n-1}}{z - \sum_{n=2}^{\infty} a_{2n-1} z^{2n-1} + \sum_{n=1}^{\infty} b_{2n-1} \bar{z}^{2n-1}} \right|
\]

If the condition (6) does not hold then

\[
(1 - \alpha) - \sum_{n=2}^{\infty} 2(n - 1) a_{2n-2} + (2n - 1 - \alpha) a_{2n-1} - \sum_{n=1}^{\infty} 2(n - 1) b_{2n-2} + (2n - 1 + \alpha) b_{2n-1} \leq 0
\]

that is,

\[
(1 - \alpha) - \sum_{n=2}^{\infty} 2(n - 1) a_{2n-2} + (2n - 1 - \alpha) a_{2n-1} - \sum_{n=1}^{\infty} 2(n - 1) b_{2n-2} + (2n - 1 + \alpha) b_{2n-1} \leq 0
\]

Hence \( f \notin \mathcal{FH}(\alpha) \) and so the proof is complete.
Theorem 2. If \( f \in \mathcal{FH}(\alpha) \) then

\[
|f(z)| \leq (1 + b_1)r + \left( \frac{1 - \alpha}{2} - \frac{1 + \alpha}{2}|b_1| \right) r^2, \quad |z| = r < 1,
\]

and

\[
|f(z)| \geq (1 - b_1)r - \left( \frac{1 - \alpha}{2} - \frac{1 + \alpha}{2}|b_1| \right) r^2, \quad |z| = r < 1.
\]

Proof. Let \( f \in \mathcal{FH}(\alpha) \). Taking the absolute value of \( f(z) \) we obtain

\[
|f(z)| \leq (1 + b_1)r + \sum_{n=2}^{\infty} (a_{2n-2} + b_{2n-2}) r^{2n-2} + \sum_{n=2}^{\infty} (a_{2n-1} + b_{2n-1}) r^{2n-1} \]

\[
\leq (1 + b_1)r + r^2 \sum_{n=2}^{\infty} (a_{2n-2} + b_{2n-2}) + \sum_{n=2}^{\infty} (a_{2n-1} + b_{2n-1}) r
\]

\[
= (1 + b_1)r + \frac{1 - \alpha}{2} r^2 \sum_{n=2}^{\infty} \frac{2}{1 - \alpha} (a_{2n-2} + b_{2n-2}) + \sum_{n=2}^{\infty} (a_{2n-1} + b_{2n-1}) r
\]

\[
\leq (1 + b_1)r + \frac{1 - \alpha}{2} r^2 \sum_{n=2}^{\infty} \frac{2(n - 1)}{1 - \alpha} (a_{2n-2} + b_{2n-2})
\]

\[
+ r^2 \sum_{n=2}^{\infty} \frac{2n - 1 - \alpha}{1 - \alpha} a_{2n-1} + \frac{2n - 1 + \alpha}{1 - \alpha} b_{2n-1}
\]

\[
\leq (1 + b_1)r + \frac{1 - \alpha}{2} r^2 \left( 1 - \frac{1 + \alpha}{1 - \alpha} b_1 \right), \quad \text{by (6)}
\]

\[
= (1 + b_1)r + \left( \frac{1 - \alpha}{2} - \frac{1 + \alpha}{2} b_1 \right) r^2,
\]

and

\[
|f(z)| \geq (1 - b_1)r - \sum_{n=2}^{\infty} (a_{2n-2} + b_{2n-2}) r^{2n-2} + \sum_{n=2}^{\infty} (a_{2n-1} + b_{2n-1}) r^{2n-1}
\]
\[ \geq (1 - b_1)r - r^2 \sum_{n=2}^{\infty} (a_{2n-2} + b_{2n-2}) + r^2 \sum_{n=2}^{\infty} (a_{2n-1} + b_{2n-1})r \]

\[ = (1 - b_1)r - \frac{1 - \alpha}{2} r^2 \sum_{n=2}^{\infty} \frac{2}{1 - \alpha} (a_{2n-2} + b_{2n-2}) + r^2 \sum_{n=2}^{\infty} (a_{2n-1} + b_{2n-1})r \]

\[ \geq (1 - b_1)r - \frac{1 - \alpha}{2} r^2 \sum_{n=2}^{\infty} \frac{2(n-1)}{1 - \alpha} (a_{2n-2} + b_{2n-2}) \]

\[ + r^2 \sum_{n=2}^{\infty} \frac{2n - 1 - \alpha}{1 - \alpha} a_{2n-1} + \frac{2n - 1 + \alpha}{1 - \alpha} b_{2n-1} \]

\[ \geq (1 - b_1)r - \frac{1 - \alpha}{2} r^2 \left( 1 - \frac{1 + \alpha}{1 - \alpha} b_1 \right), \text{ by (6)} \]

\[ = (1 - b_1)r - \left( \frac{1 - \alpha}{2} - \frac{1 + \alpha}{2} b_1 \right) r^2, \]

The following covering result follows from the left hand inequality in Theorem 2.

**Corollary** If \( f \in \mathcal{FH}(\alpha) \) then

\[ \left\{ w : |w| < \frac{1 + \alpha - (3 + \alpha)b_1}{2} \right\} \subset f(\mathbb{U}). \]

**References**


**Received: April 26, 2006**