Int. J. Contemp. Math. Sci., Vol. 2, 2007, no. 10, 459 - 463

Sakaguchi - Type Harmonic Univalent Functions with Negative Coefficients

H. Özlem Güney

University of Dicle, Faculty of Science and Art
Department of Mathematics,21280 Diyarbakır, Turkey
ozlemg@dicle.edu.tr

Abstract

In [1], a sufficient condition had been given for the class $SH(\alpha)$. We show that these coefficient conditions are also necessary when h has negative and g has positive coefficients. These lead to distortion bounds.

Mathematics Subject Classification: 30C45, 30C50, 30C55
Keywords: harmonic univalent function, Sakaguchi function, negative coefficient

1. Introduction

Let $\mathbb{U} = \{z : |z| < 1\}$ denote the open unit disc and S_H denote the class of all complex-valued, harmonic and sense-preserving univalent functions f in $\mathbb{U}$ normalized by $f(0) = f_z(0) - 1 = 0$. Each $f \in S_H$ can be expressed as $f = h + \overline{g}$, where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n \tag{1}$$

are analytic in $\mathbb{U}$.A necessary and sufficient condition for f to be locally univalent and sense-preserving in $\mathbb{U}$ is that $|h'(z)| > |g'(z)|$ in $\mathbb{U}$. Clunie and Sheil-Small [2] studied S_H together with some geometric subclasses of S_H. Observe that S_H reduces to S, the class of normalized univalent analytic functions, if the co-analytic part of f is zero.

On the other hand, Sakaguchi [3] introduced the class S_S of analytic univalent functions in $\mathbb{U}$ which are starlike with respect to symmetrical points. An analytic function f is said to be starlike with respect to symmetrical points if and only if

$$\mathfrak{Re}\frac{2zf'(z)}{f(z)-f(-z)} > 0. \tag{2}$$

Extending the definition (2) to include the harmonic functions in [1], Ahuja and Jahangiri denoted the class $SH(\alpha)$ of complex-valued, sense-preserving, harmonic starlike functions f of the form (1) which satisfy the condition

$$\mathfrak{Im}\left\{\frac{2\frac{\partial}{\partial\theta}f(re^{i\theta})}{f(re^{i\theta})-f(-re^{i\theta})}\right\} \geq \alpha, \tag{3}$$

where $z = re^{i\theta}$, $0 \leq r < 1$ and $0 \leq \theta < 2\pi$.

We further denote by $\mathcal{FH}(\alpha)$ of the subclasses of $SH(\alpha)$ such that the functions h and g in $f = h + \overline{g}$ are of the form

$$h(z) = z - \sum_{n=2}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n \qquad a_n, b_n \geq 0 \tag{4}$$

In this paper, we provide that coefficient condition which had proved sufficient in [1] is necessary when h has negative and g has positive coefficients. Furthermore we give distortion bounds for functions in this class.

2. Main Results

Firstly, we shall need the following lemma which include a sufficient condition for harmonic functions in $SH(\alpha)$ in [1].

Lemma For h and g as in (1), let the harmonic function $f = h + \bar{g}$ satisfy

$$\sum_{n=1}^{\infty}\left\{\frac{2(n-1)}{1-\alpha}(|a_{2n-2}|+|b_{2n-2}|)+\frac{2n-1-\alpha}{1-\alpha}|a_{2n-1}|+\frac{2n-1+\alpha}{1-\alpha}|b_{2n-1}|\right\} \leq 2 \tag{5}$$

where $a_1 = 1$ and $0 \leq \alpha < 1$. Then f is sense-preserving harmonic univalent in $\mathbb{U}$, and $f \in SH(\alpha)$.

Theorem 1 Let $f = h + \bar{g}$ be given by (4). Then $f \in \mathcal{FH}(\alpha)$ if and only if

$$\sum_{n=1}^{\infty}\left\{\frac{2(n-1)}{1-\alpha}(|a_{2n-2}|+|b_{2n-2}|)+\frac{2n-1-\alpha}{1-\alpha}|a_{2n-1}|+\frac{2n-1+\alpha}{1-\alpha}|b_{2n-1}|\right\} \leq 2 \tag{6}$$

where $a_1 = 1$ and $0 \le \alpha < 1$.

Proof . The if part follows from Lemma upon noting that if the analytic and co-analytic parts of $f = h+\bar{g} \in SH(\alpha)$ are of the form (4) then $f = h+\bar{g} \in \mathcal{FH}(\alpha)$. For the only part, we show that $f \notin \mathcal{FH}(\alpha)$ if the condition (6) does not hold. Note that a necessary and sufficient condition for $f = h + \bar{g}$ given by (4) to be starlike of order α, $0 \le \alpha < 1$, is that $\mathfrak{Re}\left\{\frac{2\frac{\partial}{\partial\theta}f(re^{i\theta})}{f(re^{i\theta})-f(-re^{i\theta})}\right\} - \alpha \ge 0, \quad 0 \le \alpha < 1$. This is equivalent to

$$0 \le \mathfrak{Re}\left\{\frac{2(zh'(z) - \overline{zg'(z)})}{(h(z) - h(-z)) + \overline{(g(z) - g(-z))}}\right\} - \alpha$$

$$= \mathfrak{Re}\left\{\frac{2(1-\alpha)z - \sum_{n=2}^{\infty}(2n-\alpha+(-1)^n\alpha))a_n z^n - \sum_{n=1}^{\infty}(2n+\alpha-(-1)^n\alpha))b_n\overline{z}^n}{2z - \sum_{n=2}^{\infty}(1-(-1)^n)a_n z^n + \sum_{n=1}^{\infty}(1-(-1)^n)b_n\overline{z}^n}\right\}$$

$$= \mathfrak{Re}\{\frac{2(1-\alpha)z - \sum_{n=2}^{\infty} 2(2n-2)a_{2n-2}z^{2n-2} + (2(2n-1)-2\alpha)a_{2n-1}z^{2n-1}}{2z - \sum_{n=2}^{\infty} 2a_{2n-1}z^{2n-1} + \sum_{n=1}^{\infty} 2b_{2n-1}\overline{z}^{2n-1}}$$

$$-\frac{\sum_{n=1}^{\infty}(2(2n-2)b_{2n-2}\overline{z}^{2n-2} + (2(2n-1)+2\alpha))b_{2n-1}\overline{z}^{2n-1}}{2z - \sum_{n=2}^{\infty} 2a_{2n-1}z^{2n-1} + \sum_{n=1}^{\infty} 2b_{2n-1}\overline{z}^{2n-1}}\}$$

$$\le |\frac{(1-\alpha)z - \sum_{n=2}^{\infty} 2(n-1)a_{2n-2}z^{2n-2} + (2n-1-\alpha)a_{2n-1}z^{2n-1}}{z - \sum_{n=2}^{\infty} a_{2n-1}z^{2n-1} + \sum_{n=1}^{\infty} b_{2n-1}z^{2n-1}}$$

$$-\frac{\sum_{n=1}^{\infty} 2(n-1)b_{2n-2}z^{2n-2} + (2n-1+\alpha)b_{2n-1}z^{2n-1}}{z - \sum_{n=2}^{\infty} a_{2n-1}z^{2n-1} + \sum_{n=1}^{\infty} b_{2n-1}z^{2n-1}}|.$$

If the condition (6) does not hold then

$$\le \frac{(1-\alpha) - \sum_{n=2}^{\infty} 2(n-1)a_{2n-2} + (2n-1-\alpha)a_{2n-1} - \sum_{n=1}^{\infty} 2(n-1)b_{2n-2} + (2n-1+\alpha)b_{2n-1}}{1 + \sum_{n=2}^{\infty} a_{2n-1} + \sum_{n=1}^{\infty} b_{2n-1}} \le 0$$

that is,

$$(1-\alpha) - \sum_{n=2}^{\infty} 2(n-1)a_{2n-2} + (2n-1-\alpha)a_{2n-1} - \sum_{n=1}^{\infty} 2(n-1)b_{2n-2} + (2n-1+\alpha)b_{2n-1} \le 0$$

Hence $f \notin \mathcal{FH}(\alpha)$ and so the proof is complete.

Theorem 2 . If $f \in \mathcal{FH}(\alpha)$ then

$$|f(z)| \leq (1+b_1)r + \left(\frac{1-\alpha}{2} - \frac{1+\alpha}{2}|b_1|\right) r^2, \qquad |z| = r < 1,$$

and

$$|f(z)| \geq (1-b_1)r - \left(\frac{1-\alpha}{2} - \frac{1+\alpha}{2}|b_1|\right) r^2, \qquad |z| = r < 1.$$

Proof . Let $f \in \mathcal{FH}(\alpha)$. Taking the absolute value of $f(z)$ we obtain

$$|f(z)| \leq (1+b_1)r + \sum_{n=2}^{\infty}(a_{2n-2}+b_{2n-2})r^{2n-2} + \sum_{n=2}^{\infty}(a_{2n-1}+b_{2n-1})r^{2n-1}$$

$$\leq (1+b_1)r + r^2\sum_{n=2}^{\infty}(a_{2n-2}+b_{2n-2}) + r^2\sum_{n=2}^{\infty}(a_{2n-1}+b_{2n-1})r$$

$$= (1+b_1)r + \frac{1-\alpha}{2}r^2\sum_{n=2}^{\infty}(\frac{2}{1-\alpha})(a_{2n-2}+b_{2n-2}) + r^2\sum_{n=2}^{\infty}(a_{2n-1}+b_{2n-1})r$$

$$\leq (1+b_1)r + \frac{1-\alpha}{2}r^2\sum_{n=2}^{\infty}\frac{2(n-1)}{1-\alpha}(a_{2n-2}+b_{2n-2})$$

$$+r^2\sum_{n=2}^{\infty}\frac{2n-1-\alpha}{1-\alpha}a_{2n-1} + \frac{2n-1+\alpha}{1-\alpha}b_{2n-1}$$

$$\leq (1+b_1)r + \frac{1-\alpha}{2}r^2\left(1 - \frac{1+\alpha}{1-\alpha}b_1\right), \quad by (6)$$

$$= (1+b_1)r + \left(\frac{1-\alpha}{2} - \frac{1+\alpha}{2}b_1\right) r^2,$$

and

$$|f(z)| \geq (1-b_1)r - \sum_{n=2}^{\infty}(a_{2n-2}+b_{2n-2})r^{2n-2} + \sum_{n=2}^{\infty}(a_{2n-1}+b_{2n-1})r^{2n-1}$$

$$\geq (1-b_1)r - r^2\sum_{n=2}^{\infty}(a_{2n-2}+b_{2n-2}) + r^2\sum_{n=2}^{\infty}(a_{2n-1}+b_{2n-1})r$$

$$= (1-b_1)r - \frac{1-\alpha}{2}r^2\sum_{n=2}^{\infty}(\frac{2}{1-\alpha})(a_{2n-2}+b_{2n-2}) + r^2\sum_{n=2}^{\infty}(a_{2n-1}+b_{2n-1})r$$

$$\geq (1-b_1)r - \frac{1-\alpha}{2}r^2\sum_{n=2}^{\infty}\frac{2(n-1)}{1-\alpha}(a_{2n-2}+b_{2n-2})$$

$$+r^2\sum_{n=2}^{\infty}\frac{2n-1-\alpha}{1-\alpha}a_{2n-1} + \frac{2n-1+\alpha}{1-\alpha}b_{2n-1}$$

$$\geq (1-b_1)r - \frac{1-\alpha}{2}r^2\left(1-\frac{1+\alpha}{1-\alpha}b_1\right), \quad by(6)$$

$$= (1-b_1)r - \left(\frac{1-\alpha}{2} - \frac{1+\alpha}{2}b_1\right)r^2,$$

The following covering result follows from the left hand inequality in Theorem 2.

Corollary *If* $f \in \mathcal{FH}(\alpha)$ *then*

$$\left\{w : |w| < \frac{1+\alpha-(3+\alpha)b_1}{2}\right\} \subset f(\mathbb{U}).$$

References

[1] Om P. Ahuja and J.M. Jahangiri, Sakaguchi-typeharmonic univalent functions,Scientiae Math.Japonica 59, No. 1(2004), 239-244.

[2] J.Clunie and T.Sheil-Small, Harmonic univalent functions, ann.Acad.Sci.Fenn.Ser.A I Hath. 9(1984), 3-25.

[3] K.Sakaguchi, On a certain univalent mapping, J.Math.Soc.Japan. 2(1)(1959), 72-75.

Received: April 26, 2006