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Sectional Curvature of CR-submanifolds of Maximal CR-dimension

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Abstract

In this paper we study n-dimensional compact CR-submanifolds of $(n-1)$ CR-dimension immersed in a complex projective space $CP^{(n+p)/2}$. Especially we provide necessary conditions in order for such a submanifold to be a geodesic hypersphere in $CP^{(n+p)/2}$ in terms with sectional curvature.

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1 Introduction

Let M be an n-dimensional CR-submanifold of maximal CR-dimension isometrically immersed in a complex space form $M^{(n+p)/2}(c)$. Denoting by $(J, \overline{g})$

the Kählerian structure of $M^{(n+p)/2}(c)$, it follows by definition (cf. [2,5,6,9,12]) that the dimension of the maximal J-invariant subspace

$$\mathcal{D}_x := T_xM \cap JT_xM$$

of the tangent space T_xM of M is $(n-1)$ at any point $x \in M$. So there exists a unit vector field U_1 tangent to M such that

$$\mathcal{D}_x^{\perp} = Span\{U_1\}, \quad {}^{\forall}x \in M,$$

where $\mathcal{D}_x^{\perp}$ denotes the subspace of T_xM complementary orthogonal to $\mathcal{D}_x$. Moreover, the vector field ξ defined by

$$\xi := JU_1 \tag{1.1}$$

is normal to M and satisfies

$$JTM \subset TM \oplus Span\{\xi\}.$$

Hence we have, for any tangent vector field X and for a local orthonormal basis $\{\xi_\alpha \,;\, \alpha = 1, \ldots, p\}$ $(\xi_1 := \xi)$ of normal vectors to M, the following decomposition in tangential and normal components :

$$JX = FX + u^1(X)\xi_1, \tag{1.2}$$

$$J\xi_\alpha = -U_\alpha + P\xi_\alpha, \quad \alpha = 1, \ldots, p. \tag{1.3}$$

Since the structure $(J, \overline{g})$ is hermitian and $J^2 = -I$, we can easily see from (1.2) and (1.3) that F and P are skew-symmetric linear endomorphisms acting on T_xM and $T_x^{\perp}M$, respectively, and that

$$g(FU_\alpha, X) = -u^1(X)\overline{g}(\xi_1, P\xi_\alpha), \tag{1.4}$$

$$g(U_\alpha, U_\beta) = \delta_{\alpha\beta} - \overline{g}(P\xi_\alpha, P\xi_\beta), \tag{1.5}$$

where $T_x^{\perp}M$ denotes the normal space of M at x and g the metric on M induced from $\overline{g}$. Furthermore, we also have

$$g(U_\alpha, X) = u^1(X)\delta_{1\alpha} \tag{1.6}$$

and consequently

$$g(U_1, X) = u^1(X), \quad U_\alpha = 0, \quad \alpha = 2, \ldots, p. \tag{1.7}$$

Next, applying J to (1.2) and using (1.3) and (1.7), we have

$$F^2X = -X + u^1(X)U_1, \quad u^1(X)P\xi_1 = -u^1(FX)\xi_1, \tag{1.8}$$

from which, taking account of the skew-symmetry of P and (1.4),

$$u^1(FX) = 0, \quad FU_1 = 0, \quad u^1(U_1) = 1, \quad P\xi_1 = 0. \tag{1.9}$$

Thus (1.3) may be written in the form

$$J\xi_1 = -U_1, \quad J\xi_\alpha = P\xi_\alpha, \quad \alpha = 2, \ldots, p. \tag{1.10}$$

Those equations tell us that (F, g, U_1, u^1) defines an almost contact metric structure on M (cf. [2,5,6,8,10,12]). Recently Okumura and Vanhecke [12] studied CR-submanifolds of maximal CR-dimension admitting normal almost contact metric structure when $M^{(n+p)/2}(c)$ is a complex projective space $CP^{(n+p)/2}$ and proved the following theorem as an improvement of a theorem provided by Okumura [10, Theorem 4.4, p.363] :

Theorem O-V. *Let M be an n-dimensional CR-submanifold of $(n-1)$ CR-dimension isometrically immersed in $CP^{(n+p)/2}$ and let the distinguished normal vector field ξ is parallel with respect to the normal connection. If the almost contact metric structure of M is normal, or equivalently if $FA_1 = A_1F$, then M is locally isometric to $M^C_{n_1,n_2}(r_1, r_2)$, where A_1 denotes the shape operator in direction of ξ and*

$$M^C_{n_1,n_2}(r_1, r_2) := \pi(S^{2n_1+1}(r_1) \times S^{2n_2+1}(r_2))$$

for some portion (n_1, n_2) of $(n-1)/2$ and some r_1, r_2 such that $r_1^2 + r_2^2 = 1$ (π is the Hopf fibration : $S^{n+p+1}(1) \to CP^{(n+p)/2}$).

On the other hand, in his paper [4], Kon provided

Theorem K. *Let M be a compact, minimal real hypersurface of $CP^{(n+1)/2}$. If the minimum of sectional curvatures of M is $1/n$, then M is the geodesic minimal hypersphere $M^C_{0,(n-1)/2} := M^C_{0,(n-1)/2}(\sqrt{\frac{1}{n+1}}, \sqrt{\frac{n}{n+1}})$.*

In this paper we shall study CR-submanifolds of $(n-1)$ CR-dimension isometrically immersed in $CP^{(n+p)/2}$ and improve Theorem K by using Theorem O-V.

2 Preliminaries

We first let M be as in §1 and use the same notations as shown in that section. We denote by $\overline{\nabla}$ and ∇ the Levi-Civita connection on $M^{(n+p)/2}(c)$ and M, respectively. Then Gauss and Weingarten equations are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.1}$$

$$\overline{\nabla}_X \xi_\alpha = -A_\alpha X + \nabla^\perp_X \xi_\alpha, \quad \alpha = 1, \ldots, p \tag{2.2}$$

for any vector fields X, Y tangent to M. Here $\nabla^\perp$ denotes the normal connection induced from $\overline{\nabla}$ on the normal bundle $TM^\perp$ of M, and h and A_α the second fundamental form and the shape operator in direction of ξ_α, respectively. It is clear that h and A_α are related by

$$h(X, Y) = \sum_{\alpha=1}^{p} g(A_\alpha X, Y)\xi_\alpha.$$

Especially, we put

$$\nabla^\perp_X \xi_\alpha = \sum_{\beta=1}^{p} s_{\alpha\beta}(X)\xi_\beta. \tag{2.3}$$

Then $(s_{\alpha\beta})$ is the skew-symmetric matrix of connection forms of $\nabla^\perp$.

Now, by using (2.1)-(2.3) and taking account of the Kähler condition $\overline{\nabla} J = 0$, we differentiate (1.2) and (1.3) covariantly and compare the tangential and normal parts. Then we can easily find that

$$(\nabla_X F)Y = u^1(Y)A_1 X - g(A_1 Y, X)U_1, \tag{2.4}$$

$$(\nabla_X u^1)Y = g(FA_1 X, Y), \tag{2.5}$$

$$\nabla_X U_1 = FA_1 X, \tag{2.6}$$

$$g(A_\alpha U_1, X) = -\sum_{\beta=2}^{p} s_{1\beta}(X)P_{\beta\alpha}, \quad \alpha = 2, \ldots, p \tag{2.7}$$

for any X, Y tangent to M, where we have put $P\xi_\alpha = \sum_{\beta=2}^{p} P_{\alpha\beta}\xi_\beta$ for $2 \leq \alpha \leq p$.

In the rest of this paper *we assume that the distinguished normal vector field* $\xi_1 := \xi$ *is parallel with respect to the normal connection* $\nabla^\perp$. Then (2.3) gives

$$s_{1\alpha} = 0, \quad \alpha = 2, \ldots, p, \tag{2.8}$$

which together with (2.7) yields

$$A_\alpha U_1 = 0, \quad \alpha = 2, \ldots, p. \tag{2.9}$$

On the other hand the ambient manifold $M^{(n+p)/2}(c)$ is of constant holomorphic sectional curvature c and consequently its Riemannian curvature tensor $\overline{R}$ satisfies

$$\begin{aligned}\overline{R}(\overline{X},\overline{Y})\overline{Z} = \frac{c}{4}\{\overline{g}(\overline{Y},\overline{Z})\overline{X} - \overline{g}(\overline{X},\overline{Z})\overline{Y} + \overline{g}(J\overline{Y},\overline{Z})J\overline{X}\\ - \overline{g}(J\overline{X},\overline{Z})J\overline{Y} - 2\overline{g}(J\overline{X},\overline{Y})J\overline{Z}\}\end{aligned}$$

for any $\overline{X},\overline{Y},\overline{Z}$ tangent to $M^{(n+p)/2}(c)$ (cf. [1,13]). So, equations of Gauss, Codazzi and Ricci imply

$$\begin{aligned}R(X,Y)Z =& \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y\\ &+ g(FY,Z)FX - g(FX,Z)FY - 2g(FX,Y)FZ\}\\ &+ \sum_{\alpha}\{g(A_\alpha Y,Z)A_\alpha X - g(A_\alpha X,Z)A_\alpha Y\},\end{aligned} \tag{2.10}$$

$$\begin{aligned}&(\nabla_X A_1)Y - (\nabla_Y A_1)X\\ &= \frac{c}{4}\{g(X,U_1)FY - g(Y,U_1)FX - 2g(FX,Y)U_1\},\end{aligned} \tag{2.11_a}$$

$$\begin{aligned}&(\nabla_X A_\alpha)Y - (\nabla_Y A_\alpha)X\\ &= \sum_{\beta=2}^{p}\{s_{\beta\alpha}(Y)A_\beta X - s_{\beta\alpha}(X)A_\beta Y\}, \quad \alpha = 2,\dots,p,\end{aligned} \tag{2.11_b}$$

$$[A_1, A_\alpha] = 0, \quad \alpha = 2,\dots,p \tag{2.12}$$

for any X, Y, Z tangent to M with the aid of (2.8), where R denotes the Riemannian curvature tensor of M with respect to g.

From now on we prepare some algebraic identities for later use. Putting

$$V := \nabla_{U_1}U_1 - (divU)U, \tag{2.13}$$

it follows from (1.8), (1.9) and (2.6) that

$$V = FA_1U_1, \tag{2.14}$$

$$g(U_1, V) = 0, \tag{2.15}$$

$$FV = -A_1U_1 + \lambda U_1, \tag{2.16}$$

$$g(V,V) = \mu - \alpha^2, \tag{2.17}$$

where here and in the sequel we put

$$\lambda = g(A_1U_1, U_1) = u^1(A_1U_1), \quad \mu = g(A_1^2U_1, U_1). \tag{2.18}$$

It follows easily from (2.15) and (2.16) that

$$g(A_1U_1, V) = 0. \tag{2.19}$$

Also (2.6), (2.15) and (2.16) imply

$$u^1(\nabla_X V) = g(\lambda A_1U_1 - A_1^2U_1, X). \tag{2.20}$$

Taking an orthonormal basis

$$\{e_i\}_{i=1,\ldots,n} = \{e_a, e_{a^*}, e_n = U_1\}_{a=1,\ldots,m} \quad (m := (n-1)/2)$$

of tangent vectors to M such that $e_{a^*} := Fe_a$, we can obtain that

$$\begin{aligned} divV = \sum_{i=1}^{n} g(\nabla_{e_i}V, e_i) =& \sum_{a=1}^{m} g((\nabla_{Fe_a}A_1)e_a - (\nabla_{e_a}A_1)Fe_a, U_1) \\ &+ \lambda(trA_1) - \mu - \sum_{a=1}^{m} g(A_1Fe_a, FA_1e_a) \end{aligned} \tag{2.21}$$

with the aid of (1.8), (2.14) and (2.18). Since (2.11) implies

$$\sum_{a=1}^{m} g((\nabla_{Fe_a}A_1)e_a - (\nabla_{e_a}A_1)Fe_a, U_1) = \frac{c}{4}(n-1),$$

(2.21) reduces to

$$divV = \lambda(trA_1) - \mu + \frac{c}{4}(n-1) - \sum_{a=1}^{m} g(A_1Fe_a, FA_1e_a). \tag{2.22}$$

On the other hand, (2.6) yields

$$(\mathcal{L}_{U_1}g)(X, Y) = g((FA_1 - A_1F)X, Y)$$

and consequently

$$\begin{aligned} \frac{1}{2}\|\mathcal{L}_{U_1}g\|^2 &= \frac{1}{2}\sum_{i=1}^{n} g((FA_1 - A_1F)(FA_1 - A_1F)^t e_i, e_i) \\ &= trA_1^2 - \mu - \sum_{a=1}^{m} g(A_1Fe_a, FA_1e_a). \end{aligned} \tag{2.23}$$

Hence (2.22) and (2.23) yield

$$\frac{1}{2}\|\mathcal{L}_{U_1}g\|^2 = divV - \{\lambda(trA_1) - trA_1^2 + \frac{c}{4}(n-1)\} \tag{2.24}$$

or equivalently

$$divV = \frac{1}{2}\|FA_1 - A_1F\|^2 + \lambda(trA_1) - trA_1^2 + \frac{c}{4}(n-1). \tag{2.24$'$}$$

Finally, we introduce several lemmas provided in [5] for later use.

Lemma 2.1. *Let M be an n-dimensional CR-submanifold of $(n-1)$ CR-dimension in a complex space form $\overline{M}^{(n+p)/2}(c)$ of constant holomorphic sectional curvature c and let the distinguished normal vector field ξ be parallel with respect to the normal connection. Then*

$$\|\nabla A_1\|^2 \geq \{(n-1)/8\}c^2.$$

Lemma 2.2. *Let M be as in Lemma 2.1 with $c \neq 0$. If $FA_1 = A_1F$, then $\|\nabla A_1\|^2 = \{(n-1)/8\}c^2$ and*

$$A_1^2X = \lambda A_1X + \frac{c}{4}\{X - u^1(X)U_1\}. \tag{2.25}$$

Moreover, in this case

$$A_1U_1 = \lambda U_1 \tag{2.26}$$

and the function $\lambda = u(A_1U)$ is locally constant.

Lemma 2.3. *Let M be as in Lemma 2.1 with $c \neq 0$. If $FA_1 = A_1F$, then*

$$FA_\alpha + A_\alpha F = 0, \quad trA_\alpha = 0, \quad \alpha = 2, \ldots, p. \tag{2.27}$$

Proof. Differentiating (2.9) covariantly and using (2.6), we have

$$(\nabla_X A_\alpha)U_1 + A_\alpha FA_1X = 0,$$

or equivalently

$$g((\nabla_X A_\alpha)Y, U_1) + g(A_\alpha FA_1X, Y) = 0 \tag{2.28}$$

for any vector fields X, Y tangent to M. By means of (2.9), (2.11_b), (2.12) and the assumption $FA_1 = A_1F$, it can be easily verified from (2.28) that

$$(A_\alpha F + FA_\alpha)A_1X = 0. \tag{2.29}$$

Inserting A_1X into (2.29) instead of X and using (2.9), (2.25) and (2.29) itself, we have the first equation of (2.27). The second equation of (2.27) can be easily followed from the fact that $A_\alpha = FA_\alpha F$, which is a direct consequence of (1.8), (2.9) and the first equation of (2.27). □

3 Proof of main theorems

In this section we assume that the ambient manifold is a complex projective space $CP^{(n+p)/2}$ of constant holomorphic sectional curvature 4. Suppose that the distinguished normal vector field ξ is parallel with respect to the normal connection and that the trace of the shape operator A_1 corresponding to ξ vanishes, that is,

$$trA_1 = 0. \tag{3.1}$$

Then, from (2.11_a) with $c = 4$ and $s_{1\beta} = 0$, we have

$$\sum(\nabla_i A_1)e_i = 0, \tag{3.2}$$

where $\{e_i\}_{i=1,\ldots,n}$ is the orthonormal basis given in §2 of tangent vectors to M and $\nabla_i := \nabla_{e_i}$.

Using (2.4)-(2.6) and (3.2), we can easily obtain

$$\sum(\nabla_i\nabla_i A_1)X = \sum(R(e_i, X)A_1)e_i - 3\{g(A_1U, X)U - FA_1FX\}$$

for any vector field X tangent to M. Hence we have

$$\begin{aligned} g(\nabla^2 A_1, A_1) = &\sum_{i,j} g((R(e_i, e_j)A_1)e_i, A_1e_j) \\ &- 3\{\mu - \sum g(A_1FA_1Fe_i, e_i)\}. \end{aligned} \tag{3.3}$$

Taking account of the Laplacian of trA_1^2, we have

$$\int_M \|\nabla A_1\|^2 * 1 = -\int_M g(\nabla^2 A_1, A_1) * 1. \tag{3.4}$$

On the other hand, it follows from Lemma 2.1 that

$$0 \le \int_M \{\|\nabla A_1\|^2 - 2(n-1) + \frac{1}{2}\|FA_1 - A_1F\|^2\} * 1,$$

Moreover, (3.3) and (3.4) yield

$$\begin{aligned} &\int_M \{\|\nabla A_1\|^2 - 2(n-1) + \frac{1}{2}\|FA_1 - A_1F\|^2\} * 1 \\ &= \int_M \{\frac{1}{2}\|FA_1 - A_1F\|^2 - 2(n-1) + 3(\mu - \sum g(A_1FA_1Fe_i, e_i)) \\ &\quad - \sum_{i,j} g((R(e_i, e_j)A_1)e_i, A_1e_j)\} * 1, \end{aligned}$$

from which together with (2.23) and (2.24′) with $c = 4$, we obtain

$$\begin{aligned} 0 &\leq \int_M \{\|\nabla A_1\|^2 - 2(n-1) + \frac{1}{2}\|FA_1 - A_1F\|^2\} * 1 \\ &= \int_M \{trA_1^2 - \sum_{i,j} g((R(e_i, e_j)A_1)e_i, A_1e_j)\} * 1. \end{aligned} \tag{3.5}$$

Now we choose an orthonormal frame $\{e_j\}$ of M such that

$$A_1e_j = \lambda_j e_j \quad (j = 1, \dots, n).$$

Then it is clear that

$$\begin{aligned} \sum_{i,j} g((R(e_i, e_j)A_1)e_i, A_1e_j) &= \sum_{i,j}\{g((R(e_i, e_j)A_1e_i, A_1e_j) - g(A_1R(e_i, e_j)e_i, A_1e_j)\} \\ &= \frac{1}{2}\sum_{i,j}(\lambda_i - \lambda_j)^2 K_{ij}, \end{aligned}$$

where K_{ij} denotes the sectional curvature of the plane section spanned by $\{e_i, e_j\}$. Hence, if the minimum of the sectional curvature of M is $1/n$, the above equation implies

$$\sum_{i,j} g((R(e_i, e_j)A_1)e_i, A_1e_j) \quad \geq \quad \frac{1}{2n}\sum_{i,j}(\lambda_i - \lambda_j)^2 \quad = \quad trA_1^2. \tag{3.6}$$

Comparing (3.5) with (3.6), we have

$$FA_1 = A_1F.$$

Combining those results and Lemma 2.3, we have

Theorem 3.1. *Let M be an n-dimensional compact CR-submanifold of $(n-1)$ CR-dimension in $CP^{(n+p)/2}$ and let the distinguished normal vector field ξ be parallel with respect to the normal connection. If the trace of the shape operator A_1 in direction of ξ vanishes and if the minimum of sectional curvatures of M is $1/n$, then*

$$FA_1 = A_1F \tag{3.7}$$

at any point in M. Hence M is minimal and

$$FA_\alpha + A_\alpha F = 0, \quad \alpha = 2, \dots, p. \tag{3.8}$$

By means of Theorem 3.1 we can obtain the following theorem under additional condition :

Theorem 3.2. *Let M be an n-dimensional compact CR-submanifold of $(n-1)$ CR-dimension in $CP^{(n+p)/2}$ and assume that there exists an orthonormal basis $\{\xi, \xi_\alpha\}_{\alpha=2,\dots,p}$ of normal vectors to M each of which is parallel with respect to the normal connection. If the trace of the shape operator A_1 in direction of ξ vanishes and if the minimum of sectional curvatures of M is $1/n$, then there is an $(n+1)$-dimensional totally geodesic complex projective space $CP^{(n+1)/2}$ of $CP^{(n+p)/2}$ such that $M \subset CP^{(n+1)/2}$.*

Proof. Under our assumptions it follows from Theorem 3.1 that

$$trA_\alpha = 0, \quad \alpha = 2, \dots, p.$$

Moreover, it is clear from (2.11_b) that, for any vector fields X, Y tangent to M,

$$(\nabla_X A_\alpha)Y - (\nabla_Y A_\alpha)X = 0$$

since $s_{\alpha\beta} = 0, \;\; 1 \le \alpha, \beta \le p.$, and consequently

$$\sum (\nabla_i A_\alpha) e_i = 0,$$

where $\{e_i\}_{i=1,\dots,n}$ is the orthonormal basis given in §2 of tangent vectors to M. Hence we have

$$\sum (\nabla_i \nabla_i A_\alpha) X = \sum (R(e_i, X) A_\alpha) e_i$$

for any vector field X tangent to M, and so

$$g(\nabla^2 A_\alpha, A_\alpha) = \sum_{i,j} g((R(e_i, e_j) A_\alpha e_i, A_\alpha e_j).$$

Taking account of the Laplacian of trA_α^2, we have

$$\int_M \|\nabla A_\alpha\|^2 * 1 = -\int_M g(\nabla^2 A_\alpha, A_\alpha) * 1,$$

which implies

$$0 \le \int_M \|\nabla A_\alpha\|^2 * 1 = -\int_M \sum_{i,j} g((R(e_i, e_j) A_\alpha) e_i, A_\alpha e_j) * 1. \tag{3.9}$$

As was shown in the proof of Theorem 3.1, we choose an orthonormal frame $\{e_j\}$ of M such that

$$A_\alpha e_j = \mu_j e_j \quad (j = 1, \dots, n).$$

Then we have

$$\sum_{i,j} g((R(e_i, e_j) A_\alpha) e_i, A_\alpha e_j) = \frac{1}{2} \sum_{i,j} (\mu_i - \mu_j)^2 K_{ij}.$$

Hence, if the minimum of sectional curvatures of M is $1/n$, the above equation and (3.9) yield

$$\nabla_X A_\alpha = 0, \quad \alpha = 2, \dots, p \tag{3.10}$$

for any vector field X tangent to M.

On the other hand, differentiating (3.8) covariantly and using (2.4), (2.9) and (3.10), we have

$$u^1(X)A_\alpha A_1 Y - g(A_1 A_\alpha X, Y)U_1 = 0$$

for any vector fields X, Y tangent to M. Putting $X = U$ in this equation and using (1.9) and (2.9), we have

$$A_\alpha A_1 Y = 0,$$

from which, inserting $A_1 Y$ instead of Y and taking account of (2.25) with $c = 4$,

$$A_\alpha = 0, \quad \alpha = 2, \dots, p.$$

Now we put $N_0(x) := \{\eta \in T_x^\perp M | A_\eta = 0\}$ and let $H_0(x)$ be the maximal J-invariant subspace of $N_0(x)$, that is, $H_0(x) = JN_0(x) \cap N_0(x)$. Then the orthogonal complement $H_1(x)$ of $H_0(x)$ in $T_x^\perp M$ is $\mathrm{Span}\{\xi\}$, which is invariant under parallel translation with respect to the normal connection under our assumption. Therefore we can apply Okumura's codimension reduction theorem provided in [11, Theorem 4.1, p. 579], which completes the proof of our theorem. $\square$

Combining Theorem 3.2 and Theorem O-V, we have

Theorem 3.3. *Let M be as in Theorem 3.2. If the trace of the shape operator A_1 in direction of ξ vanishes and if the minimum of sectional curvatures of M is $1/n$, then M is isometric to the geodesic minimal hypersphere $M^C_{0,(n-1)/2}$ of $CP^{(n+1)/2}$.*

Proof. By means of Theorem 3.1 and Theorem 3.2, M is a minimal real hypersurface in a complex projective space $CP^{(n+1)/2}$ and, moreover, satisfies (3.7). Hence Theorem O-V or [10, Theorem 4.4, p.363] yields that M is isometric to $M^C_{n_1,n_2}(\sqrt{\frac{2n_1+1}{n+1}}, \sqrt{\frac{2n_2+1}{n+1}})$ for some portion (n_1, n_2) of $(n-1)/2$. On the other hand, using (2.25) with $c = 4$ and (2.26), we can easily see that the minimum of sectional curvatures of $M^C_{0,(n-1)/2}$ is $1/n$ and that of $M^C_{n_1,n_2}(\sqrt{\frac{2n_1+1}{n+1}}, \sqrt{\frac{2n_2+1}{n+1}})$ $(n_1, n_2 \geq 1)$ is zero, which completes the proof of our theorem. $\square$

From Theorem 3.3, we immediately have

Corollary 3.4. ([4]) *Let* M *be a compact, minimal real hypersurface of a complex projective space* $CP^{(n+1)/2}$. *If the minimum of sectional curvatures of* M *is* $1/n$. *Then* M *is isometric to the geodesic minimal hypersphere* $M^C_{0,(n-1)/2}$.

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