Int. J. Contemp. Math. Sciences, Vol. 2, 2007, no. 9, 423 - 429

Sobolev-type Inequality for Spaces $L^{p(x)}(\mathbb{R}^N)$

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Abstract

Our aim in this paper is to prove Sobolev type inequality for measurable functions by using serial expansion for the variable exponent Lebesgue spaces and the corresponding Sobolev spaces. We also determine pre-limit exponent for measurable (may be discontinuous) functions in Sobolev type inequality.

Mathematics Subject Classification: 46E35, 26D10

Keywords: Variable exponent, Sobolev-type inequality

1. Introduction

The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ and the corresponding Sobolev space $W^{1,p(x)}(\Omega)$ have been a subject of active research stimulated by development of the studies of problems in elasticity, fluid dynamics, calculus of variations, and differential equations with $p(x)-$growth over the last decades [7]. We refer to [6,8] for fundamental properties of these spaces, to [1,2,3,4,5] for Sobolev type inequalities.

O.Kováčik and J.Rákosník [6] obtained Sobolev-type inequality

$$\| u \|_{q(x),\Omega} \leq C \| \nabla u \|_{p(x),\Omega} \quad ; \quad u \in C_0^\infty(\Omega) \tag{1}$$

for continuous $p(x)$ function, where a positive constant C is independent of u and

$$1 \leq q(x) \leq \frac{Np(x)}{N-p(x)} - \varepsilon = p^*(x) - \varepsilon \quad , \quad 0 < \varepsilon < \frac{1}{N-1},$$

and they also showed that, in general, the Sobolev space $W^{1,p(x)}(\Omega)$ is not embedding in $L^{p^*(x)}(\Omega)$.

Later D.E.Edmunds and J.Rákosník [3,4] proved $W^{1,p(x)}(\Omega) \to L^{p^*(x)}(\Omega)$ for bounded lipschitz domain, where $p(x) \in C^{0,1}(\overline{\Omega})$ and $1 \leq p(x) \leq q < N$.

X.L.Fan, J.Shen and D.Zhao [5] investigated Sobolev-type embedding for $W^{k,p(x)}(\Omega)$, where Ω is an open domain in $\mathbb{R}^N$ with cone property, and $p(x)$ is a Lipschitz continuous function defined on satisfying $1 < p^- \leq p^+ < \frac{N}{k}$.

Recently, L.Diening [1] has obtained embedding above for unbounded domain assuming that $p(x)$ is constant at infinity and satisfies (w-Lip.)

$$|p(x) - p(y)| \leqslant \frac{C}{-\log|x-y|} \quad ; \quad x, y \in \mathbb{R}^N, \ \ |x-y| \leqslant \frac{1}{2}.$$

2. Definitions and Basic Facts

Let $p : \mathbb{R}^N \to [1, \infty)$ be a measurable function called the variable exponent on $\mathbb{R}^N (N \geq 2)$. We write $p^+ := \sup\limits_{x\in\mathbb{R}^N} p(x)$ and $p^- := \inf\limits_{x\in\mathbb{R}^N} p(x)$. We define the variable exponent Lebesgue space $L^{p(x)}(\mathbb{R}^N)$ to consist of measurable functions $f : \mathbb{R}^N \to \mathbb{R}$ such that the modular $I_p(f) := \int\limits_{\mathbb{R}^N} |f(x)|^{p(x)}\, dx$ is finite. If $p^+ < \infty$, then $\| f \|_{p(x)} = \inf\left\{\lambda > 0 : I_p\left(\frac{f}{\lambda}\right) \leq 1\right\}$, defines a norm on $L^{p(x)}(\mathbb{R}^N)$. This makes $L^{p(x)}(\mathbb{R}^N)$ a Banach space. Moreover, if $\| f \|_{p(x)} \leq 1$ and only if $I_p(f) \leq 1$. If p is constant, then $L^{p(x)}$ coincides with the classical Lebesgue space L^p and so the notation can give rise to no confusion.

The corresponding Sobolev space $W^{1,p(x)}(\mathbb{R}^N)$ is the class of all functions $f \in L^{p(x)}(\mathbb{R}^N)$ such that all generalized derivatives $D_i f$, $i = 1, \dots, n$, belongs to $L^{p(x)}(\mathbb{R}^N)$. Endowed with the norm $\|f\|_{1,p(x)} = \|f\|_{p(x)} + \|\nabla f\|_{p(x)}$ it forms a Banach space.

Let w be a locally integrable function on $\mathbb{R}^N$ such that $w(x) > 0$ for *a.e.* $x \in \mathbb{R}^N$. Suppose further that w satisfies

1. $\int\limits_{2B_r^x} w(x)dx \leq C \int\limits_{B_r^x} w(x)dx$ for $w \in A_\infty$, $\exists\, C > 0$ and ball $B_r^x \subset \mathbb{R}^N$

2. $\sup\limits_{\substack{r>0 \\ x\in\mathbb{R}^N}} r^{1-N}\left[\int\limits_{B_r^x} w(y)dy\right]^{\frac{N-1}{N}} \leq C \mathop{inf}\limits_{B_r^x} w^{\frac{N-1}{N}}(x)$

where $B_r^x = \left\{y \in \mathbb{R}^N : |y - x| < r\right\}$. Then Gagliardo type inequality, can be derived from [9],

$$\left(\int\limits_{\mathbb{R}^N} \left|u(x)^{\frac{N}{N-1}}\right| w(x)\, dx\right)^{\frac{N-1}{N}} \leqslant C \int\limits_{\mathbb{R}^N} |\nabla u(x)|\, w(x)^{\frac{N-1}{N}} dx \qquad (2)$$

holds where $u \in C_0^\infty(\mathbb{R}^N)$ and C is independent of the function u.

3. Main Result

Theorem 1. *Let* $p : \mathbb{R}^N \to [1,\infty)$ *be a measurable function such that* $1 < p^- \leqslant p(x) \leqslant p^+ < N$. *Define* $q : \mathbb{R}^N \to [1,\infty)$ *by*

$$q(x) = \frac{Np(x)}{\beta p(x)(N-1) - N(p(x)-1)} < p^*(x)$$

where $1 < \beta \leqslant \frac{N}{N-1}$. *Suppose further that the set of functions* $\left\{q^{k+1}(x)\right\}$ *satisfies*

$$i) \quad \int\limits_{B_{2r}^x} q^k(y)dy \leqslant \sigma^k \int\limits_{B_r^x} q^k(y)dy \ , \ \sigma \geqslant 1,$$

$$ii) \quad \left[\frac{1}{|B_r^x|}\int\limits_{B_r^x} q^k(y)dy\right]^{\frac{1}{k}} \leqslant \beta^{N^*} \inf_{B_r^x} q(y) \quad , \ k = 1, 2, ...; \ \beta > 1.$$

Then there exists $C > 0$ *such that (1) holds for* $u \in C_0^\infty(\mathbb{R}^N)$.

Proof. Let $u \in C_0^\infty(\mathbb{R}^N)$ and assume that $\|u\|_{q(x)} = 1$. Then

$$\begin{aligned} 1 &= \int\limits_{\mathbb{R}^N} |u(x)|^{q(x)}\, dx \\ &= \int\limits_{\mathbb{R}^N} \left(\int\limits_0^{|u(x)|} q(x) t^{q(x)-1} dt \right) dx = \int\limits_0^\infty dt \left(\int\limits_{G_t} q(x)\, t^{q(x)-1} dx \right) \\ &= \int\limits_0^\infty \frac{dt}{t} \left(\int\limits_{G_t} q(x)\, t^{q(x)} dx \right) \qquad (3) \end{aligned}$$

where $G_t = \left\{x \in \mathbb{R}^N : |u(x)| > t\right\}$. Now, using Taylor expansion $t^{q(x)} = e^{q(x)\ln t} = \sum\limits_{k=0}^{\infty} \frac{(q(x)\ln t)^k}{k!}$ $(x \in \mathbb{R}^N,\ t > 0)$ for function $t^{q(x)}$ and substituting into (3), we obtain

$$\begin{aligned} 1 &= \int\limits_0^\infty \frac{dt}{t} \left[\int\limits_{G_t} q(x) \sum_{k=0}^{\infty} \frac{(q(x)\, \ln t)^k}{k!}\, dx \right] \\ &= \int\limits_0^\infty \frac{dt}{t} \left[\sum_{k=0}^{\infty} \frac{\ln^k t}{k!} \left(\int\limits_{G_t} q^{k+1}(x)\, dx \right) \right] \qquad (4) \end{aligned}$$

From the assumptions of theorem and the inequality (2) for the set of the functions $q^{k+1}(x)$, we obtain

$$\int\limits_{G_t} q^{k+1}(x)\,dx \leqslant \frac{C\beta^{kN^*}}{t^{N^*}}\Bigg(\int\limits_{E_t} q^{\frac{k+1}{N^*}}(x)\ |\nabla u|\ dx\Bigg)^{N^*}, \tag{5}$$

where $E_t = \left\{x \in \mathbb{R}^N \ :\ \frac{t}{2} < |\,u(x)\,| \leqslant t\right\}$, $N^* = \frac{N}{N-1}$ and $C > 0$ is independent of k. Using the following test function

$$z_t(x) = \min\left\{\frac{(\,|\,u\,(x)| - \frac{t}{2}\,)_+}{\frac{t}{2}}\ ,\ 1\right\} \tag{6}$$

in the inequality (2), where

$$(g)_+ = \begin{cases} g(x) & ,\quad g(x) \geqslant 0 \\ 0 & ,\quad g(x) < 0 \end{cases}$$

we have

$$1 \leqslant C\int\limits_0^\infty \frac{dt}{t}\ \sum_{k=0}^{\infty}\frac{\beta^{k\,N^*}(\ln t)^k}{t^{N^*}k\,!}\Bigg(\int\limits_{E_t} q^{\frac{k+1}{N^*}}(x)\ |\nabla u\,|\,dx\Bigg)^{N^*}$$

from inequalities (4) and (5). Applying Minkowski's inequlity as form:

$$\left[\sum_{k=0}^{\infty} c_k\Big(\int |\,f_k(x)|\ dx\Big)^{a}\right]^{\frac{1}{a}} \leqslant \int\Bigg(\sum_{k=0}^{\infty} c_k|\,f_k(x)|^{\,a}\Bigg)^{\frac{1}{a}} dx$$

we obtain

$$1 \leqslant c\int\limits_0^\infty dt\left[\ \int\limits_{E_t}\Bigg(\sum_{k=0}^{\infty}\frac{\beta^{kN^*}(\ln t)^k q^k(x)}{t^{1+N^*}k!}\,|\,\nabla u\,|^{N^*}\,q(x)\Bigg)^{\frac{1}{N^*}} dx\right]^{N^*} \tag{7}$$

and since $e^{\beta N^*\ln t\,q(t)} = \sum\limits_{k=0}^{\infty}\frac{\beta^{kN^*}(\ln t)^k q^k(t)}{k!}$ we have

$$1 \leqslant c\int\limits_0^\infty dt\left[\ \int\limits_{E_t} t^{\beta\frac{q(x)}{N^*}-1-\frac{1}{N^*}}\,q^{\frac{1}{N^*}}(x)\,|\,\nabla u\,|\,dx\right]^{N^*}. \tag{8}$$

Using Young's inequality

$$a.b \leqslant \frac{a^{p(x)}}{p(x)} + \frac{b^{p'(x)}}{p'(x)}\quad ,$$

where $a, b > 0$, $p(x) \geqslant 1$, $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ is used in (8), then we have

$$t^{\frac{\beta\, q(x)}{N^*} - 1 - \frac{1}{N^*}}\, q^{\frac{1}{N^*}}(x)\, |\nabla u| \leqslant \frac{|\nabla u|^{\,p(x)}}{\varepsilon^{p(x)}\, p(x)\, t^{\frac{1}{N^*}}} + \\ + \varepsilon^{p'(x)}\, t^{(\frac{\beta\, q(x)}{N^*} - 1 - \frac{1}{N^*})\, p'(x) + \frac{p'(x)}{p(x)\, N^*}} \quad q^{\frac{p'(x)}{N^*}}(x), \quad \varepsilon > 0 \ .$$

Therefore, inequality (8) can be rewritten as

$$\begin{aligned} 1 \ &\leq \ c \int_0^\infty dt \left[\int_{E_t} \varepsilon^{\frac{1}{p(x)}} \frac{|\nabla u|^{p(x)}}{t^{\frac{1}{N^*}} p(x)}\, dx + \int_{E_t} \varepsilon^{p'(x)}\, t^{\gamma(x)}\, q^{\frac{p'(x)}{N^*}}(x)\, dx \right]^{N^*} \\ &\leq \ C_1 \int_0^\infty \frac{dt}{t} \left(\int_{E_t} \varepsilon^{\frac{1}{p(x)}} \ \frac{|\nabla u|^{p(x)}}{p(x)}\, dx \right)^{N^*} + \\ &+ \ C_1 \int_0^\infty dt \left(\int_{E_t} \varepsilon^{p'(x)}\, t^{\gamma(x)}\, q^{\frac{p'(x)}{N^*}}(x)\, dx \right)^{N^*} \end{aligned} \tag{9}$$

where $\gamma(x) = (\frac{\beta q(x)}{N^*} - 1 - \frac{1}{N^*} + \frac{1}{p(x)N^*})p'(x)$ and we have applied the well known inequality

$$(a + b)^p \leqslant 2^{p-1}(a^p + b^p) \ , \quad a, b > 0, \quad p \geqslant 1.$$

Now, if we apply Minkowski's inequality to each integral above, then we get

$$\begin{aligned} 1 \ &\leq \ C_1 \left\{ \int_{\mathbb{R}^N} \varepsilon^{\frac{1}{p(x)}}\, dx \left[\int_{|u(x)|}^{2|u(x)|} \frac{dt}{t} \left(\frac{|\nabla u(x)|^{p(x)}}{p(x)} \right)^{N^*} \right]^{\frac{1}{N^*}} \right\}^{N^*} + \\ &+ \ C_1 \left\{ \int_{\mathbb{R}^N} \varepsilon^{p'(x)} \left[\int_{|u(x)|}^{2|u(x)|} t^{\gamma(x)N^*}\, dt \right]^{\frac{1}{N^*}} q^{\frac{p'(x)}{N^*}}(x)\, dx \right\}^{N^*} \end{aligned}$$

or

$$\begin{aligned} 1 \quad \leq \quad & C_1 \ln 2 \int_{\mathbb{R}^N} \varepsilon^{\frac{1}{p(x)}} \frac{|\nabla u|^{p(x)}}{p(x)}\, dx + \\ & + C_2 \left\{ \int_{\mathbb{R}^N} \varepsilon^{p'(x)}\, |u(x)|\, t^{\gamma(x)N^*} q^{\frac{p'(x)}{N^*}}(x)\, dx \right\}^{N^*} . \end{aligned} \tag{10}$$

It is easy to see that

$$\left[\left(\beta\, q(x) - N^* - 1 + \frac{1}{p(x)} \right) p'(x) + 1 \right] \frac{1}{N^*} = q(x)$$

from definition of $q(x)$. Since $p(x) \geq 1$ and $0 < \varepsilon < 1$, it follows $\varepsilon^{p(x)} \leqslant \varepsilon$. Using these results in (10) and taking into account of boundedness of function $q(x)$ and $\int\limits_{\mathbb{R}^N} |u(x)|^{q(x)}\,dx = 1$, the inequality (10) can be rewritten as

$$1 \leqslant C_1(\varepsilon) \int\limits_{\mathbb{R}^N} |\nabla u|^{p(x)}\,dx + C_2\,\varepsilon^{N^*} \int\limits_{\mathbb{R}^N} q^{\frac{p'(x)}{N^*}}(x)\,|u(x)|^{q(x)}\,dx$$

and by choosing $C_2\varepsilon^{N^*} < 1$ we find that

$$1 \leqslant \frac{C_1(\varepsilon)}{1 - C_2\,\varepsilon^{N^*}} \int\limits_{\mathbb{R}^N} |\nabla u(x)|^{p(x)}\,dx = C_3 \int\limits_{\mathbb{R}^N} |\nabla u(x)|^{p(x)}\,dx \tag{11}$$

where $C_3 = \frac{C_1(\varepsilon)}{1-C_2\varepsilon^{N^*}}$ is a positive number.

Hence, since $p(x) \geqslant 1$ and $C_3 > 1$ from inequality (11) for the function $\frac{u(x)}{\|u(x)\|_{q(x)}}$, we have

$$1 \leqslant C_3 \int\limits_{\mathbb{R}^N} \left(\frac{|\nabla u|}{\|u(x)\|_{q(x)}} \right)^{p(x)} dx$$

which yields (1). The theorem is proved. □

Remark 2. *If the function q satisfies $\sup\limits_{\mathbb{R}^N} q(x) \leqslant \beta^{N^*} \inf\limits_{\mathbb{R}^N} q(x)$, then above conditions i) and ii) is held.*

Remark 3. *If p is a Lipschitz continuous function, then by taking $\beta = 1$, we obtain*

$$\left(\int\limits_{\mathbb{R}^N} \left| u\, q^{\frac{k+1}{N^*}}(x) \right|^{N^*} dx \right)^{\frac{1}{N^*}} \leqslant C \int\limits_{\mathbb{R}^N} \left| \nabla \left(u\, q^{\frac{k+1}{N^*}}(x) \right) \right| dx$$

or

$$\left(\int\limits_{\mathbb{R}^N} |u|^{N^*} q^{k+1}(x)dx \right)^{\frac{1}{N^*}} \leqslant \left(\int\limits_{\mathbb{R}^N} |\nabla u|\, q^{\frac{k+1}{N^*}}(x)dx + \int\limits_{\mathbb{R}^N} |u|\, q^{\frac{k+1}{N^*}-1}(x)\, |\nabla q(x)|\, dx \right)$$

where $C > 0$ is independent of k. Hence we have limit case $q(x) = \frac{N\,p(x)}{N-p(x)}$ as $\beta = 1$.

ACKNOWLEDGEMENTS

The authors are thankful to the referees for their helpful suggestions and valuable contributions. This research was supported by DUAPK grant No. 04 FF 40.

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Received: April 5, 2006