

# Elementary Results on Special Number Sequences and Representation of Binary Quadratic Forms

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**Abstract.** In this paper some interesting results of special number sequences such as Fibonacci, Pythagorean and  $\beta$ -numbers are given. Elementary results of representation of numbers by binary quadratic forms and  $\beta$ -numbers are discovered.

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## 1. INTRODUCTION

A homogenous polynomial  $f = ax^2 + bxy + cy^2$  over  $\mathbb{Z}$  in two variables of degree 2 is called a binary quadratic form over  $\mathbb{Z}$  provided  $b^2 - 4ac$  is not a square of any integer. [1]

The theory of binary quadratic forms has been in the subject of number theory from Euler. We owe the development of the theory sistematically to J. L. Lagrange, C. F. Gauss and P. L. Dirichlet. [2]

The basic problem is the representation of an integer  $m$  by the quadratic form in the integral variables  $x$  and  $y$

$$\mathbb{Q}(x, y) = Ax^2 + Bxy + Cy^2 = m$$

The problem is twofold. First of all, we must decide if such a representation is possible or if the diophantine equation in two unknowns

$$(1) \quad \mathbb{Q}(x, y) = m$$

is solvable, and then we must find out how to characterize all solutions, i.e. how to write general  $(x, y)$  satisfying (1). [3]

## 2. PRELIMINARY NOTES

The following notations and definitions are used in this paper:

$\mathbb{Z}$  : Set of integers:  $\{\dots - 2, -1, 0, 1, 2, \dots\}$

$\mathbb{Z}^+$  : Set of positive integers:  $\{1, 2, \dots\}$

$\mathbb{Z}^*$  : Set of non-negative integers:  $\{0, 1, 2, \dots\}$

$\mathbb{Q}$  : Set of rational numbers

$\mathbb{Q}^*$  : Set of non-negative rational number

$\mathbb{R}$  : Set of real numbers

**Definition 1.** A  $\beta$ -representation is the form  $a^2 + ab + b^2$ , where  $a, b \in \mathbb{Z}^*$  and  $a \geq b$

**Definition 2.** Two  $\beta$ -representations  $a^2 + ab + b^2$  and  $c^2 + cd + d^2$  are distinct if either  $a \neq c$  or  $b \neq d$

**Definition 3.** An integer is a  $\beta$ -number if it has at least one  $\beta$ -representation.

**Definition 4.** If a  $\beta$ -number is a prime, it is a  $\beta$ -prime. [4]

## 3. MAIN RESULTS

**Theorem 1.** All  $\beta$ -primes other than 3 are of the form  $6k + 1$ . [4]

*Proof.*  $p = a^2 + ab + b^2$  be a prime. Let  $a \equiv m \pmod{6}$   $b \equiv n \pmod{6}$  and  $a^2 + ab + b^2 \equiv z \pmod{6}$ . Now,  $m^2 + mn + n^2 \equiv z \pmod{6}$  using the basic properties of congruences. For  $m = 0..5$  and  $n = 0..5$ ,  $z$  can take only the values  $6k, 6k + 1, 6k + 3, 6k + 4$ . Here  $6k$  and  $6k + 4$  are always composite.  $6k + 3$  is composite except for  $k = 0$ , i.e., when  $p = 3$ . So, the only prime values  $p$  can take 3 and  $6k + 1$ . ■

**Corollary 2.** There is no pythagorean number which is  $\beta$ -prime.

*Proof.* According to the theorem 1 all  $\beta$ -primes other than 3 are of the form  $6k + 1$ . Also every pythagorean numbers are divisible by 6. As a conclusion there is no pythagorean number which is  $\beta$ -number. ■

**Corollary 3.** The product of two  $\beta$ - numbers is a  $\beta$ - number.

*Proof.* It is convenient to prove this corollary by using the ideal theory. Let  $K = \mathbb{Q}(\sqrt{d})$  and  $R_k = \mathbb{Z}[\sqrt{d}]$  Every two classes of primitive quadratic forms arising from ideals in  $R_k$  in a given field have representatives of this type:

$$(2) \quad \mathbb{Q}_1(a, b) = x_1 a^2 + x_2 ab + x_3 b^2 \quad x_1 > 0$$

$$(3) \quad \mathbb{Q}_2(a, b) = x_2 a^2 + x_3 ab + x_4 b^2 \quad x_2 > 0$$

■

with  $(x_1, x_2) = 1$  Since the discriminants are equal, then, it is known that these forms have the same middle coefficient; like,

$$(4) \quad \mathbb{Q}_1(a, b) = x_1a^2 + xab + z_1b^2 \quad x_1 > 0$$

$$(5) \quad \mathbb{Q}_2(a, b) = x_2a^2 + xab + z_2b^2 \quad x_2 > 0$$

with  $(x_1, x_2) = 1$  So,

$$x^2 - 4x_1z_1 = x^2 - 4x_2z_2 = d$$

and  $x_1z_1 = x_2z_2$ . Since  $(x_1, x_2) = 1$ ,  $x_1$  divides  $z_2$ ,  $x_2$  divides  $z_1$ . This can be shown as  $d = x^2 - 4x_1x_2z_0$ .

We note that the forms in (2)and (3) are generated by

$$(6) \quad \dot{I}_1 = [x_1, \lambda]$$

$$(7) \quad \dot{I}_2 = [x_2, \lambda]$$

where  $\lambda = (x - \sqrt{d})/2$  so (6) and (7) will be

$$\dot{I}_1 = [1, (1 - \sqrt{(-3)})/2]$$

$$\dot{I}_2 = [1, (1 - \sqrt{(-3)})/2]$$

Here we restrict to the case  $x_1 > 0$  and  $x_2 > 0$  for any  $d$ , and  $\lambda$  satisfies the equation

$$\lambda^2 = x\lambda - x_1x_2z_0$$

Now we note that  $\dot{I}_1$ and  $\dot{I}_2$  are aggregate of

$$\alpha_1 = x_1a_1 + \lambda b_1 \quad \alpha_1 \in \dot{I}_1$$

$$\alpha_2 = x_2a_2 + \lambda b_2 \quad \alpha_2 \in \dot{I}_2$$

respectively where  $a_i$  and  $b_i$  are arbitrary ratioanal integers, Thus

$$\alpha_1\alpha_2 = x_1x_2a_1a_2 + x_1\lambda a_1b_2 + x_2\lambda a_2b_1 + b_1b_2\lambda^2$$

$$\alpha_1\alpha_2 = x_1x_2(a_1a_2 - z_0b_1b_2) + \lambda(x_1a_1b_2 + x_2a_2b_1 + xb_1b_2)$$

Otherwise expressed;

$$(8) \quad \alpha_1\alpha_2 = x_1x_2a_3 + \lambda b_3$$

These new variables are defined by special bilinear expressions in integral coefficients

$$(9) \quad a_3 = a_1a_2 - z_0b_1b_2$$

$$(10) \quad b_3 = x_1a_1b_2 + x_2a_2b_1 + xb_1b_2$$

Finally we infer from that

$$\dot{I}_1\dot{I}_2 = [a_1a_2, \lambda]$$

First we see that by the property of products  $\dot{I}_1\dot{I}_2$  contains  $x_1x_2$  and  $x_1\lambda, x_2\lambda$ , hence  $\lambda$  (since  $(x_1, x_2) = 1$ ) Thus  $\dot{I}_1\dot{I}_2 \supseteq [x_1x_2, \lambda]$ , But the index of the module  $[x_1x_2, \lambda]$  in  $R_k = [1, \lambda]$  is clearly  $x_1x_2 (= N[\dot{I}_1]N[\dot{I}_2])$  which is the index of  $\dot{I}_1\dot{I}_2$  in  $R_k$  by the index definition). Thus since  $\dot{I}_1\dot{I}_2$ , in  $R_k$  and its subset  $[x_1x_2, \lambda]$  have the same index in  $R_k$ ; they must be the same. If  $\mathbb{Q}(\dot{I}_1\dot{I}_2) = \mathbb{Q}_3$  then,

$$\mathbb{Q}_3(a, b) = x_1x_2a^2 + xab + z_0b^2$$

Indeed in the terminology of (2) (3) and (9) (10)

$$\mathbb{Q}_1(a_1, b_1)\mathbb{Q}_2(a_2, b_2) = \mathbb{Q}_3(a_3, b_3)$$

As a result the product of two  $\beta$ -numbers is a  $\beta$ -number.

**Corollary 4.** *The product of two  $\beta$ -numbers that are also Fibonacci numbers is a  $\beta$ -number but not a Fibonacci number.*

*Proof.* It is just proved that the product of two  $\beta$ -numbers is a  $\beta$ -number. If these  $\beta$ -numbers are Fibonacci numbers, It can be easily understand the product doesn't satisfy  $F_0 = 0, F_1 = 1, F_{n+1} = F_n + F_{n+1} (n = 0, \pm 1, \pm 2, \pm 3, \dots)$ . ■

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