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Associated Prime Ideals of Enveloping Algebras: (Differential Operator Rings)

V. K. Bhat* and Ravi Raina

School of Applied Physics and Mathematics, SMVD University
P/o Kakryal, Udhampur, J and K, India-182121
*e-mail: vijaykumarbhat2000@yahoo.com

Om Prakash

Department of Mathematics
Banasthali Vidyapith, Rajasthan, India-304022

Abstract. Let R be a Noetherian Q-algebra (Q the field of rational numbers) and δ be a derivation of R. An ideal P is an associated prime ideal of differential operator ring $R[x, \delta]$ if and only if $P = (P \cap R)[x, \delta]$ and $P \cap R$ is an associated prime ideal of R.

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1. INTRODUCTION

A ring R means with identity and any R-module unitary. Let R be a ring. The set of prime ideals of R is denoted by $Spec(R)$, the set of associated prime ideals of R is denoted by $Ass(R)$ and the set of minimal prime ideals of R is denoted by $Min.Spec(R)$. $Assas(M)$ denotes the assassinator of a uniform R-module M and for any subset J of an R-module M, the annihilator of J is denoted by $Ann(J)$. $C(0)$ denotes the set of regular elements of a ring R and C(I) denotes the set of elements regular modulo I, where I is an ideal of R. For any two ideals I, J of a ring R ; $I \subset J$ means that I is strictly contained in J. $\Re$ and Q denote the fields of real numbers and rational numbers respectively. Z denotes the ring of integers unless otherwise stated.

We know that a Lie algebra over a field K is a vector space L over K equipped with a non-associative product [..] satisfying the usual distributive laws and the rules [xx] = 0, [x[yz]] + [y[zx]] + [z[xy]] = 0. For example $\Re^3$ equipped with the usual cross product is a real Lie algebra. One can see that an associative K-algebra equipped with the operation [.,.] such that [x,y] = xy - yx becomes a Lie algebra over K.

Conversely starting with a Lie algebra, one can construct an associative K-algebra U(L) using the elements of L as generators, together with the relations xy-yx = [xy] for all $x, y \in L$. The algebra U(L) is called the Universal enveloping algebra of L.

The simplest Lie algebra L with a non-zero product is two-dimensional, with a basis $\{x, y\}$ such that [yx] = x. The elements of U(L) in this case can be put in the form $\sum f_i(x)y^i$, i = 1, 2, ..., n; where each $f_i(x)$ is a polynomial in the variable x. In U(L) the relation [yx] = x becomes [y,x] = x and so it follows easily that $[y, f(x)] = x(d/dx)(f(x))$ for all polynomials f(x). We note that U(L) is very similar to the first complex Weyl algebra $A_1(C)$. In $A_1(C)$ we have [D,f(x)] = (d/dx)(f(x)).

Keeping all this in mind, we start with a ring R and a map $\delta : R \to R$ which is a derivation and then construct a large ring T using an indeterminate y such that $[y, a] = \delta(a)$ for all $a \in R$. The elements of T look like differential operators $\sum \delta^i a_i$ on R, except that it may be possible for $\sum \delta^i a_i$ to be zero operator without all the coefficients a_i being zero. Thus the elements $\sum y^i a_i$ in T are called formal differential operators and T is called formal differential operator ring.

By Hilbert Basis Theorem, we see that formal differential operator rings with Noetherian coefficient rings are Noetherian and we shall view them as representative analogs of enveloping algebras.

Let now R be a Noetherian Q-algebra and δ be a derivation of R. In this paper we deal with the differential operator ring $R[x, \delta]$. We denote this ring by D(R). We mention that the coefficients of polynomials are taken to be on right, and thus $D(R) = \{\sum x^i a_i,\ a_i \in R\}$ subject to the relation $ax = xa + \delta(a)$. Some authors take coefficients on the left.

We give a structure of associated and minimal prime ideals of the differential operator ring D(R), where R is right Noetherian Q-algebra. This is proved in 2.9. Before giving the structure of associated prime ideals of $D(R)$, we first prove for a Noetherian Q-algebra R, a result of Seidenberg, namely Theorem (1) of [5] and a result of Gabriel, namely Lemma (3.4) of [2] in one go. This is proved in 2.7.

2. Associated Prime Ideals

We begin this section with the following definition:

Definition 2.1. Let R be ring and δ be a derivation of R. Then $D(R) = R[x, \delta]$ is the usual differential operator ring. Let I be a δ-invariant ideal of R. Then $D(I) = I[x, \delta]$.

Proposition 2.2. *Let R be a Noetherian Q-algebra and δ be a derivation of R. Then $e^{t\delta}$ is an automorphism of $T = R[[t]]$.*

Proof. The proof is same as given by Seidenberg in [5] and a sketch in the non-commutative case is provided by Blair and Small in [1]. ☐

Remark 2.3. Let R be a Noetherian ring and δ be a derivation of R. Let I be an ideal of R. Then we know that $I.R[[t]] = \{b_0 + tb_1 + t^2 b_2 + ...,\text{ with } b_i \in I\}$ and denote it by $I[[t]]$.

Lemma 2.4. *Let R be a Noetherian Q-algebra and δ be a derivation of R. Then an ideal I of R is δ-invariant if and only if $I.R[[t]]$ is $e^{t\delta}$-invariant.*

Proof. Let $T = R[[t]]$. Let IT be $e^{t\delta}$-invariant. Let $a \in I$. Then $a \in IT$, which implies that $e^{t\delta}(a) \in IT$; i.e. $a + t\delta(a) + (t^2\delta^2/2!)(a) + ... \in IT$. Therefore we have $\delta(a) \in I$.
Conversely suppose that $\delta(I) \subseteq I$ and let $f = \sum t^i a_i \in IT$. Then $e^{t\delta}(f) = f + t\delta(f) + (t^2\delta^2/2!)(f) + ... \in IT$ as $\delta(a_i) \in I$. Thus we have $e^{t\delta}(IT) \subseteq IT$. Replacing $e^{t\delta}$ by $e^{-t\delta}$, we get that $e^{t\delta}(IT) = IT$. ☐

We now quote the following Proposition, the proof of which is routine.

Proposition 2.5. *Let R be a ring and $T = R[[t]]$. Then:*

1. *$Q \in Ass(R)$ implies that $QT \in Ass(T)$.*
2. *$P \in Ass(T)$ implies that $P \cap R \in Ass(R)$ and $P = (P \cap R)T$.*

Proposition 2.6. *Let R be a ring and $T = R[[t]]$. Then:*

1. *$P \in Min.Spec(T)$ implies that $P \cap R \in Min.Spec(R)$ and $P = (P \cap R)T$.*
2. *$Q \in Min.Spec(R)$ implies that $QT \in Min.Spec(T)$.*

Proof. **(1)** Let $P \in Min.Spec(T)$. Then $P \cap R \in Spec(R)$. Suppose $(P \cap R) \notin Min.Spec(R)$, and $S \subset P \cap R$ is a minimal prime ideal of R. Then $ST \subset (P \cap R)T \subseteq P$, which is a contradiction as $ST \in Spec(R)$. Therefore $(P \cap R) \in Min.Spec(R)$. Now it is easy to see that $(P \cap R)T = P$.

(2) Let $Q \in Min.Spec(R)$. Then $QT \in Spec(T)$. Suppose $QT \notin Min.Spec(T)$, and $J \subset QT$ is a minimal Prime ideal of T. Then $(J \cap R) \subset QT \cap R = Q$ which is a contradiction as $(J \cap R) \in Spec(R)$. Therefore $QT \in Min.Spec(T)$. ☐

Theorem 2.7. *Let R be a Noetherian Q-algebra and δ be a derivation of R. Let $P \in (Ass(R) \cup Min.Spec(R))$. Then $\delta(P) \subseteq P$.*

Proof. Let $T = R[[t]]$. Now by 2.2 $e^{t\delta}$ is an automophism of T. Let $P \in (Ass(R) \cup Min.Spec(R))$. Then by 2.5 and 2.6 $PT \in (Ass(T) \cup Min.Spec(T))$. Therefore there exists an integer $n \geq 1$ such that $(e^{t\delta})^n(PT) = PT$; i.e. $e^{nt\delta}(PT) = PT$. Now R is a Q-algebra implies that $e^{t\delta}(PT) = PT$ and, therefore 2.4 implies that $\delta(P) \subseteq P$. ☐

Proposition 2.8. *Let R be a semiprime Noetherian ring. Let δ be a derivation of R. If $f \in D(R)$ is a regular element. Then there exists $g \in D(R)$ such that gf has leading co-efficient regular in R.(The proof of this Proposition is obvious and is left to the reader)*

Theorem 2.9. *Let R be a Noetherian Q-algebra and δ be a derivation of R. Then:*

1. $P \in Ass(D(R))$ *if and only if* $P = D(P \cap R)$ *and* $(P \cap R) \in Ass(R)$.
2. $P \in Min.Spec(D(R))$ *if and only if* $P = D(P \cap R)$ *and* $(P \cap R) \in Min.Spec(R)$.

Proof. **(1)** Let $P \in Ass(R)$. Then $\delta(P) \subseteq P$ by 2.7. Let $P = Ann(cR) = Assas(cR)$, $c \in R$. Since $cP = 0$, so $\delta^k(c)P = 0$ for all $k \geq 0$. Now let $h = \sum x^i b_i \in D(R)$ with leading coefficient b_u. Then for all $r \in R$, $crhP = 0$; i.e. $cRhP = 0$ and therefore $cRh.D(R) = 0$. Now by (14.2.5) (ii) of [4] $D(P) \in Spec(D(R))$. Suppose $D(P) \neq Ann(ch.D(R))$. Then there exists an ideal K of $D(R)$ such that $D(P) \subset K$ and $K = Ann(ch.D(R))$, $ch \neq 0$. Therefore by 2.8 there exists $g = \sum x^i a_i \in K$ with leading coefficient a_t such that $a_t \in C(P)$. Now $ch.D(R)K = 0$, which implies that $chRg = 0$. So for all $r \in R$, $chrg = 0$; i.e. $c(x^u b_u + ... + b_0)r(x^t a_t + ... + a_0) = 0$. Therefore we have $(x^{u+t}c + ... + a_t(c))b_u r a_t + ... + cb_0 r d_0 = 0$. Thus $cb_u r a_t = 0$ for all $r \in R$; i.e. $cb_u R a_t = 0$. Therefore we have $a_t \in Ann(cb_u R) = P$, which is a contradiction as $a_t \in C(P)$. Therefore $D(P) = Ann(ch.D(R))$ for all $h \in D(R)$. Hence $D(P) = Assas(ch.D(R))$.

Conversely suppose that $P \in Ass(D(R))$. Choose $f = \sum x^i a_i$ (with leading coefficient a_n) of least degree such that $P = Ann(f.D(R)) = Assas(f.D(R))$. Now $a_i R(P \cap R) = 0$ for all i, $1 \leq i \leq n$. Let $P_1 \in Ass(a_n R)$. Then there exists $s \in R$ such that $a_n s \neq 0$ and $P_1 = Ann(a_n sR) = Assas(a_n sR)$. Now $\delta(P_1) \subseteq P_1$ by 2.7 and by above paragraph $D(P_1) \in Ass(D(R))$. Now $(a_n sRP_1) = 0$ so $d^t(a_n s)RP_1 = 0$ for all integers $t \geq 1$ and for any $h = \sum x^i b_i \in D(R)$, $a_n sRhP_1 = 0$, which implies that $a_n sR.D(R).P_1 = 0$. Therefore we have $P_1 \subseteq (P \cap R)$. Also $(P \cap R) \subseteq P_1$, as $a_n sR(P \cap R) = 0$. Thus we have $P_1 = P \cap R$, which means that $D(P_1 = D(P \cap R) \in Ass(D(R))$. Therefore as in first paragraph $D(P \cap R) = P$.

(2). Let $P_1 \in Min.Spec(R)$. Then 2.7 implies that $\delta(P_1) \subseteq P_1$. So by (14.2.5) (ii) of [4] $D(P_1) \in Spec(D(R))$. Suppose $D(P_1) \notin Min.Spec(D(R))$, and $P_2 \subset D(P_1)$ is a minimal prime ideal of $D(R)$. Then $P_2 = D(P_2 \cap R) \subset D(P_1) \subseteq Min.Spec(D(R))$, which implies that $(P_2 \cap R) \subset P_1$, which is a contradiction as $(P_2 \cap R) \in Spec(R)$. Therefore $D(P_1) \in Min.Spec(D(R))$.

Conversely if $P \in Min.spec(D(R))$, then $(P \cap R) \in Spec(R)$ by Lemma (2.21) of [3] and $D(P \cap R) \in Spec(D(R))$. Therefore $D(P \cap R) = P$. We now show that $(P \cap R) \in Min.Spec(R)$. Suppose $(P \cap R) \notin Min.Spec(R)$, and $P_1 \subset (P \cap R)$ is a minimal prime ideal of R. Then $D(P_1) \subset D(P \cap R)$ and as in first paragraph $D(P_1) \in Spec(D(R))$, which is a contradiction. Hence $(P \cap R) \in Min.spec(R)$. □

References

[1] W. D. Blair and L. W. Small, Embedding differential and skew-polynomial rings into artinain rings, Proc. Amer. Math. Soc. 109(4) 1990, 881-886.

[2] P. Gabriel, Representations Des Algebres De Lie Resoulubles (D Apres J. Dixmier. In Seminaire Bourbaki, 1968-69, Pp 1-22, Lecture Notes In Math. No 179, Berlin 1971 Springer Verlag.

[3] K. R. Goodearl and R. B. Warfield Jr., An introduction to non-commutative Noetherian rings, Cambridge Uni. Press 1989.

[4] J. C. McConnell and J. C. Robson, Noncommutative Noetherian Rings, Wiley(1987); revised edition: American Math. Society (2001).

[5] A. Seidenberg, Differential ideals in rings of finitely generated Type, Amer. J. Math. 89 (1967), 22-42

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