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Real Space Curves

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Abstract. Here consider the existence of real space curves with prescribed degree, genus, and topological type of their real locus.

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1. Introduction

For any a smooth and connected projective curve X of genus $g \geq 0$ defined over $\mathbb{R}$ let $X(\mathbb{R})$ denote its set of real points and $n(X)$ the number of the connected component of $X(\mathbb{R})$. Hence $X(\mathbb{R})$ is the disjoint union of $n(X)$ circles. Set $a(X) = 1$ if $X(\mathbb{C})\backslash X(\mathbb{R})$ is connected and $a(X) = 0$ if $X(\mathbb{C})\backslash X(\mathbb{R})$ is not connected, i.e. if $X(\mathbb{C})\backslash X(\mathbb{R})$ has two connected components. The topological pair $(X(\mathbb{C}), X(\mathbb{R}))$ is uniquely determined by the triple of integers $(g, n(X), a(X))$ and such a triple of integers (g, n, a) is associated to some smooth real genus g curve if and only if either $a = 0$, $n \equiv g + 1 \pmod 2$ and $1 \leq n \leq g+1$ or $a = 1$ and $0 \leq n \leq g$ ([3], Prop. 3.1). See [2] or [7] for the list of all pairs (d, g) such that there is a smooth, connected and non-degenerate complex space curve C with $\deg(C) = d$ and $p_a(C) = g$. In section 2 we will prove the following results.

Theorem 1. *Fix integers g, d such that $d \geq 9$ and $0 < g \leq (d-1)^2/8$. Then there exists a real smooth curve $X \subset \mathbf{P}^3$ such that $\deg(X) = d$, $p_a(X) = g$, $n(X) = 1$ and $a(X) = 1$.*

Theorem 2. *Fix integers g, d such that $d \geq 9$ and $3 \leq g \leq (d-1)^2/8$. Then there exists a real smooth curve $X \subset \mathbf{P}^3$ such that $\deg(X) = d$, $p_a(X) = g$, $n(X) \geq \lfloor g/2 \rfloor$ and $a(X) = 1$.*

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Remark 1. For the construction of real curves in a smooth cubic, use a real smooth cubic such that all its 27 lines. Using Example 1 it is is easy to follow step by step the proofs of [2] and [7] to show that for all pairs (d, g) covered in those papers there is a smooth and connected curve real curve $X \subset \mathbf{P}^3$ such that $\deg(X) = d$, $p_a(X) = g$ and $X(\mathbb{R}) \neq \emptyset$.

In section 3 we will look at real space curves without real points. Obviously, any such curve must have even degree. We will see that there are other restrictions.

2. Proof of Theorems 1 and 2

Remark 2. Let $X \subset \mathbf{P}^3$ a smooth, connected and non-degenerate complex curve contained in a quadric Q. Set $d := \deg(X)$ and $g := p_a(X)$. If $d \geq 5$, then Q is unique and hence it is defined over $\mathbb{R}$ if both X and the embedding $X \subset \mathbf{P}^3$ are defined over $\mathbb{R}$. First assume that Q is smooth. Then there are integers $a > 0$, $b > 0$ such that $d = a + b$ and $g = ab - a - b + 1$. Conversely, for all integers $a > 0$, $b > 0$, there is a smooth, connected and non-degenerate curve $D \subset Q$ with degree $a + b$ and genus g. The set of all such curves is a non-empty Zariski open subset of the $(ab + a + b)$-dimensional projective space parametrizing the all polynomials $\mathbb{C}[w_0, w_1, z_0, z_1]$ which are homogeneous of degree a in the variables w_0, w_1 and homogeneous of degree b in the variables z_0, z_1. We may also find such a curve defined over $\mathbb{R}$ taking polynomials with real coefficients. Here we use the real form $\{x_0^2 + x_1^2 - x_2^2 - x_3^2 = 0\}$ of Q. There is a real smooth curve with empty real locus and of type (a, b) if and only if both a and b are even. Hence we cover in this way the range $d = a + b$, $g = ab - a - b + 1$ with both a, b even. If $d = 4k$, $k \geq 2$. we do not cover (for real curves with empty real locus) the pair $(d, g) = (4k, (2k+1)(2k-1) - 4k + 1) = (4k, 4k^2 - 4k)$. The real forms $\{x_0^2 + x_1^2 + x_2^2 - x_3^2 = 0\}$ and $\{x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0\}$ give no new pair (d, g), because in both cases we have to take a even and b even. Now assume that Q is a quadric cone and that d is even (resp. odd). In this case X is the complete intersection of Q with a surface of degree $a \geq 3$ and hence $d = 2a$ and $g = a^2 - 2a + 1$ (resp. $d = 2a + 1$ and $g = a^2 - a$ for some integer a) ([6], Ex. V.2.9). We only cover the pairs (d, g) previously obtained when $a = b$ (case d even) or $|a - b| = 1$ (case d odd). If $c(X) = 0$ and $Q(\mathbb{R})$ is a cone with as a basis a real conic with real points, then we must take a even and hence we do not get new pairs (d, g). However, if $Q(\mathbb{R})$ is a cone with as a basis a real conic without real points, then every integer $a \geq 3$ is allowed, and hence we cover all pairs $(2a, a^2 - 2a + 1)$, $a \in \mathbb{N}$, $a \geq 3$. The classical papers on real space curves X on a quadric (mainly interested in the case $X(\mathbb{R}) \neq \emptyset$) are quoted in [4], §5.

Example 1. Here we follow [2], §1, and [7], §3, to describe real curves on a quartic surface $Y \subset \mathbf{P}^3$ with a double line. Let $T \subset \mathbf{P}^2$ a smooth degree 3 curve defined over $\mathbb{R}$. Fix 9 points $P_1, \dots, P_9 \in T(\mathbb{R})$ such that the classes $\mathcal{O}_T(1)$ and $\mathcal{O}_T(P_i)$, $1 \leq i \leq 9$, are $\mathbb{Z}$-independent in $\mathrm{Pic}(T)$. This is possible, because

$T(\mathbb{R})$ is infinite. Let $\pi : S \to \mathbf{P}^2$ the blowing-up of the points $P_1, \dots, P_9$. S and π are defined over $\mathbb{R}$. Set $E_i := \pi^{-1}(P_i)$, $1 \le i \le 9$, and $\Delta := \pi^*(\mathcal{O}_T(1))$. Hence Pic($S$ has a basis defined over $\mathbb{R}$. Set $\bar{c} := \Delta - E_1$ and $\bar{b} := E_8 + E_9$. Let E be the strict transform of T in S. Hence $E \cong T$ as real curves and $\{E\} = |3\Delta - \sum_{i=1}^{9} E_i| = |-\omega_S|$. From now on we we assume $n(T) = 1$ and hence $a(T) = 1$.

(a) Here we check the existence of a real smooth and connected $X \in |\bar{c}|$ such that $n(X) = 1$ and $a(X) = 0$. By [2], proof of Prop. 1.11, or [5], Prop. 3.2, any smooth element of $|\bar{c}|$ has genus 0. It is sufficient to take as X the strict transform of a real line through P_1.

(b) Here we prove that for all integers $n > 0$ there is a real smooth and connected $X \in |\bar{c} - n\omega_S|$ such that $n(X) = 1$ and $a(X) = 1$. By [2], proof of Prop. 1.11, or [5], Prop. 3.2, any smooth element of $|\bar{c} - n\omega_S|$ has genus $2n$. There is a real line $L \subset \mathbf{P}^2$ passing through P and another point of $T(\mathbb{R})$, but not tangent to T. Hence $L \cap T = \{P, Q, Q'\}$ with $Q' \in T(\mathbb{R})$ and $Q' \neq Q$. Let D be the strict transform of L. Set $Y := D \cup E$. Y is a nodal, connected and real and $Y \in |\bar{c} - \omega_S|$. By [8], Prop. 2.11 and Cor. 2.14, and the proof of the plane case given in [4], pp. 12–13, we may independently do any real smoothing of each of the 2 singular points, say Q_1 and Q_2, of Y. Let $u : S' \subset S$ be the blowing-up of Q_2 and Y' the strict transform of Y in S'. Smoothing the only singular point of Y' we get a real and geometrically connected smooth curve Y'' with genus 1, $n(Y'') = 1$ and $a(Y'') = 1$. Then we make a real smoothing of the only singular point (i.e. Q_1) of $u(Y'')$. There are two potentially different real smoothing of Q_1. It is quite obvious that one of them has connected real locus, while the other one has real locus with 2 connected components. Since $a(E) = 1$, it is clear that in the first case we get a curve X with $a(X) = 1$. Now assume $n \ge 2$ and that the result is true for the integer $n' := n - 1$. Take a smooth and real $W \in |\bar{c} - n\omega_S|$ such that $n(W) = a(W) = 1$. Since $|\bar{c} - n\omega_S|$ is base point free ([2], Prop. 1.7, or [7], Prop. 3.2), we may take W as above and intersecting transversally E. Since $E(\mathbb{R})$ is infinite, we may take W as above with the additional condition that $W \cap E$ contains at least one real point of E. Call it P. Set $Z := W \cup E$. Let $v : S'' \to S$ be the blowing-up of S along $W \cap E \backslash \{P\}$. Use (Z, v, S'') instead of (Y, u, S') to repeat the proof of the case $n = 1$.

(c) Here we check the existence of a real smooth and connected $X \in |\bar{b} - \omega_S|$ such that $n(X) = 1$ and $a(X) = 1$. By [2], proof of Prop. 1.11, or [5], Prop. 3.2, any smooth element of $|\bar{b} - \omega_S|$ has genus 1.

(d) Here we prove that for all integers $n > 0$ there is a real smooth and connected $X \in |\bar{b} - n\omega_S - n\omega_S|$ such that $n(X) = 1$ and $a(X) = 1$. By [2], proof of Prop. 1.11, or [7], Prop. 3.2, any smooth element of $|\bar{b} - n\omega_S|$ has genus $2n - 1$. The case $n = 1$ is done in part (c). The cases with $n \ge 2$ may be proved by induction on n as in part (b).

Proof of Theorem 1. Use Example 1 instead of [2], Cor. 1.8, or [7], Prop. 1.11, and then use verbatim the proof of [2], Th. 1.1, or [7],§3 and §4. □

Proof of Theorem 2. Use Example 1 instead of [2], Cor. 1.8, or [7], Prop. 1.11, except that in steps (b) and (d) each step we increase by one the number of the connected components of the real locus. Then use verbatim the proof of [2], Th. 1.1, or [7], §3, 4. □

3. Real space curves with no real points

Fix an odd integer $r > 0$. There are two real forms of $\mathbf{P}^r_{\mathbb{C}}$, the standard one, called here $\mathbf{P}^r_{\mathbb{R}}$, with $\mathbf{P}^r(\mathbb{R})$ as its real locus, and the one with empty real locus. Let N_r denote the latter real structure of $\mathbf{P}^r_{\mathbb{C}}$. Let X be a real curve such that there exists a morphism $\phi : X \to N_r$ defined over $\mathbb{R}$. Since $N_r(\mathbb{R}) = \emptyset$, we have $X(\mathbb{R}) = \emptyset$. For some embeddings of some X into N_r, $r \geq 3$ and odd, see [1]. in the case $r = 3$ no gap is covered in this way. All real surfaces of N_3 have even degree. See [2], Appendice B, for the discussion of all pairs (d, g), $d \geq 10$, which can occur only as curves in a cubic surface and hence that are gaps for real embeddings of real curves with no real point. From now on we consider real embeddings of real smooth curves without real points into $\mathbf{P}^3$ with the usual real structure.

Remark 3. Let S be a geometrically integral projective surface defined over $\mathbb{R}$ which is isomorphic over $\mathbb{C}$ to an integral degree 3 surface of $\mathbf{P}^3$. Hence ω_S is locally free, the anticanonical system $|-\omega_S|$ is very ample and $\dim(|-\omega_S|) = 3$. Since ω_S is defined over $\mathbb{R}$, S is isomorphic over $\mathbb{R}$ to an integral degree 3 surface of $\mathbf{P}^3$. Since 3 is odd, $S_{reg}(\mathbb{R}) \neq \emptyset$.

The list of all real type of integral cubic surfaces is too long to get nice lists of gaps for real space curves with empty real locus and which are contained in a degree 3 surface. We only list the case of smooth cubic surfaces such that all its 27 lines are real.

Example 2. Let S be a smooth cubic surface such that all its 27 lines are real, i.e. let S be the blowing-up of $\mathbf{P}^2$ at 6 real points, no 3 of them on a line and not all on a conic. Every complex line bundle on S is real. There is a smooth and connected complex curve $X \subset S$ with degree d and genus g if an only if there is a smooth and geometrically connected real curve $X \subset S$ with degree d and genus g if and only if there are integers $a, m_1, \dots, m_6$ such that $m_1 \geq \cdots \geq m_6 \geq 0$, $a \geq m_1 + m_2 + m_3$,

$$d = 3a - \sum_{i=1}^{6} m_i \tag{1}$$

$$g = 1 + (a^2 - \sum_{i=1}^{6} m_i^2 - d)/2 \tag{2}$$

and $(a, m_1, \dots, m_6) \neq (n, n, 0, \dots, 0)$ with $n \geq 2$ ([2], §2, or [7], p. 303. If X is real and $X(\mathbb{R}) = \emptyset$, the d is even and each m_i is even, because the intersection number of X with each real line of S must be even. By (2) we get that if $d \equiv 2 \pmod 2$, then g must be even, while if $d \equiv 0 \pmod 4$, then g must be odd. This is a very strong restriction.

Now we briefly consider gaps for smooth real curves in $\mathbf{P}^r$, $r \geq 4$, with empty real locus. Fix an integer $r \geq 4$. For all integers $d \geq 2r = 1$ set $m := \lfloor (d-1)/(r-1) \rfloor$, $\epsilon := d(r-1) - m - 1$ and $\pi(d,r) := \binom{m}{2}(r-1) + m\epsilon$, $m_1 := \lfloor (d-1)/r \rfloor$, $\epsilon_1 := d - m_1 r - 1$, $\mu_1 := 1$ if $\epsilon_1 = r-1$, $\mu_1 := 0$ if $\epsilon_1 \neq r-1$ and $\pi_1(d,r) := \binom{m_1}{2}r + m_1(\epsilon_1 + 1) + \mu_1$. Let $X \subset \mathbf{P}^r$ be a mooth, connected and non-degenerate curve with degree d and genus g. Then $g \leq \pi(d,r)$ and if $\pi_1(d,r) < g \leq \pi(d,r)$, then X lies in a surface S of degree $r-1$ ([5], Cor. 3.12 and Th. 3.15). If $r \neq 5$, then S is either an embedding of F_e, $e \equiv r-1 \pmod 2$, $0 \leq e \leq r-2$, by the complete linear system $|h + ((e+r-1)/f|$ (with the notations of Example 3 below) or a cone over a rational normal curve of $\mathbf{P}^{r-1}$. If $r = 5$, then S may also be the Veronese surface.

Example 3. Let F_e, $e \geq 0$, denote the complex Hirzebruch surface with a section with minimal self-intersection $-e$. If $e = 0$ we fix one of its two rulings. Hence we have a ruling $\pi : F_e \to \mathbf{P}^1$ and call h a section of π with minimal self-intersection and f a fiber of π. Thus $\mathrm{Pic}(F_e)$ is freely generated by the classes of h and f, $h^2 = -e$, $h \cdot f = 1$ and $f^2 = 0$. There is a smooth and connected $C \in |ah + bf|$ if and only if either $a = 0$ and $f = 1$, or $a = 1$, $b = 0$ and $e = 0$, or $a > 0$, $e = 0$, and $b > 0$, or $e > 0$, $a > 0$ and $b \geq ea$. Fix any such C. We have $\omega_{F_e} \cong \mathcal{O}_{F_e}(-2h + (-2-e)f)$. Thus $\omega_C \cong \mathcal{O}_C((a-2)h + (b-2-e)f)$ (adjunction formula). Thus $p_a(C) = ba - ea^2/2 + ea/2 - a + 1$. F_e has a unique real structure for which π is a real morphism when we take the real structure of $\mathbf{P}^1$ with non-empty real locus. There is a real $C \in |ah + bf|$ as above if and only if a and $b - ea$ are even. There is a real structure on F_e with empty real locus if and only if e is even. When e is even, we may also take $C \in |ah + bf|$ with a odd.

Remark 4. Let $S \subset \mathbf{P}^5$ be a Veronese surface with the real structure with non-empty real locus. There is a smooth, connected and non-degenerate real curve $X \subset S$ with $X(\mathbb{R}) = \emptyset$, degree d and genus g if and only if d is even, $d \geq 4$, and $g = (d-2)(d-4)/4$ (genus formula for plane curves). Let $S' \subset \mathbf{P}^5$ a Veronese surface with the real structure with empty real locus. There is a smooth, geometrically connected and non-degenerate real curve $X \subset S$ with $X(\mathbb{R}) = \emptyset$, degree d and genus g if and only if d is even, $d \geq 4$, and $g = (d/2-1)(d/2-2)/2$.

Remark 5. Let $S \subset \mathbf{P}^r$ be a degree $r-1$ non-degenerate surface cone defined over $\mathbb{R}$. Let P be the vertex of S. Hence $P \in X(\mathbb{R})$ and S is a cone over a rational normal curve $C \subset \mathbf{P}^{r-1}$ defined over $\mathbb{R}$. Let $X \subset S$ be a smooth and connected curve defined over $\mathbb{R}$ and such that $X(\mathbb{R}) = \emptyset$. The latter condition implies $P \notin X$. First assume $C(\mathbb{R}) \neq \emptyset$. This is always the case if r is even. We use the notations of Example 3. The blowing-up of S at its vertex is isomorphic to F_{r-1} and the strict transform Y of X in F_{r-1} is a real member of $|ah + a(r-1)f|$. The condition $X(\mathbb{R}) = \emptyset$ implies a even. Now assume $C(\mathbb{R}) = \emptyset$ and hence r odd. In this case we have again $Y \in |ah + a(r-1)f|$, but the case a odd and $a \geq 2$ is admissible.

Remark 6. Now we fix an integer $r \geq 3$ and consider smooth and non-degenerate real curves $X \subset \mathbf{P}^r$ which are $\mathbb{R}$-irreducible, but not geometrically irreducible. Thus $X = C_1 \sqcup C_2$ (disjoint union) with each C_i a smooth complex curve and C_1, C_2 are exchanged by the complex conjugation of $\mathbf{P}^r$. The smoothness of X is equivalent to $C_1 \cap \mathbf{P}^3(\mathbb{R}) = \emptyset$. Conversely, any such C_1 uniquely determines one X as above. Set $x := \deg(C_1)$, $q := p_a(C_1)$, $d := \deg(X)$ and $g := \chi(\mathcal{O}_X) - 1$. Hence $d = 2x$ and $g = 2q - 1$.Set $a := \dim(\langle C_1 \rangle) = \dim(\langle C_1 \rangle)$. Since X spand $\mathbf{P}^r$, we have $2a + 1 \geq r$. Fix any integer a such that $1 \leq a \leq r$ and $2a + 1 \geq a$. We may take as C_1 any smooth and connected curve spanning an a-dimensional linear subspace. If $a = 1$, then $x = 1$ and $q = 0$. If $a = 2$, then $q = (x-2)(x-1)/2$. If $a \geq 3$, then $q \leq \pi(d/2, a)$ and the equality may hold. For $d \gg r$ we have $2 \cdot \pi(d, \lceil (r+1)/2 \rceil) - 1 \ll \pi(d, r)$ and hence we have large gaps.

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