Int. J. Contemp. Math. Sci., Vol. 2, 2007, no. 11, 519 - 521

Curves in a Projective Surface with Prescribed Ordinary Singularities

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Abstract. Here we give an existence theorem for integral curves C contained in a smooth projective surface S and with as only singularities prescribed ordinary multiple points at general points of S. The proof heavily use the proof of the case $S = \mathbf{P}^2$ given by T. Mignon.

Mathematics Subject Classification: 14H50; 14H10; 14J99

Keywords: singular curve; curve in a surface; curve with prescribed singularities; ordinary multiple point

Let S be a smooth and connected projective. Fix $R, H \in \mathrm{Pic}(S)$ such that H is ample. Here we generalize the case $S = \mathbf{P}^2$ of [3] and prove the following result.

Theorem 1. *Let S be a smooth and connected projective. Fix $R, H \in Pic(S)$ such that H is ample and an integer $m \ge 2$. There is an integer η such that for all integers $d \ge \eta$, all integers $s > 0$, all integers m_i, $1 \le i \le s$, with $2 \le m_i \le m$, then*

- (i) *$h^2(S, \mathcal{I}_Z \otimes R \otimes H^{\otimes d}) = 0$ and either $h^0(S, \mathcal{I}_Z \otimes R \otimes H^{\otimes d}) = 0$ or $h^1(S, \mathcal{I}_Z \otimes R \otimes H^{\otimes d}) = 0$ for a general union $Z = \cup_{i=1}^{s} m_i P_i$ of s fat points of multiplicities $m_1, \dots, m_s$;*
- (ii) *If $h^0(S, \mathcal{I}_Z \otimes R \otimes H^{\otimes d}) > 0$, then a general $X \in |\mathcal{I}_Z \otimes R \otimes H^{\otimes d}|$ is integral, $Sing(X) = \{P_1, \dots, P_s\}$, and each P_i is an ordinary point with multiplicity m_i of X.*

Proof of Theorem 1. Part (i) is just the 2-dimensional case of [1]. We fix an integer d_0 (depending om m, S, H, R) such that part (i) is true for the triples (m, R, H) and $(m, \mathcal{O}_S, H)$ for all integers $d \ge d_0$.

[1]The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

(a) Here we will assume that H is very ample, that $h^i(S, H^{\otimes t}) = h^i(S, R\otimes H^{\otimes t}) = 0$ for $i = 1, 2$ and all $t > 0$ and that the following Condition (+) is satitfied:

Condition (+): For all $A_i \in S$, $1 \le i \le 3$, such that $A_i \ne A_j$ for all $i \ne j$ and every non-zero tangent vector v of $T_{A_1}S$ there are smooth $D, D' \in H$ such that $\{A_1, A_2\} \subset D$, $A_3 \notin D$, $v \subset D'$ and $A_2 \notin D'$.

Set $c := H^2$, $e = H \cdot R$, $v := R^2$, $\epsilon := \omega_S \cdot H$, and $\epsilon' := \omega_S \cdot R$. We will often use the additive notation for line bundles on S and on the blowing-ups of S. We have $\chi(\mathcal{O}_S(R + th)) = (v + ct^2 + 2te - \epsilon t - \epsilon')/2 + \chi(\mathcal{O}_S)$ and $\chi(\mathcal{O}_S(tH)) = (ct^2 - \epsilon t)/2 + \chi(\mathcal{O}_S)$ for all $t \in \mathbb{Z}$.If $C \in |tH|$, then $p_a(C) = 1 + (t^2c + \epsilon t)/2$. If $A \in |R + th|$, then $p_a(A) = 1 + (v + ct^2 + 2te + \epsilon t + \epsilon')/2$. Let $\pi_r : S_r \to S$ denote the blowing-up of S at r distinct points $Q_1, \dots, Q_r$. Let $E_i := \pi_r^{-1}(Q_i)$ be the exceptional divisors. For all $(r + 1)$-ples of integers $\underline{d} = (d, d_1, \dots, d_r)$ set $(\underline{d}) := \pi_r^*(dH)(-d_1E_1 - \dots - d_rE_r) \in \mathrm{Pic}(S_r)$ and $(R + \underline{d}) := \pi_r^*(R + dH)(-d_1E_1 - \dots - d_rE_r) \in \mathrm{Pic}(S_r)$. Fix $\underline{d} = (d, d_1, \dots, d_r)$ and $\underline{a} = (a, a_1, \dots, a_r)$. We have $(\underline{d}) \cdot (\underline{a}) = cad - \sum_{i=1}^r a_id_i$, $(R + \underline{d}) \cdot (\underline{a}) = ea + cad - \sum_{i=1}^r a_id_i$ and $(R + \underline{d}) \cdot (R + \underline{a}) = ea + ed + v + cad - \sum_{i=1}^r a_id_i$. Set $\chi(\underline{d}) := \chi(\mathcal{O}_{S_r}(\underline{d}) = (cd^2 - \epsilon d)/2 + \chi(\mathcal{O}_S) - \sum_{i=1}^r d_i(d_i + 1)/2$ and $\chi(R + \underline{d}) := \chi(R + \mathcal{O}_{S_r}(\underline{d}) = (cd^2 - \epsilon d + 2ed + v - \epsilon')/2 + \chi(\mathcal{O}_S) - \sum_{i=1}^r d_i(d_i + 1)/2$. Set $g(\underline{d}) := (cd^2 + \epsilon d)/2 + \chi(\mathcal{O}_S) - \sum_{i=1}^r d_i(d_i + 1)/2$ and $g(R + \underline{d}) := (cd^2 + 2ed + v + \epsilon d + \epsilon')/2 + \chi(\mathcal{O}_S) - \sum_{i=1}^r d_i(d_i + 1)/2$. If $C \in |(\underline{a})|$, then $p_a(C) = 1 + (a^2c + \epsilon a)/2 - \sum_{i=1}^r a_i(a_i - 1)/2$, $\chi(\mathcal{O}_C(\underline{d}) = -(a^2c + \epsilon a)/2 + \sum_{i=1}^r a_i(a_i - 1)/2 + cad - \sum_{i=1}^r a_id_i$, and $\chi(R + \mathcal{O}_C(\underline{d}) = -(a^2c + \epsilon a)/2 + \sum_{i=1}^r a_i(a_i - 1)/2 + cad - \sum_{i=1}^r a_id_i + ea$. We copy [3]. The proof of the key Lemma [3], Lemma 2.1, given in [2] does not use the assumption $S = \mathbf{P}^2$. In our set-up $g(\underline{a}) := 1 + (a^2c + \epsilon a)/2 - \sum_{i=1}^r a_i(a_i - 1)/2$ and we use $\chi(\mathcal{O}_C(\underline{d})$ instead of $da + 1 - g$ in the numerical computation in [3], p. 291. We use $R + \underline{d}$ (resp. $R + \underline{d} - \underline{a}$, resp. $\underline{a}$) instead of the line bundles $\underline{d}$ (resp. $\underline{d} - \underline{a}$, resp. $\underline{a}$) used in [3]. Notice that $h^1(C, \mathcal{O}_C(\underline{d})) = 0$ when $\underline{d} \gg \underline{a}$. The computations in [3], pp. 291–295, and the check of the assumptions 2) and 3) of [2], Lemma 3.1, are OK because they require only the quadratic part of the Rieman-Roch formula for $\chi(R + \underline{d})$, $\chi(R + \underline{d} - \underline{a})$ and $\chi(\underline{a})$. Similar numerical problems were checked in arbitrary dimension in [1]. Condition (+) is used to copy the proof of [3], Prop. 4.1, using a smooth curve $L \in |H|$ instead of a line.

(b) For every very ample $L \in \mathrm{Pic}(S)$ and every integer $t \ge 2$ the line bundle $L^{\otimes t}$ is very ample and satisfies Condition (+). Take H ample. There is an integer $x > 0$ such that $H^{\otimes t}$ is very ample, satisfies Condition (+) and $h^i(S, H^{\otimes t}) = h^i(S, R \otimes H^{\otimes t}) = 0$ for $i = 1, 2$ and all $t \ge x + 1$. We may apply the previous proof taking $\widetilde{H} := H^{\otimes x}$ with respect to the x line bundles $R \otimes H^{\otimes i}$, $0 \le i \le x - 1$. □

We work over an algebraically closed field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = 0$.

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Received: November 17, 2006