Curves in a Projective Surface
with Prescribed Ordinary Singularities

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Abstract. Here we give an existence theorem for integral curves $C$ contained in a smooth projective surface $S$ and with as only singularities prescribed ordinary multiple points at general points of $S$. The proof heavily use the proof of the case $S = \mathbb{P}^2$ given by T. Mignon.

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Let $S$ be a smooth and connected projective. Fix $R, H \in \text{Pic}(S)$ such that $H$ is ample. Here we generalize the case $S = \mathbb{P}^2$ of [3] and prove the following result.

Theorem 1. Let $S$ be a smooth and connected projective. Fix $R, H \in \text{Pic}(S)$ such that $H$ is ample and an integer $m \geq 2$. There is an integer $\eta$ such that for all integers $d \geq \eta$, all integers $s > 0$, all integers $m_i, 1 \leq i \leq s$, with $2 \leq m_i \leq m$, then

(i) $h^2(S, I_Z \otimes R \otimes H^{\otimes d}) = 0$ and either $h^0(S, I_Z \otimes R \otimes H^{\otimes d}) = 0$ or $h^1(S, I_Z \otimes R \otimes H^{\otimes d}) = 0$ for a general union $Z = \bigcup_{i=1}^{s} m_i P_i$ of $s$ fat points of multiplicities $m_1, \ldots, m_s$;
(ii) If $h^0(S, I_Z \otimes R \otimes H^{\otimes d}) > 0$, then a general $X \in |I_Z \otimes R \otimes H^{\otimes d}|$ is integral, $\text{Sing}(X) = \{P_1, \ldots, P_s\}$, and each $P_i$ is an ordinary point with multiplicity $m_i$ of $X$.

Proof of Theorem 1. Part (i) is just the 2-dimensional case of [1]. We fix an integer $d_0$ (depending on $m, S, H, R$) such that part (i) is true for the triples $(m, R, H)$ and $(m, \mathcal{O}_S, H)$ for all integers $d \geq d_0$.

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(a) Here we will assume that \( H \) is very ample, that \( h^i(S, H^{\otimes t}) = h^i(S, R \otimes H^{\otimes t}) = 0 \) for \( i = 1, 2 \) and all \( t > 0 \) and that the following Condition (+) is satisfied:

Condition (+): For all \( A_i \in S, 1 \leq i \leq 3 \), such that \( A_i \neq A_j \) for all \( i \neq j \) and every non-zero tangent vector \( v \) of \( T_{A_i}S \) there are smooth \( D, D' \in H \) such that \( \{A_1, A_2\} \subset D, A_3 \notin D, v \subset D' \) and \( A_2 \notin D' \).

Set \( c := H^2, e = H \cdot R, v := R^2, e := \omega_S \cdot H, \) and \( e' := \omega_S \cdot R \). We will often use the additive notation for line bundles on \( S \) and on the blowing-ups of \( S \). We have \( \chi(O_S(R + th)) = (v + ct^2 + 2te - et' + e'/2 + \chi(O_S) \) and \( \chi(O_S(tH)) = (ct^2 - et')/2 + \chi(O_S) \) for all \( t \in \mathbb{Z} \). If \( C \in [tH] \), then \( p_a(C) = 1 + (t^2c + et)/2 \). If \( A \in [R + th] \), then \( p_a(A) = 1 + (v + ct^2 + 2te + et')/2 \). Let \( \pi_r : S_r \to S \) denote the blowing-up of \( S \) at \( r \) distinct points \( Q_1, \ldots, Q_r \). Let \( E_i := \pi_r^{-1}(Q_i) \) be the exceptional divisors. For all \( (r + 1) \)-plies of integers \( d = (d, d_1, \ldots, d_r) \) set \( (d) := \pi_r^*(dH)(-d_1E_1 - \cdots - d_rE_r) \in \text{Pic}(S_r) \) and \( (R + d) := \pi_r^*(R + dH)(-d_1E_1 - \cdots - d_rE_r) \in \text{Pic}(S_r) \). Fix \( d = (d, d_1, \ldots, d_r) \) and \( a = (a, a_1, \ldots, a_r) \). We have \( (d) \cdot (a) = cad - \sum_{i=1}^r a_i d_i \) and \( (R + d) \cdot (R + a) = ea + cad - \sum_{i=1}^r a_i d_i \). Set \( \chi(d) := \chi(O_S, d) = (cd^2 - cd)/2 + \chi(O_S) - \sum_{i=1}^r d_i(d_i + 1)/2 \) and \( \chi(R + d) := \chi(R + O_S, d) = (cd^2 - cd + 2ed + v + e')/2 + \chi(O_S) - \sum_{i=1}^r d_i(d_i + 1)/2 \). Set \( g(d) := (cd^2 + ed)/2 + \chi(O_S) - \sum_{i=1}^r d_i(d_i + 1)/2 \) and \( g(R + d) := (cd^2 + 2ed + v + ed + e')/2 + \chi(O_S) - \sum_{i=1}^r d_i(d_i + 1)/2 \). If \( C \in [(a)] \), then \( p_a(C) = 1 + (a^2c + ea)/2 - \sum_{i=1}^r a_i(a_i - 1)/2 \). \( \chi(O_C(d) = -(a^2c + ea)/2 + \sum_{i=1}^r a_i(a_i - 1)/2 + cad - \sum_{i=1}^r a_i a_i d_i \) and \( \chi(R + O_C(d) = -(a^2c + ea)/2 + \sum_{i=1}^r a_i(a_i - 1)/2 + cad - \sum_{i=1}^r a_i a_i d_i + ea \). We copy [3], Lemma 2.1, given in [2] does not use the assumption \( S = P^2 \). In our set-up \( g(a) := 1 + (a^2c + ea)/2 - \sum_{i=1}^r a_i(a_i - 1)/2 \) and we use \( \chi(O_C(d) \) instead of \( da + 1 - g \) in the numerical computation in [3], p. 291. We use \( R + d \) (resp. \( R + d - a \), resp. \( a \)) instead of the line bundles \( d \) (resp. \( d - a \), resp. \( a \)) used in [3]. Notice that \( h^i(C, \mathcal{O}_C(d)) = 0 \) when \( d \gg a \).

The computations in [3], pp. 291–295, and the check of the assumptions 2) and 3) of [2], Lemma 3.1, are OK because they require only the quadratic part of the Riemann-Roch formula for \( \chi(R + d) \). \( \chi(R + d - a) \) and \( \chi(a) \). Similar numerical problems were checked in arbitrary dimension in [1]. Condition (+) is used to copy the proof of [3], Prop. 4.1, using a smooth curve \( L \in |H| \) instead of a line.

(b) For every very ample \( L \in \text{Pic}(S) \) and every integer \( t \geq 2 \) the line bundle \( L^{\otimes t} \) is very ample and satisfies Condition (+). Take \( H \) ample. There is an integer \( x > 0 \) such that \( H^{\otimes x} \) is very ample, satisfies Condition (+) and \( h^i(S, H^{\otimes t}) = h^i(S, R \otimes H^{\otimes t}) = 0 \) for \( i = 1, 2 \) and all \( t \geq x + 1 \). We may apply the previous proof taking \( \tilde{H} := H^{\otimes x} \) with respect to the \( x \) line bundles \( R \otimes H^{\otimes t}, 0 \leq i \leq x - 1 \). \( \square \)

We work over an algebraically closed field \( K \) with \( \text{char}(K) = 0 \).
Multiple points

REFERENCES


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