Int. J. Contemp. Math. Sciences, Vol. 2, 2007, no. 6, 291 - 296

Convergence of external rays in parameter spaces of symmetric polynomials

Ahmad Zireh

Department of Mathematics, Shahrood University
of Technology, P.O.Box 316-36155 Shahrood, Iran
azireh@gmail.com

Abstract

We consider the complex dynamics a one parametric family of polynomials $f_c(z) = z(z^d + c)$, where $d \geq 1$ is a given integer and $c \in \mathbb{C}$. In the dynamics of quadratic polynomials $P_c(z) = z^2 + c$, Douady-Hubbard[1, 2] have proved that periodic rational external rays land on a parameter in the Mandelbrot set. This result has been extended to the parameter space of uni-critical polynomials $g_c(z) = z^d + c$[6]. We extend special case of the Douady-Hubbard Theorem to parameter space of *symmetric polynomials* $f_c(z) = z^d + cz$. Note that when $d = 1$, f_c is just a quadratic polynomial.

Mathematics Subject Classification: Primary: 32H50, Secondary: 37F45, 30D05

Keywords: Mandelbrot set, Multibrot set, connectedness locus, polynomial dynamics, symmetric polynomials, external rays.

1 Introduction

We first recall some terminology and definitions in holomorphic dynamics. let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial self-map of the complex plan. For each $z \in \mathbb{C}$, the orbit of z is

$$\mathrm{Orb}_f(z) = \{z, f(z), f(f(z)), \cdots, f^n(z), \cdots\}.$$

The dynamical plane $\mathbb{C}$ is decomposed into two complementary sets: the *filled Julia set*

$$K(f) = \{c \in \mathbb{C} : \quad \mathrm{Orb}_f(z) \quad \text{is bounded}\},$$

and its complementary, the *basin of infinity*

$$A_f(\infty) = \mathbb{C} - K(f).$$

The boundary of $K(f)$, called the *Julia set*, is denoted by $J(f)$.
When f is a quadratic polynomial, say $f(z) = P_c = z^2 + c$, in the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the point at infinity is a super-attracting fixed point and hence there exists U_c a neighborhood of infinity and a conformal map(the Böttcher map),

$$\varphi_c : U_c \to \{z \in \mathbb{C}; |z| > 1\},$$

such that $\varphi_c(\infty) = \infty, \varphi_c'(\infty) = 1$ and $\varphi_c \circ P_c = (\varphi_c)^2$. For $\theta \in \mathbb{R}/\mathbb{Z}$, the external ray of argument θ is defined as

$$R_c(\theta) = \varphi_c^{-1}\{re^{2\pi i\theta}; r > 1\}.$$

A classical result shows that the inverse, Riemann map, φ^{-1} extends continuously to a map from the closed disk if and only if the Julia set J_c is locally connected.
The *Mandelbrot set* $\mathcal{M}_2$ is defined as the set of parameter value c, for which $K(P_c)$ is connected

$$\mathcal{M}_2 = \{c \in \mathbb{C} : K(P_c) \quad \text{is connected}\}.$$

Let $\Phi : \hat{\mathbb{C}} - \mathcal{M}_2 \to \hat{\mathbb{C}} - \overline{D}$ denote the unique *Riemann map* tangent to the identity at infinity(D is open unit disk), then $\Phi(c) = \varphi_c(c)$. Given $\theta \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $c \in \mathbb{C}$: the dynamical(external) ray $R_c(\theta)$ of J_c starts at ∞ and either ends at a precritical point $z_0 \in \bigcup_{n\geq 0} P_c^{-n}(c)$ or ends by accumulating on some subset of the Julia set J_c. The parameter(external) ray $R_{M_2}(\theta)$ of $\mathcal{M}_2$ is the analytic arc $\Phi^{-1}(re^{2\pi i\theta})$. A ray is called a rational ray if θ is rational, i.e. $\theta \in \mathbb{Q}/\mathbb{Z}$. A ray R is said to land or converge, if the accumulation set $\overline{R} - R$ is a singleton subset of J.
The conjecture that the Mandelbrot set is locally connected is equivalent to the continuous landing of all external rays.

The following substantial result have been obtained:

Douady-Hubbard's Theorem[1, 2]. *If $\theta \in \mathbb{Q}/\mathbb{Z}$ is rational, then the external ray $R_{M_2}(\theta)$ for the Mandelbrot set lands on a parameter $c \in \partial M_2$.*

The extension of the theorem to other classes of polynomials constitutes part of today's research in this area. For instance, Douady-Hubbard's Theorem has been extended to uni-critical polynomials $g_c(z) = z^d + c$ [6].

We assume that $\mathcal{C}_d = \{c \in \mathbb{C}; c_0 \notin A_c(\infty)\}$ is the *connectedness locus* of the family f_c. We shall see some properties of $\mathcal{C}_d$, among them the following main result:

Main result. *Let $\omega = e^{2\pi i/d}$ then the external rays*

$$R_0 = R_{\mathcal{C}_d}(\pm\frac{1}{2(d+1)}), \omega R_0, \cdots, \omega^{d-1}R_0$$

land at root points of the connectedness locus $\mathcal{C}_d$.

2 Symmetric polynomials

To better understand the major contribution, we provide in this section the necessary preliminaries, including a symmetry condition, for the family of polynomials of degree d. This condition is similar to one investigated by Milnor [5] who introduced quadratic rational maps with symmetries and derived the one-parameter family $f_k(z) = k(z + z^{-1})$ instead of the general family of all quadratic rational maps.

From now on, $d \geq 2$ will be a fixed integer. In the study of monic centered polynomials $f(z) = z^{d+1} + a_{d-1}z^{d-1} + \cdots + a_0$, with the parameter space $\mathbb{C}^d$, we can restrict ourselves to a one parameter family, just as Milnor did. For this purpose, we define an *automorphism of* $f(z) = z^{d+1} + a_{d-1}z^{d-1} + \cdots + a_0$, as a non-constant linear map $R(z) = az + b; \quad a, b \in \mathbb{C}$ which commutes with f, i.e., $R \circ f \circ R^{-1} = f$. The collection of all automorphisms of f, denoted by Aut (f), forms a finite subgroup of the rotation group $\Sigma_d = \{\gamma \in \mathbf{C}; \gamma^d = 1\}$. In some particular cases, the groups Aut(f) and Σ_d are equal. The following lemma characterizes all such polynomials.

Lemma 2.1 *For a monic centered polynomial $f(z) = z^{d+1} + a_{d-1}z^{d-1} + \cdots + a_0$, the following are equivalent:*
(i) Aut(f)=$\sum_d$
(ii) There exists $c \in \mathbb{C}$ such that $f(z) = f_c(z) = z(z^{d-1} + c)$ for all, $z \in \mathbf{C}$.
(iii) The polynomial f vanishes at the origin and the critical set $Z(f') = \{z \in \mathbb{C}; f'(z) = 0\}$ is stable under the action of $\sum_d$.

The proof is straightforward and is omitted.

Definition 2.2 *A $d-$symmetric polynomial, or it symmetric polynomial if there is no confusion, is a polynomial of the form $f_c(z) = z(z^d + c), c \in \mathbb{C}$.*

Remark. A $d-$symmetric polynomial $f_c(z) = z(z^d + c), c \in \mathbb{C}$ has d symmetric critical points (counting with multiplicity when $c = 0$):

$$c_0, c_1 = \omega c_0, \cdots, c_{d-1} = \omega^{d-1}c_0$$

where $\omega = e^{2\pi i/d}$ and c_0 is one of the solutions of $(d+1)z^d + c = 0$, denoted

$$c_0 = (\frac{-c}{d+1})^{1/d}; \tag{1}$$

accordingly, it has d symmetric critical values

$$\upsilon_0 = f_c(c_0) =, \upsilon_1 = \omega \upsilon_0, \ldots, \upsilon_{d-1} = \omega^{d-1} \upsilon_0. \tag{2}$$

Notations. We have already fixed the number $d \geq 2$. For each $c \in \mathbb{C}$, let $A_c = A_c(\infty) = \{z \in \mathbb{C}; \lim_{n\to\infty} f_c^n(z) = \infty\}$ is the domain of attraction at infinity of f_c, K_c is the *filled Julia set* of f_c,i.e.

$$K_c = \{z \in \mathbb{C}; \{f_c^n(z)\}_{n\in\mathbb{N}} \quad \text{is bounded}\},$$

and $J_c = \partial K_c$ is the *Julia set* of f_c. denote $\overline{\mathbf{D}}(\mathbf{r})$ is a closed disk with radius r around origin, and $\overline{\mathbf{D}}$ is the closed unit disk. Let $\psi_c : \mathbb{C} - K_c \to \mathbb{C} - \overline{D}$ denote the Böttcher coordinate for f_c, i.e. conjugating f_c to $z \to z^{d+1}$ and tangent to the identity at infinity.

3 Properties of the connectedness locus

We first observe that the connectedness locus is bounded and symmetric with respect to the action of Σ_d. More precisely, with the notations above, we have the following.

Proposition 3.1 *For each $d \geq 2$, there exists a real number $1 < \alpha < 2$ such that*

$$\mathcal{C}_d = \{c \in \mathbb{C}; \quad |f_c^n(c_0)| \leq (1 + \frac{d+1}{d}\alpha)^{1/d} \quad \textit{for every} \quad n \in \mathbb{N}\}.$$

Proof. For $|z| > (1 + |c|)^{1/d}$, we have $|f_c(z)| \geq |z|(|z|^d - |c|) > |z|$. It follows that

$$\{z; |z| > (1 + |c|)^{1/d}\} \subset A_c(\infty).$$

Now if α is the unique positive root of the polynomial $g(t) = t^{d+1} - (d+1)t - d$, then for $|c| > \frac{d+1}{d}\alpha$, we have

$$|f_c(c_0)| = \frac{d|c|}{d+1}(\frac{|c|}{d+1})^{1/d} > (1 + |c|)^{1/d}.$$

Henceforth, $c_0 \in A_c(\infty)$. □

Proposition 3.2 *The set $\mathcal{C}_d$ is invariant under the action of the group Σ_d.*

Proof. Let $c \in \mathcal{C}_d, \omega = e^{2\pi i/d}$ and c_0 a critical point of f_c. Then $\omega c \in \mathcal{C}_d$. Indeed, $\omega^{1/d} c_0$ is a critical point of $f_{\omega c}$, the corresponding critical values being related by

$$f_{\omega c}(\omega^{1/d} c_0) = \omega^{1/d} c_0(\omega c_0^d + \omega c) = \omega^{\frac{1}{d}+1} c_0(c_0^d + c) = \omega^{\frac{1}{d}+1} f_c(c_0),$$

hence $|f_{\omega c}(\omega^{1/d} c_0)| = |f_c(c_0)|$. In view of the proposition 3.1, the value α depends only on degrees and the polynomials f_c and $f_{\omega c}$ have the same degree. This completes the proof. □

4 Main Result

It is proved [3] that the connectedness locus $\mathcal{C}_d$ is connected, precisely there exist the Riemann map,

$$\Psi : \hat{\mathbb{C}} - \mathcal{C}_d \to \hat{\mathbb{C}} - \overline{D}$$

$$\Psi(c) = \psi_c(\upsilon_0).$$

Now we are going to prove the landing theorem in the connectedness locus.

Theorem 4.1 *The external rays $R_{\mathcal{C}_d}(\pm\frac{1}{2(d+1)})$ land at $c = 1$, the root point of the connectedness locus.*

Proof. Note that if the ray $R_c(0)$ does not branch then it must land at a fixed point. The situation is stable in c if the fixed point is repelling. It is unstable either if the fixed point is indifferent in which case it is parabolic of multiplier 1, or if the ray $R_c(0)$ branches at the critical points. The stable set is open in the parameter space and the unstable set is closed in the same space.

The ray $R_c(0)$ passes through critical values if and only if

$$Arg(\psi_c(\upsilon_0)) = 0.$$

From equation(1) it follows that this is equivalent to

$$c \in R_{\mathcal{C}_d}(\frac{n}{d} \pm \frac{1}{2(d+1)}), \quad n \in \mathbb{Z}$$

i.e., restricted to the one part of the connectedness locus we have $c \in R_{\mathcal{C}_d}(\pm\frac{1}{2(d+1)})$.

For $c = 1 + \frac{1}{d}$ the polynomial f_c has a superattracting fixed point and $\mathbb{R}_+ \subset \mathbb{C} - K_c$. By symmetry, $R_c(0) = \mathbb{R}_+$ and $R_c(0)$ lands at fixed point $z = 0$.

There exists $\rho > 1$ so that the polynomials f_c for $c = \pm\rho e^{(\pi i/d)} \in \partial\mathcal{C}_d$ have a superattracting cycle of period two with $c_0 < 0 < \upsilon_0$. By symmetry, $R_c(0) \subset \mathbb{R}_+$ but $R_c(0)$ does not land at $z = 0$.

For c in the parameter space the situation can change only if $c = 1$ for which 0 is a parabolic fixed point of multiplier 1, or if $c \in R_{\mathcal{C}_d}(\pm\frac{1}{2(d+1)})$ for which the ray $R_c(0)$ branches at c_0. □

Now from theorem 4.1 and proposition 3.2, we have:

Corollary 4.2 *Let* $\omega = e^{2\pi i/d}$ *then the external rays*

$$R_0 = R_{\mathcal{C}_d}(\pm\frac{1}{2(d+1)}), \omega R_0, \cdots, \omega^{d-1} R_0$$

land at root points of the connectedness locus $\mathcal{C}_d$*.*

References

[1] A. Douady and J. Hubbard, Iterations des Polynomes Quadratiques Complexes , *C.R. Acad. Sci., Paris, Ser. I,* **29** (1982), 123-126.

[2] A. Douady and J. Hubbard, Etude dynamique des polynomes complexes, *Prepublications mathemathiques d'Orsay,* no. 2/4 (1984 / 1985).

[3] A. Chademan and A. Zireh, Extension of the Douady-Hubbard's Theorem on Connecteness of the Mandelbrot set to Symmetric Polynomials, *Bull. of Iranian Math. Soc,* **31**, no. 1 (2005), 77-84.

[4] J. Milnor, *Dynamics in one complex variable*, American Mathematical Society, Providence 2000.

[5] J. Milnor, Geometry and dynamics of quadratic rational maps, *Experimental Math*, **2** , no. 1 (1993), 37-83.

[6] J. Peterson and G. Ryd, Convergence of rational rays in parameter spaces, *The Mandelbrot set, theme and variations, London Math. Soc. Lecture Note Ser.,***274**, Cambridge Univ. Press, Cambridge,(2000), 161-172.

Received: July 26, 2006