Convergence of external rays in parameter spaces of symmetric polynomials

Ahmad Zireh

Department of Mathematics, Shahrood University of Technology, P.O.Box 316-36155 Shahrood, Iran
azireh@gmail.com

Abstract

We consider the complex dynamics a one parametric family of polynomials \( f_c(z) = z(z^d + c) \), where \( d \geq 1 \) is a given integer and \( c \in \mathbb{C} \). In the dynamics of quadratic polynomials \( P_c(z) = z^2 + c \), Douady-Hubbard[1, 2] have proved that periodic rational external rays land on a parameter in the Mandelbrot set. This result has been extended to the parameter space of uni-critical polynomials \( g_c(z) = z^d + c \)[6]. We extend special case of the Douady-Hubbard Theorem to parameter space of symmetric polynomials \( f_c(z) = z^d + cz \). Note that when \( d = 1 \), \( f_c \) is just a quadratic polynomial.

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1 Introduction

We first recall some terminology and definitions in holomorphic dynamics. Let \( f : \mathbb{C} \to \mathbb{C} \) be a polynomial self-map of the complex plan. For each \( z \in \mathbb{C} \), the orbit of \( z \) is

\[
\text{Orb}_f(z) = \{ z, f(z), f(f(z)), \cdots, f^n(z), \cdots \}.
\]

The dynamical plane \( \mathbb{C} \) is decomposed into two complementary sets: the filled Julia set

\[
K(f) = \{ c \in \mathbb{C} : \text{Orb}_f(z) \text{ is bounded} \},
\]

and its complementary, the basin of infinity

\[
A_f(\infty) = \mathbb{C} - K(f).
\]
The boundary of $K(f)$, called the Julia set, is denoted by $J(f)$.
When $f$ is a quadratic polynomial, say $f(z) = P_c = z^2 + c$, in the extended complex plane $\hat{C} = \mathbb{C} \cup \{\infty\}$, the point at infinity is a super-attracting fixed point and hence there exists $U_c$ a neighborhood of infinity and a conformal map (the Böttcher map),

$$\varphi_c : U_c \to \{ z \in \mathbb{C}; |z| > 1 \},$$

such that $\varphi_c(\infty) = \infty, \varphi'_c(\infty) = 1$ and $\varphi_c \circ P_c = (\varphi_c)^2$. For $\theta \in \mathbb{R}/\mathbb{Z}$, the external ray of argument $\theta$ is defined as

$$R_c(\theta) = \varphi_c^{-1}\{ re^{2\pi i\theta}, r > 1 \}.$$

A classical result shows that the inverse, Riemann map, $\varphi^{-1}$ extends continuously to a map from the closed disk if and only if the Julia set $J_c$ is locally connected.

The Mandelbrot set $M_2$ is defined as the set of parameter value $c$, for which $K(P_c)$ is connected

$$M_2 = \{ c \in \mathbb{C} : K(P_c) \text{ is connected} \}.$$

Let $\Phi : \hat{C} - M_2 \to \hat{C} - \bar{\mathcal{T}}$ denote the unique Riemann map tangent to the identity at infinity ($\mathcal{T}$ is open unit disk), then $\Phi(c) = \varphi_c(c)$. Given $\theta \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $c \in \mathbb{C}$: the dynamical (external) ray $R_c(\theta)$ of $J_c$ starts at $\infty$ and either ends at a precritical point $z_0 \in \bigcup_{n \geq 0} P_c^{-n}(c)$ or ends by accumulating on some subset of the Julia set $J_c$. The parameter (external) ray $R_{M_2}(\theta)$ of $M_2$ is the analytic arc $\Phi^{-1}(re^{2\pi i\theta})$. A ray is called a rational ray if $\theta$ is rational, i.e. $\theta \in \mathbb{Q}/\mathbb{Z}$. A ray $R$ is said to land or converge, if the accumulation set $\mathcal{T} - R$ is a singleton subset of $J$.

The conjecture that the Mandelbrot set is locally connected is equivalent to the continuous landing of all external rays.

The following substantial result have been obtained:

**Douady-Hubbard’s Theorem**[1, 2]. If $\theta \in \mathbb{Q}/\mathbb{Z}$ is rational, then the external ray $R_{M_2}(\theta)$ for the Mandelbrot set lands on a parameter $c \in \partial M_2$.

The extension of the theorem to other classes of polynomials constitutes part of today’s research in this area. For instance, Douady-Hubbard’s Theorem has been extended to uni-critical polynomials $g_c(z) = z^d + c$ [6].

We assume that $C_d = \{ c \in \mathbb{C}; c_0 \notin A_c(\infty) \}$ is the connectedness locus of the family $f_c$. We shall see some properties of $C_d$, among them the following main result:
Main result. Let $\omega = e^{2\pi i/d}$ then the external rays

$$R_0 = R_{C_d}(\pm \frac{1}{2(d+1)}), \omega R_0, \ldots, \omega^{d-1}R_0$$

land at root points of the connectedness locus $C_d$.

2 Symmetric polynomials

To better understand the major contribution, we provide in this section the necessary preliminaries, including a symmetry condition, for the family of polynomials of degree $d$. This condition is similar to one investigated by Milnor [5] who introduced quadratic rational maps with symmetries and derived the one-parameter family $f_k(z) = k(z + z^{-1})$ instead of the general family of all quadratic rational maps.

From now on, $d \geq 2$ will be a fixed integer. In the study of monic centered polynomials $f(z) = z^{d+1} + a_{d-1}z^{d-1} + \cdots + a_0$, with the parameter space $C^d$, we can restrict ourselves to a one parameter family, just as Milnor did. For this purpose, we define an automorphism of $f(z) = z^{d+1} + a_{d-1}z^{d-1} + \cdots + a_0$, as a non-constant linear map $R(z) = az + b; a, b \in C$ which commutes with $f$; i.e., $R \circ f \circ R^{-1} = f$. The collection of all automorphisms of $f$, denoted by $\text{Aut}(f)$, forms a finite subgroup of the rotation group $\Sigma_d = \{ \gamma \in C; \gamma^d = 1 \}$. In some particular cases, the groups $\text{Aut}(f)$ and $\Sigma_d$ are equal. The following lemma characterizes all such polynomials.

Lemma 2.1 For a monic centered polynomial $f(z) = z^{d+1} + a_{d-1}z^{d-1} + \cdots + a_0$, the following are equivalent:

(i) $\text{Aut}(f) = \Sigma_d$

(ii) There exists $c \in C$ such that $f(z) = f_c(z) = z(z^{d-1} + c)$ for all, $z \in C$.

(iii) The polynomial $f$ vanishes at the origin and the critical set $Z(f') = \{ z \in C; f'(z) = 0 \}$ is stable under the action of $\Sigma_d$.

The proof is straightforward and is omitted.

Definition 2.2 A $d-$symmetric polynomial, or it symmetric polynomial if there is no confusion, is a polynomial of the form $f_c(z) = z(z^d + c), c \in C$.

Remark. A $d-$symmetric polynomial $f_c(z) = z(z^d + c), c \in C$ has $d$ symmetric critical points (counting with multiplicity when $c = 0$):

$$c_0, c_1 = \omega c_0, \ldots, c_{d-1} = \omega^{d-1}c_0$$

where $\omega = e^{2\pi i/d}$ and $c_0$ is one of the solutions of $(d+1)z^d + c = 0$, denoted

$$c_0 = \left(\frac{-c}{d+1}\right)^{1/d}.$$
accordingly, it has $d$ symmetric critical values

$$v_0 = f_c(c_0) =, v_1 = \omega v_0, \ldots, v_{d-1} = \omega^{d-1} v_0.$$  

(2)

**Notations.** We have already fixed the number $d \geq 2$. For each $c \in \mathbb{C}$, let $A_c = A_c(\infty) = \{z \in \mathbb{C}; \lim_{n \to \infty} f^n_c(z) = \infty\}$ is the domain of attraction at infinity of $f_c$, $K_c$ is the filled Julia set of $f_c$, i.e.

$$K_c = \{z \in \mathbb{C}; \{f^n_c(z)\}_{n \in \mathbb{N}} \text{ is bounded}\},$$

and $J_c = \partial K_c$ is the Julia set of $f_c$. denote $\overline{D}(r)$ is a closed disk with radius $r$ around origin, and $\overline{D}$ is the closed unit disk. Let $\psi_c : \mathbb{C} - K_c \to \mathbb{C} - \overline{D}$ denote the Böttcher coordinate for $f_c$, i.e. conjugating $f_c$ to $z \to z^{d+1}$ and tangent to the identity at infinity.

## 3 Properties of the connectedness locus

We first observe that the connectedness locus is bounded and symmetric with respect to the action of $\Sigma_d$. More precisely, with the notations above, we have the following.

**Proposition 3.1** For each $d \geq 2$, there exists a real number $1 < \alpha < 2$ such that

$$C_d = \{c \in \mathbb{C}; |f^n_c(c_0)| \leq (1 + \frac{d + 1}{d - \alpha})^{1/d} \text{ for every } n \in \mathbb{N}\}.$$  

**Proof.** For $|z| > (1 + |c|)^{1/d}$, we have $|f_c(z)| \geq |z||z|^d - |c| > |z|$. It follows that

$$\{z; |z| > (1 + |c|)^{1/d}\} \subset A_c(\infty).$$

Now if $\alpha$ is the unique positive root of the polynomial $g(t) = t^{d+1} - (d+1)t - d$, then for $|c| > \frac{d+1}{d}\alpha$, we have

$$|f_c(c_0)| = \frac{d|c|}{d + 1} \left(\frac{|c|}{d + 1}\right)^{1/d} > (1 + |c|)^{1/d}.$$  

Henceforth, $c_0 \in A_c(\infty)$.

**Proposition 3.2** The set $C_d$ is invariant under the action of the group $\Sigma_d$. 

Proof. Let $c \in C_d, \omega = e^{2\pi i/d}$ and $c_0$ a critical point of $f_c$. Then $\omega c \in C_d$. Indeed, $\omega^{1/d} c_0$ is a critical point of $f_{\omega c}$, the corresponding critical values being related by

$$f_{\omega c}(\omega^{1/d} c_0) = \omega^{1/d} c_0 (\omega c_0^d + \omega c) = \omega^{1/d} c_0 (c_0^d + c) = \omega^{1/d} f_c(c_0),$$

hence $|f_{\omega c}(\omega^{1/d} c_0)| = |f_c(c_0)|$. In view of the proposition 3.1, the value $\alpha$ depends only on degrees and the polynomials $f_c$ and $f_{\omega c}$ have the same degree. This completes the proof. \qed

4 Main Result

It is proved [3] that the connectedness locus $C_d$ is connected, precisely there exist the Riemann map,

$$\Psi: \hat{C} - C_d \rightarrow \hat{C} - D$$

$$\Psi(c) = \psi_c(\nu_0).$$

Now we are going to prove the landing theorem in the connectedness locus.

Theorem 4.1 The external rays $R_{\omega c_d}(\pm \frac{1}{2(d+1)})$ land at $c = 1$, the root point of the connectedness locus.

Proof. Note that if the ray $R_c(0)$ does not branch then it must land at a fixed point. The situation is stable in $c$ if the fixed point is repelling. It is unstable either if the fixed point is indifferent in which case it is parabolic of multiplier 1, or if the ray $R_c(0)$ branches at the critical points. The stable set is open in the parameter space and the unstable set is closed in the same space.

The ray $R_c(0)$ passes through critical values if and only if

$$\text{Arg}(\psi_c(\nu_0)) = 0.$$ 

From equation (1) it follows that this is equivalent to

$$c \in R_{Cd}(\pm \frac{n}{d} \pm \frac{1}{2(d+1)}), \quad n \in \mathbb{Z}$$

i.e., restricted to the one part of the connectedness locus we have $c \in R_{Cd}(\pm \frac{1}{2(d+1)})$.

For $c = 1 + \frac{1}{d}$ the polynomial $f_c$ has a superattracting fixed point and $\mathbb{R}_+ \subset \mathbb{C} - K_c$. By symmetry, $R_c(0) = \mathbb{R}_+$ and $R_c(0)$ lands at fixed point $z = 0$.

There exists $\rho > 1$ so that the polynomials $f_c$ for $c = \pm \rho e^{(\pi i/d)} \in \partial C_d$ have a superattracting cycle of period two with $c_0 < 0 < \nu_0$. By symmetry, $R_c(0) \subset \mathbb{R}_+$ but $R_c(0)$ does not land at $z = 0$. 

Convergence of external rays
For $c$ in the parameter space the situation can change only if $c = 1$ for which $0$ is a parabolic fixed point of multiplier $1$, or if $c \in R_{C_d} \left( \pm \frac{1}{2(d+1)} \right)$ for which the ray $R_c(0)$ branches at $c_0$. □

Now from theorem 4.1 and proposition 3.2, we have:

**Corollary 4.2** Let $\omega = e^{2\pi i/d}$ then the external rays

$$R_0 = R_{C_d} \left( \pm \frac{1}{2(d+1)} \right), \omega R_0, \cdots, \omega^{d-1} R_0$$

land at root points of the connectedness locus $C_d$.

**References**


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