Fuzzy Programming Based on Interval-Valued Fuzzy Numbers and Ranking

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Abstract

In this paper, we use interval-valued fuzzy numbers to fuzzify the crisp linear programming to three cases. The first case, we use interval-valued fuzzy numbers to fuzzify the coefficients in the objective function. We get a linear programming in the fuzzy sense. The second case, we use interval-valued fuzzy numbers to fuzzify the coefficients $a_{kj}$ in the constraints about $x_j, j = 1, 2, \ldots, n$ and the constants $b_k, k = 1, 2, \ldots, m$. We also get a linear programming in the fuzzy sense. The third case, we combine the first and the second cases.

Keywords: Fuzzy programming, fuzzy objective function, interval-valued fuzzy numbers, fuzzy sense

1 Introduction

In paper [1,6,8], they use fuzzy number to fuzzify the crisp linear programming. They do not use interval-valued fuzzy numbers to fuzzify. In [1], for crisp linear programming, the constraints equations are $\sum_{j=1}^{n} a_{kj} x_j \leq b_k, k = 1, 2, \ldots, m$.

They use fuzzy number $\tilde{a}_{kj}, \tilde{b}_k$ to fuzzify and get the fuzzy numbers inequality $\sum_{j=1}^{n} \tilde{a}_{kj} x_j \preceq \tilde{b}_k, k = 1, 2, \ldots, m$. Then they use ranking of fuzzy numbers to get linear programming in the fuzzy sense. They do not defuzzify the objective function and did not use interval-valued fuzzy numbers to defuzzify. In [6], they use trapezoidal fuzzy numbers to fuzzify $c_j, a_{kj}, b_k$ as $\tilde{c}_j, \tilde{a}_{kj}, \tilde{b}_k$. Then get $\tilde{Z} = \sum_{j=1}^{n} \tilde{c}_j x_j, \sum_{j=1}^{n} \tilde{a}_{kj} x_j \preceq \tilde{b}_k, k = 1, 2, \ldots, m$. They reduce it to linear programming in the fuzzy sense. They do not use interval-valued fuzzy numbers to discuss. In this paper, we use interval-valued fuzzy numbers to consider this
problem. In §2, we discuss interval-valued fuzzy numbers and their ranking which will be used in §3, 4. In §3, for crisp linear programming under constraints is as following: \[ \sum_{j=1}^{n} a_{kj}x_j \leq b_k, k = 1, 2, \ldots, m, x_j \geq 0, j = 1, 2, \ldots, n \]
we find optimal solution of objective function \[ Z = \sum_{j=1}^{n} c_jx_j. \] In monopoly market, the price \( c_j, j = 1, 2, \ldots, n \) can be determined by the factory. If \( a_{kj}, b_k, j = 1, 2, \ldots, n, k = 1, 2, \ldots, m \) do not vary in the plan period \( T \), but \( c_j \) in the plan period \( T \) for a perfect competitive market may fluctuate a little, we need to fuzzify \( c_j \) to \( \tilde{c}_j \). In this plan period \( T \), the grade of membership is not necessary equal to 1. We suppose that the grade of membership lies in the interval \( [\lambda, 1] \), \( 0 < \lambda < 1 \). We set \( \tilde{c}_j \) to be level \( (\lambda, 1) \) i-v fuzzy number. Through this, we get the linear programming in the fuzzy sense. This is stated in theorem 1.

In §3.3, we fuzzify \( a_{kj} \) and \( b_k, j = 1, 2, \ldots, n, k = 1, 2, \ldots, m. \) in the constraints for the crisp linear programming to interval-valued fuzzy numbers \( \tilde{a}_{kj}, \tilde{b}_k \) and get \[ \sum_{j=1}^{n} \tilde{a}_{kj}x_j \preceq \tilde{b}_k, k = 1, 2, \ldots, m. \] Using ranking of the interval-valued fuzzy numbers in §2, we have linear programming in the fuzzy sense. This is stated in theorem 2. In §3.4, we combine theorem 1 and 2 and obtain fuzzy objective function \( \tilde{Z} = \sum_{j=1}^{n} \tilde{c}_jx_j \) and fuzzy constraints \[ \sum_{j=1}^{n} \tilde{a}_{kj}x_j \preceq \tilde{b}_k, k = 1, 2, \ldots, m. \] Then we have a linear programming in the fuzzy sense. This is stated in theorem 3. In §4, we give an example and §5 we give the discussions.

2 Interval-Valued Fuzzy Numbers and Ranking

For the purpose to consider fuzzy programming based on interval-valued fuzzy numbers and ranking, we first consider the following:

**Definition 1** \( \tilde{a} \) is called a fuzzy point, if its membership function on \( R= (-\infty, +\infty) \) is

\[
\mu_{\tilde{a}}(x) = \begin{cases} 
1 & x = a \\
0 & x \neq a
\end{cases}
\]

(1)

**Definition 2** \( \tilde{C} \) is called a level \( \lambda \) fuzzy number, \( 0 < \lambda \leq 1 \), if its membership function is

\[
\mu_{\tilde{C}}(x) = \begin{cases} 
\frac{\lambda(x-a)}{b-a} & a \leq x \leq b \\
\frac{\lambda(c-x)}{c-b} & b \leq x \leq c \\
0 & \text{otherwise}
\end{cases}
\]

(2)

We denote \( \tilde{C} = (a, b, c; \lambda) \). When \( a = b = c, \lambda = 1 \) then \( (a, a, a; 1) = \tilde{a} \) is a fuzzy point.
**Definition 3**  A fuzzy set is called the level $\alpha$ fuzzy interval, $0 \leq \alpha \leq 1$ and denote it by $[a, b; \alpha]$, $a < b$, if its membership function is

$$
\mu_{[a, b; \alpha]}(x) = \begin{cases} 
\alpha & a \leq x \leq b \\
0 & \text{otherwise}
\end{cases}
$$

(3)

When $a = b, \alpha = 1$ then $[a, a; 1] = \tilde{a}$ is a fuzzy point.

**Definition 4** ([3]) An interval-valued fuzzy set (i-v fuzzy set for short) $\tilde{A}$ on $R$ is given by

$$
\tilde{A} \triangleq \{(x, [\mu_{\tilde{A}L}(x), \mu_{\tilde{A}U}(x)])\}, \ x \in R \text{ where } \mu_{\tilde{A}L} \text{ and } \mu_{\tilde{A}U} \text{ maps } R \text{ into } [0, 1] \text{ and } \mu_{\tilde{A}L} \leq \mu_{\tilde{A}U}, \forall x \in R.
$$

Denote $\mu_{\tilde{A}}(x) = [\mu_{\tilde{A}L}(x), \mu_{\tilde{A}U}(x)], x \in R$ or

$$
\tilde{A} = [\tilde{A}^{L}, \tilde{A}^{U}]
$$

(4)

Then the grade of membership of i-v fuzzy set $\tilde{A}$ at $x$ belongs to the interval $[\mu_{\tilde{A}L}(x), \mu_{\tilde{A}U}(x)]$

![Fig.1 level ($\lambda, \rho$) i-v fuzzy number](image)

Let

$$
\mu_{\tilde{A}L}(x) = \begin{cases} 
\frac{\lambda(x-a)}{b-a} & a \leq x \leq b \\
\frac{\lambda(c-x)}{c-b} & b \leq x \leq c \\
0 & \text{otherwise}
\end{cases}
$$

(5)

then $\tilde{A}^{L} = (a, b, c; \lambda)$ is called level $\lambda$ fuzzy number.

Let

$$
\mu_{\tilde{A}U}(x) = \begin{cases} 
\frac{\rho(x-p)}{b-p} & p \leq x \leq b \\
\frac{\rho(x-r)}{r-b} & b \leq x \leq r \\
0 & \text{otherwise}
\end{cases}
$$

(6)

then $\tilde{A}^{U} = (p, b, r; \rho)$, where $0 < \lambda \leq \rho \leq 1, p < a < b < c < r$. We get i-v fuzzy set $\tilde{A} \triangleq \{(x, [\mu_{\tilde{A}L}(x), \mu_{\tilde{A}U}(x)])\}, x \in R$. Denote $\tilde{A} = [(a, b, c; \lambda), (p, b, r; \rho)] = [\tilde{A}^{L}, \tilde{A}^{U}]$ and call $\tilde{A}$ level ($\lambda, \rho$) i-v fuzzy number.
The family of all level \((\lambda, \rho)\) i-v fuzzy numbers is defined as following, where \(0 < \lambda \leq \rho \leq 1\),

\[
F_{IN}(\lambda, \rho) = \{(a, b, c; \lambda), (p, b, r; \rho)| p < a < b < c < r, p, a, b, c, r \in R\}
\]  \(\text{(7)}\)

Let \(\tilde{A} = [(a, b, c; \lambda), (p, b, r; \rho)] = [\tilde{A}^L, \tilde{A}^U] \in F_{IN}(\lambda, \rho)\). From eqs.(5) and (6), we get the left and right endpoint of \(\alpha\)-cut as following:

\[
\text{if } 0 \leq \alpha < \lambda \text{ then } A^L_\alpha(a) = a + (b - a)\frac{\alpha}{\lambda}, \quad A^R_\alpha(a) = c - (c - b)\frac{\alpha}{\lambda}, \quad A^L_\alpha(p) = p + (b - p)\frac{\alpha}{\rho}, \quad A^R_\alpha(p) = r - (r - b)\frac{\alpha}{\rho}
\]  \(\text{(8)}\)

and if \(\lambda \leq \alpha \leq \rho\) then \(A^L_\alpha(p) = p + (b - p)\frac{\alpha}{\rho}, \quad A^R_\alpha(p) = r - (r - b)\frac{\alpha}{\rho}\)

Let \(\tilde{B} = [(d, e, g; \lambda), (u, e, w; \rho)] = [\tilde{B}^L, \tilde{B}^U] \in F_{IN}(\lambda, \rho)\). Through operations \(\oplus\) of level \(\lambda\) fuzzy numbers and level \(\rho\) fuzzy numbers, we can get the following

\[
\tilde{A}^L \oplus \tilde{B}^L = (a + d, b + e, c + g; \lambda), \quad \tilde{A}^U \oplus \tilde{B}^U = (p + u, b + e, r + w; \rho)
\]  \(\text{(9)}\)

**Definition 5** \(\tilde{A}, \tilde{B} \in F_{IN}(\lambda, \rho), k \in R\)

\[
\tilde{A} \oplus \tilde{B} = [\tilde{A}^L \oplus \tilde{B}^L, \tilde{A}^U \oplus \tilde{B}^U]
\]  \(\text{(10)}\)

\[
k\tilde{A} = [k\tilde{A}^L, k\tilde{A}^U]
\]  \(\text{(11)}\)

From eqs.(9)~(11), we have the following

**Property 1** Let \(\tilde{A} = [(a, b, c; \lambda), (p, b, r; \rho)], \tilde{B} = [(d, e, g; \lambda), (u, e, w; \rho)] \in F_{IN}(\lambda, \rho)\) then

\[
(1^0) \quad \tilde{A} \oplus \tilde{B} = [(a + d, b + e, c + g; \lambda), (p + u, b + e, r + w; \rho)]
\]

\[
(2^0) \quad \text{when } k > 0 \text{ then } k\tilde{A} = [(ka, kb, kc; \lambda), (kp, kb, kr; \rho)]
\]

\[
(3^0) \quad \text{when } k < 0 \text{ then } k\tilde{A} = [(kc, kb, ka; \lambda), (kr, kb, kp; \rho)]
\]

\[
(4^0) \quad \text{when } k = 0 \text{ then } k\tilde{A} = [(0, 0, 0; \lambda), (0, 0, 0; \rho)]
\]

with the similarly arguments as [9], we use signed distance to consider ranking. In order to consider ranking of \(F_{IN}(\lambda, \rho)\) in \(R\), we first consider ranking on \(R\).

**Definition 6** Let \(a, 0 \in R\), we define the signed distance \(d^*\) as \(d^*(a, 0) = a\).

The meaning of \(d^*\) is that when \(a > 0\), \(d^*(a, 0) = a > 0\), i.e. \(a\) is at the right of 0 and the distance from 0 is \(a\), when \(a < 0\), \(d^*(a, 0) = -a < 0\), i.e. \(a\) is at the left of 0 and the distance from 0 is \(-a\). Therefore, \(d^*(a, 0)\) is called the signed distance of \(a\) from 0.
The signed distance on $F_{IN}(\lambda, \rho)$, by definition 6, can be defined by the following: if $\tilde{A} = [(a, b, c, \lambda), (p, b, r, \rho)] = [\tilde{A}_L, \tilde{A}_U] \in F_{IN}(\lambda, \rho)$. The $\alpha$-level set of $\tilde{A} = [\tilde{A}_L, \tilde{A}_U] \in F_{IN}(\lambda, \rho)$ is defined as $\{x | \mu_{\tilde{A}_U}(x) \geq \alpha\} - \{x | \mu_{\tilde{A}_L}(x) > \alpha\}$, then by Decomposition Theorem and Fig.2 we have

$$\tilde{A} = \bigcup_{0 \leq \alpha < \lambda} ([A^L_\alpha(\alpha), A^L_\alpha(\alpha); \alpha] \cup [A^U_\alpha(\alpha), A^U_\alpha(\alpha); \alpha]) \cup (\bigcup_{\lambda \leq \alpha \leq \rho} [A^L_\alpha(\alpha), A^U_\alpha(\alpha); \alpha])$$

(12)

![Fig.2 signed distance of i-v fuzzy numbers](image)

We have following one-one onto mapping for each $\alpha$. When $0 \leq \alpha < \lambda$,

$$[A^L_\alpha(\alpha), A^L_\alpha(\alpha); \alpha](\text{corresponding PQ}) \longleftrightarrow [A^L_\alpha(\alpha), A^L_\alpha(\alpha)] = [P', Q'],$$

$$[A^L_\alpha(\alpha), A^U_\alpha(\alpha); \alpha](\text{corresponding RS}) \longleftrightarrow [A^L_\alpha(\alpha), A^U_\alpha(\alpha)] = [R', S'],$$

and $[A^L_\alpha(\alpha), A^L_\alpha(\alpha)] \cap [A^L_\alpha(\alpha), A^U_\alpha(\alpha)] = \emptyset$,

and when $\lambda \leq \alpha \leq \rho$,

$$[A^L_\alpha(\alpha), A^U_\alpha(\alpha); \alpha](\text{corresponding TW}) \longleftrightarrow [A^L_\alpha(\alpha), A^U_\alpha(\alpha)] = [T', W'].$$

From definition 6, we obtain when $0 \leq \alpha < \lambda$, $d^*(A^U_\alpha(\alpha), 0) = A^U_\alpha(\alpha)$, $d^*(A^L_\alpha(\alpha), 0) = A^L_\alpha(\alpha)$, $d^*(A^L_\alpha(\alpha), 0) = A^L_\alpha(\alpha)$, and $d^*(A^U_\alpha(\alpha), 0) = A^U_\alpha(\alpha)$. That is to say, the signed distances of $P', Q', R', S'$ from 0 are $A^U_\alpha(\alpha), A^L_\alpha(\alpha)$, $A^L_\alpha(\alpha)$, and $A^U_\alpha(\alpha)$. Therefore, the signed distance of interval $[A^L_\alpha(\alpha), A^L_\alpha(\alpha)]$ from 0 is $d^*([A^L_\alpha(\alpha), A^L_\alpha(\alpha)], 0)$. It can be defined as

$$\frac{1}{2}[d^*(A^L_\alpha(\alpha), 0) + d^*(A^L_\alpha(\alpha), 0)] = \frac{1}{2}[A^L_\alpha(\alpha) + A^L_\alpha(\alpha)]$$

$$= \frac{1}{2}[a + p + (b - a)\frac{\alpha}{\lambda} + (b - p)\frac{\alpha}{\rho}]$$
Similarly, \( d^*(\{A^U_r(\alpha), A^L_r(\alpha)\}, 0) = \frac{1}{2}[c + r - (c - b)\frac{\alpha}{\lambda} - (r - b)\frac{\alpha}{\rho}] \)

Since, \([P',Q'] \cap [R',S'] = \emptyset\), for the \(\alpha\)-cut of \(\tilde{A}\) on \(0 \leq \alpha < \lambda\), the signed distance of \([P',Q'] \cup [R',S']\) from 0, can be defined as

\[
\begin{align*}
& d^*([A^U_r(\alpha), A^L_r(\alpha)] \cup [A^L_r(\alpha), A^U_r(\alpha)], 0) \\
& = \frac{1}{2}[d^*([A^U_r(\alpha), A^L_r(\alpha)], 0) + d^*([A^L_r(\alpha), A^U_r(\alpha)], 0)] \\
& = \frac{1}{4}[a + p + c + r + (2b - a - c)\frac{\alpha}{\lambda} + (2b - p - r)\frac{\alpha}{\rho}]
\end{align*}
\]

This function is continuous on \(0 \leq \alpha < \lambda\) with respect to \(\alpha\). It follows that, by integration, we can find the average value.

\[
\begin{align*}
& \frac{1}{\lambda} \int_0^\lambda d^*([A^U_r(\alpha), A^L_r(\alpha)] \cup [A^L_r(\alpha), A^U_r(\alpha)], 0) d\alpha \\
& = \frac{1}{8}[a + c + 2b + 2p + 2r + (2b - p - r)\frac{\lambda}{\rho}] \quad (13)
\end{align*}
\]

Similarly, when \(\lambda \leq \alpha \leq \rho\),

\[
\begin{align*}
& d^*([A^U_r(\alpha), A^L_r(\alpha)], 0) \\
& = \frac{1}{2}[d^*([A^U_r(\alpha), 0], 0) + d^*([A^L_r(\alpha), 0])] \\
& = \frac{1}{2}[A^U_r(\alpha) + A^L_r(\alpha)] \\
& = \frac{1}{2}[p + r + (2b - p - r)\frac{\alpha}{\rho}]
\end{align*}
\]

This function is also continuous on \(\lambda \leq \alpha \leq \rho\) with respect to \(\alpha\). By the same reason, by integration, find the average value, \(\lambda < \rho\).

\[
\begin{align*}
& \frac{1}{\rho - \lambda} \int_\lambda^\rho d^*([A^U_r(\alpha), A^L_r(\alpha)], 0) d\alpha \\
& = \frac{1}{4}[2b + p + r + (2b - p - r)\frac{\lambda}{\rho}] \quad (14)
\end{align*}
\]

From eqs.(12)~(14) we define the signed distance of \(\tilde{A}\) from \(\tilde{0}\).

**Definition 7** Let \(\tilde{A} = [a, b, c; \lambda], (p, b, r; \rho)] \in F_{IN}(\lambda, \rho)\). The signed distance of \(\tilde{A}\) from \(\tilde{0}\) is defined as

\[
\begin{align*}
& d(\tilde{A}, \tilde{0}) \\
& = \frac{1}{\lambda} \int_0^\lambda d^*([A^U_r(\alpha), A^L_r(\alpha)] \cup [A^L_r(\alpha), A^U_r(\alpha)], 0) d\alpha \\
& \quad + \frac{1}{\rho - \lambda} \int_\lambda^\rho d^*([A^U_r(\alpha), A^L_r(\alpha)], 0) d\alpha \\
& = \frac{1}{8}[6b + a + c + 4p + 4r + 3(2b - p - r)\frac{\lambda}{\rho}]
\end{align*}
\]
Fuzzy Programming

\[ d(\tilde{A}, \tilde{0}) = \frac{1}{8} \left[ db + a + c + p + r \right] \]

By definition 7, we can define the ranking of \( F_{IN}(\lambda, \rho) \) as following:

**Definition 8** Let \( \tilde{A} = [(a, b, c; \lambda), (p, b, r; \rho)], \) \( \tilde{B} = [(d, e, g; \lambda), (u, e, w; \rho)] \) \( \in F_{IN}(\lambda, \rho), \)

\[ \tilde{B} \prec \tilde{A} \text{ iff } d(\tilde{B}, \tilde{0}) < d(\tilde{A}, \tilde{0}) \]

\[ \tilde{B} \approx \tilde{A} \text{ iff } d(\tilde{B}, \tilde{0}) = d(\tilde{A}, \tilde{0}) \]

From linear order property of \((R, <, =)\) and definition 8, we get the following property.

**Property 2** Let \( \tilde{A}, \tilde{B}, \tilde{C} \in F_{IN}(\lambda, \rho). \)

(a) \((F_{IN}(\lambda, \rho), \approx, \prec)\) satisfies the law of trichotomy, i.e., only one of \( \tilde{A} \prec \tilde{B}, \tilde{A} \approx \tilde{B}, \tilde{B} \prec \tilde{A} \) will occur.

(b) \((F_{IN}(\lambda, \rho), \approx, \prec)\) satisfies the following ordering relation

\[ (1^0) \quad \tilde{A} \preceq \tilde{A} \]

\[ (2^0) \quad \tilde{A} \preceq \tilde{B} \text{ and } \tilde{B} \preceq \tilde{A} \Rightarrow \tilde{A} \approx \tilde{B} \]

\[ (3^0) \quad \tilde{A} \preceq \tilde{B} \text{ and } \tilde{B} \preceq \tilde{C} \Rightarrow \tilde{A} \preceq \tilde{C} \]

From property 2, we known that "\( \approx, \prec, \)" is the linear order on \( F_{IN}(\lambda, \rho). \)

**Definition 9** Let \( \tilde{A}_n, n = 1, 2, 3, \cdots \) \( \tilde{B} \in F_{IN}(\lambda, \rho). \) If \( \tilde{A}_n \preceq \tilde{B} \forall n = 1, 2, \cdots, \) then we write \( \tilde{B} = \max_{n \in \{1, 2, 3, \cdots\}} \tilde{A}_n \)

3 Fuzzy objective function in linear programming based on interval-valued fuzzy numbers

3.1 Crisp linear programming

Consider the following crisp linear programming problem.

A factory produces \( n \) products \( X_j, j = 1, 2, \cdots, n. \) Each product requires \( m \) processes \( A_k, k = 1, 2, \cdots, m. \) Product \( X_j, \) through process \( A_k \) requires \( a_{kj} \) hours, \( k = 1, 2, \cdots, m, j = 1, 2, \cdots, n. \) Each process \( A_k \) provides \( b_k \) hours, \( k = 1, 2, \cdots, m. \) Let the quantity produced for \( X_j \) be \( x_j, j = 1, 2, \cdots, n. \) Then we get the following constraint functions

\[ \sum_{j=1}^{n} a_{kj} x_j \leq b_k, \quad k = 1, 2, \cdots, m. \]
In monopoly market, the monopolist can determine the sale price $c_j (> 0), \ j = 1, 2, \cdots, n$ and can get total income $Z = \sum_{j=1}^{n} c_j x_j$. Therefore, we have the following crisp linear programming objective function

Maximize \[ Z = \sum_{j=1}^{n} c_j x_j \] (15)

subject to:

\[ \sum_{j=1}^{n} a_{kj} x_j \leq b_k, \ k = 1, 2, \cdots, m \] (16)

\[ x_j \geq 0, \ j = 1, 2, \cdots, n \] (17)

Let

\[ L = \{(x_1, x_2, \cdots, x_n) | \sum_{j=1}^{n} a_{kj} x_j \leq b_k, k = 1, 2, \cdots, m, x_j \geq 0, j = 1, 2, \cdots, n\} \]

Obvious, $L$ is a closed bounded convex set. Under condition eqs.(16) and (17), monopolist can find out $x_j, \ j = 1, 2, \cdots, n$ which maximize the total income $Z$. This is a crisp linear programming problem. We can use simplex method to find the optimal solution. Suppose that this optimal solution is the production quantity $x_j^{(0)}, j = 1, 2, \cdots, n$. The total income $Z_0 = \sum_{j=1}^{n} c_j x_j^{(0)}$ is maximized. If in a plan period, $a_{kj}, b_k, c_j, j = 1, 2, \cdots, n, k = 1, 2, \cdots, m$ do not change, The result stays the same. That is to say, in this period, the optimal solutions of the product $X_j$ is the quantity $x_j^{(0)}, j = 1, 2, \cdots, n$.

In a perfect competitive market, the price $c_j$ in a plan period may fluctuate a little. We can fuzzify to $\tilde{c}_j$. In this plan period $T$, the grade of membership of $c_j$ is not necessarily equal to 1. We let the grade of membership of $c_j$ lie in the interval $[\lambda, 1], 0 < \lambda < 1$, (see Fig.3). Set $\tilde{c}_j$ to be level $(\lambda,1)$ i-v fuzzy number, $0 < \lambda < 1$.

\[ \tilde{c}_j = [(c_j - \delta_{j2}, c_j, c_j + \delta_{j3}; \lambda), (c_j - \delta_{j1}, c_j, c_j + \delta_{j4}; 1)], \ j = 1, 2, \cdots, n \] (18)

where $0 < \delta_{j2} < \delta_{j1} < c_j, \ 0 < \delta_{j3} < \delta_{j4}, \ j = 1, 2, \cdots, n$. 
3.2 Fuzzy objective function

We denote $(x_1\tilde{c}_1) \oplus (x_2\tilde{c}_2) \oplus \cdots \oplus (x_n\tilde{c}_n)$ as $\sum_{j=1}^{n} \tilde{c}_j x_j$. In eqs.(15)~(17), if we fuzzify $c_j$, $j = 1, 2, \ldots, n$ to level $(\lambda, 1)$ i-v fuzzy numbers in a crisp linear programming, we will have the following result.

**Theorem 1** In crisp linear programming eqs.(15)~(17), we fuzzify $c_j$, $j = 1, 2, \ldots, n$ to eq.(18), then we have

(a) Fuzzy programming

Maximize \[ \tilde{Z} = \sum_{j=1}^{n} \tilde{c}_j x_j \quad \text{(by definition 9)} \] (19)

subject to:

\[ \sum_{j=1}^{n} a_{kj} x_j \leq b_k, \ k = 1, 2, \ldots, m \] (20)

\[ x_j \geq 0, \ j = 1, 2, \ldots, n \] (21)

(b) Corresponding to (a), by definition 7, 8, 9 we get linear programming in the fuzzy sense as following:

Maximize \[ Z^* = \frac{1}{2} d(\tilde{Z}, 0) \]

\[ = \sum_{j=1}^{n} c_j x_j + \frac{1}{16} \sum_{j=1}^{n} [\delta_{j3} - \delta_{j2} + (4 - 3\lambda)(\delta_{j4} - \delta_{j1})] x_j \] (22)

subject to:

\[ \sum_{j=1}^{n} a_{kj} x_j \leq b_k, \ k = 1, 2, \ldots, m \] (23)
\[ x_j \geq 0, \quad j = 1, 2, \ldots, n \quad (24) \]

Proof: (a) It follows from eqs. (15), (18) and definition 9.
(b) Since \( x_j \geq 0, \quad j = 1, 2, \ldots, n \), by property 1, we get

\[
\tilde{Z} = \left( \sum_{j=1}^{n} (c_j - \delta_{j2})x_j, \sum_{j=1}^{n} c_jx_j, \sum_{j=1}^{n} (c_j + \delta_{j3})x_j; \lambda \right), \notag
\]

\[
\left( \sum_{j=1}^{n} (c_j - \delta_{j1})x_j, \sum_{j=1}^{n} c_jx_j, \sum_{j=1}^{n} (c_j + \delta_{j4})x_j; 1 \right) \]

Through definition 7, we obtain

\[
d(\tilde{Z}, \tilde{0}) = 2 \sum_{j=1}^{n} c_jx_j + \frac{1}{8} \sum_{j=1}^{n} \left[ \delta_{j3} - \delta_{j2} + (4 - 3\lambda)(\delta_{j4} - \delta_{j1}) \right] x_j \notag
\]

Using definition 8, 9 and putting them to eq. (19), we have eq. (22). This prove (b).

Remark 1 In eq. (22), when \( \delta_{j1} = \delta_{j2} = \delta_{j3} = \delta_{j4} = 0, \quad j = 1, 2, \ldots, n \), this equation reduces eq. (15), i.e., \( Z^* = Z \). Therefore, we take \( \frac{1}{2}d(\tilde{Z}, \tilde{0}) \) in eq. (22).

Remark 2 In theorem 1(b), eqs. (22) \( \sim \) (24), the linear programming in the fuzzy sense can be found by the simplex method (or using computer package) to find the optimal solution.

3.3 fuzzy constraints
Suppose the sale price \( c_j, \quad j = 1, 2, \ldots, n \) do not vary in the plan period \( T \).

Similarly to §3.1, §3.2, we consider constraints of eq. (16) \( \sum_{j=1}^{n} a_{kj}x_j \leq b_k, \quad k = 1, 2, \ldots, m \). We fuzzify both \( a_{kj}, b_k, \quad j = 1, 2, \ldots, n, \quad k = 1, 2, \ldots, m \) as the following interval-valued fuzzy numbers, \( 0 < \lambda < 1 \)

\[
\tilde{a}_{kj} = [(a_{kj} - \delta_{kj2}, a_{kj}), (a_{kj} + \delta_{kj3}; \lambda)], \quad (a_{kj} - \delta_{kj1}, a_{kj}, a_{kj} + \delta_{kj4}; 1) \quad (25)
\]

where \( 0 < \delta_{kj2} < \delta_{kj1} < a_{kj}, \quad 0 < \delta_{kj3} < \delta_{kj4} \quad \forall j, k \)

\[
\tilde{b}_k = [(b_k - \omega_{k2}, b_k, b_k + \omega_{k3}; \lambda), (b_k - \omega_{k1}, b_k, b_k + \omega_{k4}; 1)] \quad (26)
\]

where \( 0 < \omega_{k2} < \omega_{k1} < b_k, \quad 0 < \omega_{k3} < \omega_{k4} \quad k = 1, 2, \ldots, m \).

Theorem 2 In eqs. (15) \( \sim \) (17) of the crisp linear programming, if we fuzzify \( a_{kj}, b_k, \quad k = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n \) to level \( (\lambda, 1) \) i-v fuzzy numbers eqs. (25) and (26) then we have the following:

(a) Fuzzy programming

Maximize \( Z = \sum_{j=1}^{n} c_jx_j \)
subject to:

\[ \sum_{j=1}^{n} \tilde{a}_{kj}x_j \preceq \tilde{b}_k, \ k = 1, 2, \ldots, m \]  
(27)

\[ x_j \geq 0, \ j = 1, 2, \ldots, n \]  
(28)

(b) Corresponding to (a), by definition 7, 8, 9 we get the linear programming in the fuzzy sense as

Maximize \[ Z = \sum_{j=1}^{n} c_j x_j \]  
(29)

subject to:

\[ \sum_{j=1}^{n} a_{kj} x_j + \frac{1}{16} \sum_{j=1}^{n} [\delta_{kj3} - \delta_{kj2} + (4-3\lambda)(\delta_{kj4} - \delta_{kj1})] x_j \]

\[ \leq b_k + \frac{1}{16} [\omega_{k3} - \omega_{k2} + (4-3\lambda)(\omega_{k4} - \omega_{k1})], \ k = 1, 2, \ldots, m \]  
(30)

\[ x_j \geq 0, \ j = 1, 2, \ldots, n \]  
(31)

Proof: (b) Using definition 8 and putting into eq.(27), we have

\[ d(\sum_{j=1}^{n} \tilde{a}_{kj}x_j, \tilde{0}) \leq d(b_k, \tilde{0}), \ k = 1, 2, \ldots, m. \]  
From definition 7, we get eq.(30).

3.4 Fuzzy objective function and Fuzzy constraints

Combining §3.2 and §3.3, we have the following result.

Theorem 3 In eqs.(15)~(17) of the crisp linear programming, if we fuzzify \( c_j, a_{kj}, b_k, \ j = 1, 2, \ldots, n, \ k = 1, 2, \ldots, m, \) to level \((\lambda,1)\) i-v fuzzy numbers eqs.(18)(25)(26), then we obtain

(a) Fuzzy programming

Maximize \[ \tilde{Z} = \sum_{j=1}^{n} \tilde{c}_j x_j \]  
(32)

subject to:

\[ \sum_{j=1}^{n} \tilde{a}_{kj}x_j \preceq \tilde{b}_k, \ k = 1, 2, \ldots, m \]  
(33)

\[ x_j \geq 0, \ j = 1, 2, \ldots, n \]  
(34)

(b) Corresponding to (a), by definition 7, 8, 9 we get the linear programming in the fuzzy sense as

Maximize \[ Z^* = \sum_{j=1}^{n} c_j x_j + \frac{1}{16} \sum_{j=1}^{n} [\delta_{j3} - \delta_{j2} + (4-3\lambda)(\delta_{j4} - \delta_{j1})] x_j \]  
(35)
subject to:

\[
\sum_{j=1}^{n} a_{kj}x_j + \frac{1}{16} \sum_{j=1}^{n} [\delta_{kj3} - \delta_{kj2} + (4 - 3\lambda)(\delta_{kj4} - \delta_{kj1})]x_j \\
\leq b_k + \frac{1}{16} [\omega_{k3} - \omega_{k2} + (4 - 3\lambda)(\omega_{k4} - \omega_{k1})], \quad k = 1, 2, \ldots, m \quad (36)
\]

\[
x_j \geq 0, \quad j = 1, 2, \ldots, n \quad (37)
\]

4 Examples

A factory produces automobiles and truck. Each requires three processes. The production condition are given in table 1.

<table>
<thead>
<tr>
<th>type</th>
<th>process1</th>
<th>process2</th>
<th>process3</th>
<th>profit hundred dollars</th>
</tr>
</thead>
<tbody>
<tr>
<td>automobil</td>
<td>15</td>
<td>24</td>
<td>21</td>
<td>25</td>
</tr>
<tr>
<td>truck</td>
<td>30</td>
<td>6</td>
<td>14</td>
<td>48</td>
</tr>
<tr>
<td>total hour</td>
<td>45000</td>
<td>24000</td>
<td>28000</td>
<td></td>
</tr>
</tbody>
</table>

Let the quantity of automobiles and truck produced be \(x_1\) and \(x_2\). Then we have the following crisp linear programming

Maximize \[ Z = 25x_1 + 48x_2 \ (\text{hundred dollars}) \quad (38) \]

subject to:

\[
15x_1 + 30x_2 \leq 45000 \\
24x_1 + 6x_2 \leq 24000 \quad (39)
\]

\[
21x_1 + 14x_2 \leq 28000 \\
x_1 \geq 0, \quad x_2 \geq 0 \quad (40)
\]
In this figure, A(0, 1500), B(500, 1250), C(800, 800) and D(1000, 0).

[A] Crisp case

From Fig.4, the optimal solution of the crisp linear programming eqs. (38)∼(40) are among the points A,B,C,D. Therefore, we get $x_1 = 500(\equiv x_1^{(0)})$ and $x_2 = 1250(\equiv x_2^{(0)})$ which will maximize the profit $Z = 72500(\equiv z^{(0)})$.

[B] Fuzzy case

Case 2.1. Let $\delta_{11} = 7$, $\delta_{12} = 6$, $\delta_{13} = 8$, $\delta_{14} = 9$, $\delta_{21} = 5$, $\delta_{22} = 4$, $\delta_{23} = 6$, $\delta_{24} = 8$, $\lambda = 0.9$.

(B.1) From theorem 1(b).

Maximize $Z^* = 25x_1 + 48x_2 + \frac{1}{16}[4.6x_1 + 5.9x_2]$ subject to:

\[
\begin{align*}
15x_1 + 30x_2 & \leq 45000 \\
24x_1 + 6x_2 & \leq 24000 \\
21x_1 + 14x_2 & \leq 28000 \\
x_j & \geq 0, \quad j = 1, 2
\end{align*}
\]

Since the constraints are the same as the crisp case, from Fig.4, we need only to consider points A,B,C,D where $Z^*$ is the maximum. We have $x_1 = 500(\equiv x_1^{(1)})$, $x_2 = 1250(\equiv x_2^{(1)})$ and the maximum profit $Z^* = 73104.687$ (hundred dollars).
Case 2.2. Let

\[
\begin{align*}
\delta_{111} &= 5, & \delta_{112} &= 1, & \delta_{113} &= 2, & \delta_{114} &= 3 \\
\delta_{121} &= 7, & \delta_{122} &= 5, & \delta_{123} &= 4, & \delta_{124} &= 8 \\
\delta_{211} &= 4, & \delta_{212} &= 3, & \delta_{213} &= 2, & \delta_{214} &= 9 \\
\delta_{221} &= 4, & \delta_{222} &= 2, & \delta_{223} &= 2, & \delta_{224} &= 5 \\
\delta_{311} &= 5, & \delta_{312} &= 4, & \delta_{313} &= 1, & \delta_{314} &= 5 \\
\delta_{321} &= 6, & \delta_{322} &= 2, & \delta_{323} &= 5, & \delta_{324} &= 8 \\
\omega_{11} &= 30, & \omega_{12} &= 20, & \omega_{13} &= 30, & \omega_{14} &= 70 \\
\omega_{21} &= 60, & \omega_{22} &= 20, & \omega_{23} &= 50, & \omega_{24} &= 60 \\
\omega_{31} &= 50, & \omega_{32} &= 10, & \omega_{33} &= 30, & \omega_{34} &= 40
\end{align*}
\]

(B.2) From theorem 2(b).

Maximize \[ Z = 25x_1 + 48x_2 \]  \hspace{1cm} (41)

subject to:

\[
\begin{align*}
15x_1 + 30x_2 + \frac{1}{16}(-1.6x_1 + 0.3x_2) & \leq 45000 + \frac{1}{16} \hspace{1cm} (62) \\
24x_1 + 6x_2 + \frac{1}{16}(5.5x_1 + 1.3x_2) & \leq 24000 + \frac{1}{16} \hspace{1cm} (30) \hspace{1cm} (42) \\
21x_1 + 14x_2 + \frac{1}{16}(-3x_1 + 5.6x_2) & \leq 28000 + \frac{1}{16} \hspace{1cm} (7) \\
& x_1 \geq 0, \quad x_2 \geq 0 \hspace{1cm} (43)
\end{align*}
\]

From eq.(42) and (43), the closed bounded convex set L is the following:

\[
\begin{align*}
14.9x_1 + 30.01875x_2 & \leq 45003.875 \\
24.34375x_1 + 6.08125x_2 & \leq 24001.875 \hspace{1cm} (44) \\
20.8125x_1 + 14.35x_2 & \leq 28000.4375 \\
& x_1 \geq 0, \quad x_2 \geq 0 \hspace{1cm} (45)
\end{align*}
\]
In Fig. 5, $A_1(3020.394,0)$, $B_1(0,1499.192)$, $A_2(985.956,0)$, $B_2(0.3946.865)$, $A_3(1345.366,0)$, $B_3(0,1951.25)$. The vertices of $L$ are $B_1$, $B(475.353,1261.822)$, $C(781.756,817.432)$, $A_2(985.956,0)$. Since the optimal solution must be integers, we consider the points in $L$ which are closest to point $B_1$, $B$, $C$, $A_2$. Here we take points $B_1^*(0,1499)$, $B^*(475,126)$, $C^*(781,817)$ and $A_2^*(985,0)$. The optimal solution of eq. (41) occurs when $x_1 = 475(\equiv x_1^{(2)})$, $x_2 = 1261(\equiv x_2^{(2)})$ and the maximum profit is $Z = 72403$.

5 Discussion

(A) The crisp case is a special case of the fuzzy case.

(a) In theorem 1(b), let $\delta_j = \delta_j$ and $\delta_j = \delta_j$, $j = 1, 2, \cdots, n$. Then, in theorem 1(b), eqs. (22) ~ (24) reduces to

$$\text{Maximize } Z^* = \sum_{j=1}^{n} c_j x_j$$

subject to:

$$\sum_{j=1}^{n} a_{kj} x_j \leq b_k, \quad k = 1, 2, \cdots, m$$

$$x_j \geq 0, \quad j = 1, 2, \cdots, n$$

this is the crisp case of eqs. (15) ~ (17). Therefore, the crisp case of eqs. (15) ~ (17) is a special case of theorem 1(b).
(b) In theorem 2(b), let \( \delta_{kj2} = \delta_{kj3} \) and \( \delta_{kj1} = \delta_{kj4} \), \( j = 1, 2, \cdots, n \), \( \omega_{k2} = \omega_{k3} \) and \( \omega_{k1} = \omega_{k4} \) for all \( k = 1, 2, \cdots, m \), \( j = 1, 2, \cdots, n \), then in theorem 2(b), eqs.(29)~(31) reduce to the crisp case of eqs.(15)~(17). Therefore, the crisp case of eqs.(15)~(17) is a special case of theorem 2(b).

(c) In theorem 1,2(b), each are special case of theorem 3(b).

(c1) In theorem 3(b), let \( \delta_{kj2} = \delta_{kj3} \) and \( \delta_{kj1} = \delta_{kj4} \), \( \omega_{k2} = \omega_{k3} \) and \( \omega_{k1} = \omega_{k4} \) for all \( k = 1, 2, \cdots, m \), \( j = 1, 2, \cdots, n \), then theorem 3(b) eqs.(35)~(37) reduce to theorem 1(b) eqs.(22)~(24). Therefore, theorem 1(b) is a special case of theorem 3(b).

(c2) In theorem 3(b), let \( \delta_{j2} = \delta_{j3} \) and \( \delta_{j1} = \delta_{j4} \), \( j = 1, 2, \cdots, n \), then theorem 3(b) eqs.(35)~(37) reduce to theorem 2(b) eqs.(29)~(31). Therefore, theorem 2(b) is a special case of theorem 3(b).

(B) The result of fuzzification by fuzzy numbers is a special case of fuzzification by interval-valued fuzzy numbers.

(b1) In theorem 1 eq.(18), let \( \delta_{j3} = \delta_{j2} = 0 \) for all \( j \) and \( \lambda = 0 \). From Fig.3, we have level \((\lambda,1)\) i-v fuzzy number in eq.(18) reduce to fuzzy number \( \tilde{c}_j = (c_j - \delta_{j1}, c_j, c_j + \delta_{j4}; 1), \ j = 1, 2, \cdots, n \). This implies eq.(19) in theorem 1(a) use fuzzy numbers \( \tilde{c}_j = (c_j - \delta_{j1}, c_j, c_j + \delta_{j4}; 1), \ j = 1, 2, \cdots, n \).

In theorem 1(b), eq.(22). \( Z^* = \sum_{j=1}^{n} c_j x_j + \frac{1}{4} \sum_{j=1}^{n} (\delta_{j4} - \delta_{j1}) x_j \) is the result of defuzzification by signed distance using fuzzy numbers \( \tilde{c}_j = (c_j - \delta_{j1}, c_j, c_j + \delta_{j4}; 1) \), through

\[
d(\tilde{c}_j, \tilde{0}) = \frac{1}{2} \int_0^1 (\tilde{c}_{jL}(\alpha) + \tilde{c}_{jU}(\alpha))d\alpha = c_j + \frac{1}{4}(\delta_{j4} - \delta_{j1})
\]

Therefore, the defuzzification by using fuzzy numbers is a special case of using level \((\lambda,1)\) i-v fuzzy numbers.

(b2) In theorem 2(a), let \( \delta_{kj2} = \delta_{kj1} = 0, \ \omega_{k2} = \omega_{k1} = 0 \) for all \( j, k \), and \( \lambda = 0 \). It is similarly to (b1). Level \((\lambda,1)\) i-v fuzzy numbers in eq.(25) reduce to fuzzy numbers \( \tilde{a}_{kj} = (a_{kj} - \delta_{kj1}, a_{kj}, a_{kj} + \delta_{kj4}; 1) \), and level \((\lambda,1)\) i-v fuzzy numbers in eq.(26) reduce to fuzzy number \( \tilde{b}_k = (b_k - \omega_{k1}, b_k, b_k + \omega_{k4}; 1) \) for all \( j, k \). \( \tilde{a}_{kj}, \tilde{b}_k, k = 1, 2, \cdots, m, j = 1, 2, \cdots, n \) in eq.(27) of theorem 2(a) are all fuzzy numbers. Eq.(30) in theorem 2 becomes

\[
\sum_{j=1}^{n} a_{kj} x_j + \frac{1}{4} \sum_{j=1}^{n} (\delta_{kj4} - \delta_{kj1}) x_j \leq b_k + \frac{1}{4}(\omega_{k4} - \omega_{k1}), \ k = 1, 2, \cdots, m
\]
This is the result of defuzzification of fuzzy number through the signed distance
\[ d(\tilde{a}_{kj}, \tilde{0}) = a_{kj} + \frac{1}{4} \sum_{j=1}^{n} (\delta_{kj4} - \delta_{kj1}), \]
\[ d(\tilde{b}_{k}, \tilde{0}) = b_{k} + \frac{1}{4} (\omega_{k4} - \omega_{k1}). \]
Therefore, it has the same conclusion as (b1). The defuzzification result
by using fuzzy numbers is a special case of using level \((\lambda,1)\) i-v fuzzy
numbers.

(b3) In theorem 3, the same treatments will lead to the same conclusions as
(b1) and (b2).

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