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Subclasses of p-Valent and Prestarlike Functions

M. K. Aouf

Dept. of Math., Faculty of Science
Univ. of Mansoura, Mansoura 35516, Egypt
mkaouf127@yahoo.com

H. Silverman

Dept. of Math., Univ. of Charleston
Charleston, South Carolina 29424, USA
silvermanh@cofc.edu

Abstract

The object of the present paper is to investigate coefficients for functions belonging to the subclasses $R^p[\alpha,\beta]$ and $C^p[\alpha,\beta]$ of p-valent α-prestarlike functions of order β with negative coefficients. We obtain closure theorems, integral operators, radius of starlikeness and distortion theorems for functions belonging to the classes $R^p[\alpha,\beta]$ and $C^p[\alpha,\beta]$. We also obtain several results for the modified Hadamard products of functions belonging to the classes $R^p[\alpha,\beta]$ and $C^p[\alpha,\beta]$.

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1 Introduction

Let $A(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \ (p \in N = \{1, 2, ...\}) \tag{1.1}$$

which are analytic and p-valent in the unit disc $U = \{z : |z| < 1\}$. A function $f(z) \in A(p)$ is called p-valent starlike of order $\alpha(0 \leq \alpha < p)$ if $f(z)$ satisfies the conditions

$$\mathrm{Re}\left\{\frac{zf^{'}(z)}{f(z)}\right\} > \alpha \quad (z \in U) \tag{1.2}$$

and

$$\int_{0}^{2\pi} \mathrm{Re}\left\{\frac{zf^{'}(z)}{f(z)}\right\} d\theta = 2\pi p \ (z \in U) \ . \tag{1.3}$$

We denote by $S^*(p,\alpha)$ the class of p-valent starlike functions of order α. The class $S^*(p,\alpha)$ was introduced by Patil and Thakare [8].

The function

$$s_\alpha^p(z) = \frac{z^p}{(1-z)^{2(p-\alpha)}} \ (0 \le \alpha < p; p \in N) \tag{1.4}$$

is the familiar extremal function for the class $S^*(p,\alpha)$. Setting

$$G^p(\alpha, n) = \frac{\prod\limits_{m=2}^{n} [2(p-\alpha)+m-2]}{(n-1)!} \ (n \in N/\{1\}, 0 \le \alpha < p) \, , \tag{1.5}$$

$s_\alpha^p(z)$ can be written in the form:

$$s_\alpha^p(z) = z^p + \sum_{n=1}^{\infty} G^p(\alpha, n+1) z^{p+n} \ . \tag{1.6}$$

Clearly, $s_\alpha^p(z) \in S^*(p,\alpha)$ and $G^p(\alpha, n+1)$ is a decreasing function in $\alpha(0 \le \alpha \le \frac{2p-1}{2}; p \in N)$ and satisfies

$$\lim_{n\to\infty} G^p(\alpha, n+1) = \begin{cases} \infty & (\alpha < \frac{2p-1}{2}) \\ 1 & (\alpha = \frac{2p-1}{2}) \\ 0 & (\alpha > \frac{2p-1}{2}) \ . \end{cases}$$

Let $(f * g)(z)$ denote the Hadamard product (or convolution) of the functions $f(z)$ and $g(z)$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z) = z^p + \sum_{n=1}^{\infty} b_{p+n} z^{p+n} \ , \tag{1.7}$$

then

$$(f * g)(z) = z^p + \sum_{n=1}^{\infty} a_{p+n}\, b_{p+n}\, z^{p+n}\,. \tag{1.8}$$

A function $f(z) \in A(p)$ is said to be p-valent α-prestarlike function of order $\beta(0 \leq \alpha < p; 0 \leq \beta < p; p \in N)$ if

$$(f * s_\alpha^p)(z) \in S^*(p,\beta)\ , \tag{1.9}$$

where $s_\alpha^p(z)$ is defined by (1.4). We denote by $R^p(\alpha,\beta)$ the class of all p-valent α-prestarlike functions of order β. For $\alpha = \frac{2p-1}{2}; 0 \leq \beta < p; p \in N, R^p(\frac{2p-1}{2},\beta) = S^*(p,\beta)$. Further let $C^p(\alpha,\beta)$ be the subclass of $A(p)$ consisting of functions $f(z)$ satisfying

$$f(z) \in C^p(\alpha,\beta) \text{ if and only if } \frac{zf'(z)}{p} \in R^p(\alpha,\beta). \tag{1.10}$$

We note that:

(i) $R^p(\alpha,\alpha) = R^p(\alpha)(0 \leq \alpha < p; p \in N)$, the class of p-valent prestarlike functions of order α, was studied by Kumar and Reddy [3] and Shenen et al. [13];

(ii) $R^1(\alpha,\beta) = R_{\alpha,\beta}(0 \leq \alpha < 1; 0 \leq \beta < 1)$, the class of α-prestarlike functions of order β, was introduced by Sheil-Small et al. [12];

(iii) $R^1(\alpha,\alpha) = R_\alpha(0 \leq \alpha < 1)$, the class of prestarlike functions of order α, was introduced by Ruscheweyh [10];

(iv) $C^1(\alpha,\beta) = C(\alpha,\beta)(0 \leq \alpha < 1; 0 \leq \beta < 1)$, the subclass of $A(1) = A$ consisting of functions $f(z) \in A$ satisfying $zf'(z) \in R^1(\alpha,\beta) = R(\alpha,\beta)$, was introduced by Owa and Uralegaddi [7].

Denoting by $T(p)$ the subclass of $A(p)$ consisting of functions of the form:

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n} (a_{p+n} \geq 0; p \in N)\,. \tag{1.11}$$

We denote by $S^*[p,\beta]$, $R^p[\alpha,\beta]$ and $C^p[\alpha,\beta]$ the classes obtained by taking intersections, respectively, of the classes $S^*(p,\beta)$, $R^p(\alpha,\beta)$ and $C^p(\alpha,\beta)$ with the class $T(p)$. Thus, we have

$$S^*[p,\beta] = S^*(p,\beta) \cap T(p), \tag{1.12}$$

$$R^p[\alpha,\beta] = R^p(\alpha,\beta) \cap T(p), \tag{1.13}$$

and

$$C^p[\alpha,\beta] = C^p(\alpha,\beta) \cap T(p)\,. \tag{1.14}$$

The class $S^*[p,\beta]$ was studied by Owa [5].

It follows from (1.13) and (1.14) that

$$f(z) \in C^p[\alpha,\beta] \ \text{ if and only if } \ \frac{zf'(z)}{p} \in R^p[\alpha,\beta]\,. \tag{1.15}$$

Also we note that, by specializing the parameters α, β and p, we obtain the following subclasses studied by various authors:

(i) $R^p[p\alpha, p\alpha] = R^p[\alpha](0 \le \alpha < 1; p \in N)$ (Kumar and Reddy [3]);

(ii) $R^1[\alpha,\alpha] = R[\alpha](0 \le \alpha < 1)$ (Silverman and Silvia [14] and Owa and Al-Bassam [6]);

(iii) $R^1[\alpha,\beta] = R[\alpha,\beta](0 \le \alpha < 1, 0 \le \beta < 1)$ (Silverman and Silvia [15], Uralegaddi and Sarangi [17], Aouf and Salagean [2], Aouf et al. [1], Srivastava and Aouf [16] and Rania and Srivastava [9]).

In the present paper we investigate coefficients for functions belonging to the subclasses $R^p[\alpha,\beta]$ and $C^p[\alpha,\beta]$ of p-valent α-prestarlike functions of order β with negative coefficients. We obtain closure theorems, integral operators, radii of starlikeness and convexity and distortion theorems for functions belonging to the classes $R^p[\alpha,\beta]$ and $C^p[\alpha,\beta]$. We also obtain several results for the modified Hadamard products of functions belonging to the classes $R^p[\alpha,\beta]$ and $C^p[\alpha,\beta]$.

2 Coefficient inequalities

Theorem 1 *. Let the function $f(z)$ be defined by* (1.11). *Then $f(z)$ is in the class $R^p[\alpha,\beta]$ if and only if*

$$\sum_{n=1}^{\infty}(n+p-\beta)G^p(\alpha,n+1)a_{p+n} \le (p-\beta)\,. \tag{2.1}$$

Proof. It is known [5] that a necessary and sufficient condition for $g(z) = z^p - \sum\limits_{n=1}^{\infty} b_{p+n}z^{p+n}(b_{p+n} \ge 0)$ to be in the class $S^*[p,\beta]$ is that

$$\sum_{n=1}^{\infty}(n+p-\beta)b_{p+n} \le (p-\beta)\,. \tag{2.2}$$

Since

$$(f * s_\alpha^p)(z) = z^p - \sum_{n=1}^{\infty} G^p(\alpha,n+1)a_{p+n}z^{p+n}\,, \tag{2.3}$$

where $s_\alpha^p(z)$ is given by (1.4), the result follows.

Corollary 2 *.If $f(z)$ is in the class $R^p[\alpha,\beta]$, then*

$$a_{p+n} \leq \frac{(p-\beta)}{(n+p-\beta)G^p(\alpha,n+1)}\ (p,n \in N)\,, \tag{2.4}$$

with equality for

$$f(z) = z^p - \frac{(p-\beta)}{(n+p-\beta)G^p(\alpha,n+1)} z^{p+n}(p,n \in N)\,. \tag{2.5}$$

In view of (1.15), Theorem 1 yields the following necessary and sufficient condition for the class $C^p[\alpha,\beta]$.

Theorem 3 *. The function $f(z)$, defined by* (1.11)*, is in the class $C^p[\alpha,\beta]$ if and only if*

$$\sum_{n=1}^{\infty}(\frac{p+n}{p})(n+p-\beta)G^p(\alpha,n+1)a_{p+n} \leq (p-\beta)\,. \tag{2.6}$$

3 Extreme points

From Theorem 1 and Theorem 2, we see that both $R^p[\alpha,\beta]$ and $C^p[\alpha,\beta]$ are closed under convex linear combinations, which enables us to determine the extreme points for these classes.

Theorem 4 *. Let*

$$f_p(z) = z^p \tag{3.1}$$

and

$$f_{p+n}(z) = z^p - \frac{(p-\beta)}{(n+p-\beta)G^p(\alpha,n+1)} z^{p+n}(p,n \in N). \tag{3.2}$$

Then $f(z) \in R^p[\alpha,\beta]$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=0}^{\infty} \mu_{p+n} f_{p+n}(z)\,, \tag{3.3}$$

where $\mu_{p+n} \geq 0$ and $\sum\limits_{n=0}^{\infty} \mu_{p+n} = 1$.

Proof. Suppose that

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \mu_{p+n} f_{p+n}(z) \\ &= z^p - \sum_{n=1}^{\infty} \frac{(p-\beta)}{(n+p-\beta)G^p(\alpha,n+1)} \mu_{p+n} z^{p+n} \,. \end{aligned} \tag{3.4}$$

Then it follows that

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{(n+p-\beta)G^p(\alpha,n+1)}{(p-\beta)} \cdot \frac{(p-\beta)}{(n+p-\beta)G^p(\alpha,n+1)} \mu_{p+n} \\ = \; &\sum_{n=1}^{\infty} \mu_{p+n} = 1 - \mu_p \le 1 \,. \end{aligned} \tag{3.5}$$

Therefore, by Theorem 1, $f(z) \in R^p[\alpha,\beta]$.

Conversely, assume that the function $f(z)$ defined by (1.11) belongs to the class $R^p[\alpha,\beta]$. Then

$$a_{p+n} \le \frac{(p-\beta)}{(n+p-\beta)G^p(\alpha,n+1)} (p,n \in N) \,. \tag{3.6}$$

Setting

$$\mu_{p+n} = \frac{(n+p-\beta)G^p(\alpha,n+1)}{(p-\beta)} (p,n \in N) \tag{3.7}$$

and

$$\mu_p = 1 - \sum_{n=1}^{\infty} \mu_{p+n} \;, \tag{3.8}$$

we see that $f(z)$ can be expressed in the form (3.3). This completes the proof of Theorem 3.

Corollary 5 . *The extreme points of the class* $R^p[\alpha,\beta]$ *are the functions* $f_p(z) = z^p$ *and*

$$f_{p+n}(z) = z^p - \frac{(p-\beta)}{(n+p-\beta)G^p(\alpha,n+1)} z^{p+n} (p,n \in N) \,.$$

Similarly, we have

Theorem 6 .*Let*

$$f_p(z) = z^p \tag{3.9}$$

and

$$f_{p+n}(z) = z^p - \frac{(p-\beta)}{(\frac{p+n}{p})(n+p-\beta)G^p(\alpha, n+1)} z^{p+n} (p, n \in N). \tag{3.10}$$

Then $f(z) \in C^p[\alpha, \beta]$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=0}^{\infty} \mu_{p+n} f_{p+n}(z)\,, \tag{3.11}$$

where $\mu_{p+n} \geq 0$ and $\sum\limits_{n=0}^{\infty} \mu_{p+n} = 1$.

Corollary 7 *.The extreme points of the class* $C^p[\alpha, \beta]$ *are the functions* $f_p(z) = z^p$ *and*

$$f_{p+n}(z) = z^p - \frac{(p-\beta)}{(\frac{p+n}{p})(n+p-\beta)G^p(\alpha, n+1)} z^{p+n} (p, n \in N)\,.$$

4 Distortion theorems

In view of Theorems 3 and 4, we will obtain distortion theorems for the classes $R^p[\alpha, \beta]$ and $C^p[\alpha, \beta]$.

Lemma 8 *. For* $0 \leq \alpha \leq \frac{2p-1}{2}, 0 \leq \beta < p$ *and* $p \in N$, *then* $(n+p-\beta)G^p(\alpha, n+1)$ *is an increasing function of* n *, where* $G^p(\alpha, n+1)$ *is defined by* (1.5).
Proof. *. Let* $K(\alpha, \beta, n, p) = (n+p-\beta)G^p(\alpha, n+1)$.*Since,*

$$G^p(\alpha, n+2) = \frac{2p+n-2\alpha}{n+1}\, G^p(\alpha, n+1), \tag{4.1}$$

we can see that $K(\alpha, \beta, n+1, p) \geq K(\alpha, \beta, n, p)$ if and only if

$$2(p-\alpha)(n+1+p-\beta) - (p-\beta) \geq 0\ , \tag{4.2}$$

for $0 \leq \alpha \leq \frac{2p-1}{2}$, $0 \leq \beta < p$ which holds for $p \in N$. This completes the proof of Lemma 1.

In the remainder of this section, we assume that $f(z)$ is defined by (1.11), $0 \leq \alpha \leq \frac{2p-1}{2}, 0 \leq \beta < p$ and p in N.

Theorem 9 . *If $f(z)$ is in the class $R^p[\alpha,\beta]$, then*

$$|z|^p - \frac{(p-\beta)}{2(1+p-\beta)(p-\alpha)}|z|^{p+1} \leq |f(z)| \leq |z|^p + \frac{(p-\beta)}{2(1+p-\beta)(p-\alpha)}|z|^{p+1}$$

$$(z \in U)\,. \tag{4.3}$$

Equality holds for the function $f_{p+1}(z)$ given by

$$f_{p+1}(z) = z^p - \frac{(p-\beta)}{2(1+p-\beta)(p-\alpha)}z^{p+1}\,(z \in U)\,. \tag{4.4}$$

Proof. By virtue of Theorem 3, we note that

$$|z|^p - \max_{n\in N} \tfrac{(p-\beta)}{(n+p-\beta)G^p(\alpha,n+1)}|z|^{p+n} \leq |f(z)| \leq |z|^p + \max_{n\in N} \tfrac{(p-\beta)}{(n+p-\beta)G^p(\alpha,n+1)}|z|^{p+n}\,. \tag{4.5}$$

From Lemma 1, we see that the max in (4.5) occurs when $n = 1$. This completes the proof of Theorem 5.

Theorem 10 . *If $f(z)$ is in the class $R^p[\alpha,\beta]$, then*

$$p|z|^{p-1} - \frac{(p+1)(p-\beta)}{2(1+p-\beta)(p-\alpha)}\,|z|^p \leq |f^{'}(z)| \leq p|z|^{p-1} + \frac{(p+1)(p-\beta)}{2(1+p-\beta)(p-\alpha)}|z|^p$$

$$(z \in U)\ . \tag{4.6}$$

Equality holds for $f_{p+1}(z)$ given by (4.4).
Proof. We know that

$$p|z|^{p-1} - \max_{n\in N} \tfrac{(p-\beta)(n+p)|z|^{n+p-1}}{(n+p-\beta)G^p(\alpha,n+1)} \leq |f^{'}(z)| \leq p|z|^{p-1} + \max_{n\in N} \tfrac{(p-\beta)(n+p)|z|^{n+p-1}}{(n+p-\beta)G^p(\alpha,n+1)}(z \in U). \tag{4.7}$$

From Lemma 1, we see that the max in (4.7) occurs when $n = 1$. This completes the proof of Theorem 6.

Theorem 11 .*If $f(z)$ is in the class $C^p[\alpha,\beta]$, then*

$$|f(z)| \geq |z|^p - \frac{(p-\beta)}{2(\frac{p+1}{p})(1+p-\beta)(p-\alpha)}|z|^{p+1} \tag{4.8}$$

and

$$|f(z)| \leq |z|^p + \frac{(p-\beta)}{2(\frac{p+1}{p})(1+p-\beta)(p-\alpha)}|z|^{p+1} \tag{4.9}$$

for $z \in U$. The results are sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{(p-\beta)}{2(\frac{p+1}{p})(1+p-\beta)(p-\alpha)} z^{p+1} \quad (z \in U). \tag{4.10}$$

Proof. From Theorem 4, we have that

$$|f(z)| \geq |z|^p - \max_{n \in N} \frac{(p-\beta)}{(\frac{p+n}{p})(n+p-\beta)G^p(\alpha, n+1)}|z|^{p+n} \tag{4.11}$$

and

$$|f(z)| \leq |z|^p + \max_{n \in N} \frac{(p-\beta)}{(\frac{p+n}{p})(n+p-\beta)G^p(\alpha, n+1)}|z|^{p+n} \tag{4.12}$$

for $z \in U$. From Lemma 1, we see that the max in (4.11) and (4.12) occur when $n = 1$. This completes the proof of Theorem 7.

Corollary 12 *.If $f(z)$ is in the class $C^p[\alpha,\beta]$. Then $f(z)$ is included in a disc with its center at the origin and radius r given by*

$$r = 1 + \frac{(p-\beta)}{2(\frac{p+1}{p})(1+p-\beta)(p-\alpha)} . \tag{4.13}$$

Theorem 13 *.If $f(z)$ is in the class $C^p[\alpha,\beta]$, then*

$$|f'(z)| \geq p\,|z|^{p-1} - \frac{p(p-\beta)}{2(1+p-\beta)(p-\alpha)}|z|^p \tag{4.14}$$

and

$$|f'(z)| \leq p\,|z|^{p-1} + \frac{p(p-\beta)}{2(1+p-\beta)(p-\alpha)}|z|^p \tag{4.15}$$

for $z \in U$. The bounds for (4.14) and (4.15) are sharp for the function $f(z)$ given by (4.10).

Proof. By means of Theorem 4, we note that

$$|f'(z)| \geq p\,|z|^{p-1} - \max_{n\in N}\ \frac{p(p-\beta)}{(n+p-\beta)G^p(\alpha,n+1)}\,|z|^{p+n-1} \tag{4.16}$$

and

$$|f'(z)| \leq p\,|z|^{p-1} + \max_{n\in N}\ \frac{p(p-\beta)}{(n+p-\beta)G^p(\alpha,n+1)}\,|z|^{p+n-1}\,. \tag{4.17}$$

Also by using Lemma 1, we see that the max in (4.16) and (4.17) occur when $n = 1$. This completes the proof of Theorem 8.

Remark 1 . *Making use of the relationship (1.15) between the classes* $R^p[\alpha,\beta]$ *and* $C^p[\alpha,\beta]$, *we can deduce Theorem 8 from Theorem 5.*

5 Integral operators

Theorem 14 . *Let the function* $f(z)$ *defined by* (1.11) *be in the class* $R^p[\alpha,\beta]$, *and let* c *be a real number such that* $c > -p$. *Then the function* $F(z)$ *defined by*

$$F(z) = \frac{c+p}{z^c}\int_0^z t^{c-1}\,f(t)\;dt \tag{5.1}$$

also belongs to the class $R^p[\alpha,\beta]$.
Proof. From the representation of $F(z)$, it follows that

$$F(z) = z^p - \sum_{n=1}^{\infty} b_{p+n}\,z^{p+n}, \tag{5.2}$$

where

$$b_{p+n} = (\frac{c+p}{c+p+n})a_{p+n}\ .$$

Therefore

$$\sum_{n=1}^{\infty}(n+p-\beta)G^p(\alpha,n+1)b_{p+n} = \sum_{n=1}^{\infty}(n+p-\beta)G^p(\alpha,n+1)(\tfrac{c+p}{c+p+n})a_{p+n}$$
$$\leq \sum_{n=1}^{\infty}(n+p-\beta)G^p(\alpha,n+1)a_{p+n} \leq (p-\beta)\,,$$

since $f(z) \in R^p[\alpha,\beta]$. Hence, by Theorem 1, $F(z) \in R^p[\alpha,\beta]$.

Corollary 15 . *Under the same conditions as Theorem 9, a similar proof shows that the function $F(z)$ defined by (5.1) is in the class $C^p[\alpha,\beta]$, whenever $f(z)$ is in the class $C^p[\alpha,\beta]$.*

6 Radius of starlikeness

Since $f(z)$ defind by (1.11) is p-valent in the unit disc U if

$$\sum_{n=1}^{\infty}\left(\frac{p+n}{p}\right)a_{p+n} \le 1 \ (cf.[4]) \tag{6.1}$$

we can see that $R^p[\alpha,\beta]$ is a subclass of $T(p)$ if $0 \le \alpha \le \dfrac{2p-1}{2}, 0 \le \beta < p, p \in N$ with the aid of Theorem 1. Also we can see that $C^p[\alpha,\beta]$ is a subclass of $T(p)$ if $0 \le \alpha \le \dfrac{(2p-1)(p-\beta)+2p}{2(1+p-\beta)}$, $0 \le \beta < p$, $p \in N$ with the aid of Theorem 2.

Theorem 16 . *Let the function* $f(z)$ *defined by* (1.11) *be in the class* $R^p[\alpha,\beta]$.

Then $f(z)$ is p-valently starlike of order $\delta\,(0 \le \delta < p)$ in $|z| < r_1$, where

$$r_1 = \inf_{n\in N}\left\{\frac{(p-\delta)(n+p-\beta)G^p(\alpha,n+1)}{(n+p-\delta)(p-\beta)}\right\}^{\frac{1}{n}} (n \ge 1)\,. \tag{6.2}$$

The result is sharp, with the extremal function $f(z)$ given by (2.5).

Proof. It is sufficient to show that $\left|\dfrac{zf'(z)}{f(z)} - p\right| \le p-\delta$ for $|z| < r_1$. We have

$$\left|\frac{zf'(z)}{f(z)} - p\right| \le \frac{\displaystyle\sum_{n=1}^{\infty} n\ a_{p+n}\,|z|^n}{1-\displaystyle\sum_{n=1}^{\infty} a_{p+n}\,|z|^n}\,.$$

Thus $\left|\dfrac{zf'(z)}{f(z)} - p\right| \le p-\delta$ if

$$\sum_{n=1}^{\infty}\frac{(n+p-\delta)}{(p-\delta)}\ a_{p+n}\,|z|^n \le 1. \tag{6.3}$$

Hence, by Theorem 1, (6.3) will be true if

$$\frac{(n+p-\delta)}{(p-\delta)}\,|z|^n \le \frac{(n+p-\beta)G^p(\alpha,n+1)}{(p-\beta)}$$

or if

$$|z| \leq \left\{ \frac{(p-\delta)(n+p-\beta)G^p(\alpha,n+1)}{(n+p-\delta)(p-\beta)} \right\}^{\frac{1}{n}} (n \geq 1)\ . \tag{6.4}$$

The theorem follows easily from (6.4).

Corollary 17 . *Let the function $f(z)$ defined by (1.11) be in the class $R^p[\alpha,\beta]$. Then $f(z)$ is p-valently convex of order $\delta\,(0 \leq \delta < p)$ in $|z| < r_2$, where*

$$r_2 = \inf_n \left\{ \frac{p(p-\delta)(n+p-\beta)G^p(\alpha,n+1)}{(n+p)(n+p-\delta)(p-\beta)} \right\}^{\frac{1}{n}} (n \geq 1)\,. \tag{6.5}$$

The result is sharp, with the extremal function $f(z)$ given by (2.5).

Theorem 18 . *Let the function $f(z)$ defined by* (1.11) *be in the class $C^p[\alpha,\beta]$, $0 \leq \alpha \leq \dfrac{(2p-1)(p-\beta)+2p}{2(1+p-\beta)}$, $0 \leq \beta < p$ and $p \in N$. Then $f(z)$ is p-valently starlike of order $\delta\,(0 \leq \delta < p)$ in $|z| < r_3$, where*

$$r_3 = \inf_{n\in N} \left\{ \frac{(p-\delta)(p+n)(n+p-\beta)G^p(\alpha,n+1)}{p(p-\beta)(n+p-\delta)} \right\}^{\frac{1}{n}} . \tag{6.6}$$

The result is sharp, with the extremal function $f(z)$ given by

$$f(z) = z^p - \frac{(p-\beta)}{(\frac{p+n}{p})(n+p-\beta)G^p(\alpha,n+1)} z^{p+n} (p,n \in N)\ . \tag{6.7}$$

Corollary 19 . *Let the function $f(z)$ defined by* (1.11) *be in the class $C^p[\alpha,\beta]$, $0 \leq \alpha \leq \dfrac{(2p-1)(p-\beta)+2p}{2(1+p-\beta)}$, $0 \leq \beta < p$ and $p \in N$. Then $f(z)$ is p-valently convex of order $\delta\,(0 \leq \delta < p)$ in $|z| < r_4$, where*

$$r_4 = \inf_{n\in N} \left\{ \frac{(p-\delta)(n+p-\beta)G^p(\alpha,n+1)}{(p-\beta)(n+p-\delta)} \right\}^{\frac{1}{n}} (n \geq 1). \tag{6.8}$$

The result is sharp, with the extremal function $f(z)$ given by (6.7).

7 Modified Hadamard products

Let the functions $f_j(z)(j = 1, 2)$ be defined by (3.1). The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z^p - \sum_{n=1}^{\infty} a_{p+n,1}a_{p+n,2} \, z^{p+n}. \tag{7.1}$$

Theorem 20 . *Let the functions* $f_j(z)(j = 1, 2)$ *defined by* (3.1) *be in the class* $R^p[\alpha, \beta]$. *Then* $(f_1 * f_2)(z) \in R^p[\alpha, \gamma(\alpha, \beta, p)]$, *where*

$$\gamma(\alpha, \beta, p) = p - \frac{(p-\beta)^2}{2(1+p-\beta)^2(p-\alpha) - (p-\beta)^2}. \tag{7.2}$$

The result is sharp.

Proof. . Employing the technique used earlier by Schild and Silverman [11], we need to find the largest $\gamma = \gamma(\alpha, \beta, p)$ such that

$$\sum_{n=1}^{\infty} \frac{(n+p-\gamma)G^p(\alpha, n+1)}{(p-\gamma)} a_{p+n,1} \, a_{p+n,2} \le 1. \tag{7.3}$$

Since

$$\sum_{n=1}^{\infty} \frac{(n+p-\beta)G^p(\alpha, n+1)}{(p-\beta)} a_{p+n,1} \le 1 \tag{7.4}$$

and

$$\sum_{n=1}^{\infty} \frac{(n+p-\beta)G^p(\alpha, n+1)}{(p-\beta)} a_{p+n,2} \le 1, \tag{7.5}$$

by the Cauchy - Schwarz inequality we have

$$\sum_{n=1}^{\infty} \frac{(n+p-\beta)G^p(\alpha, n+1)}{(p-\beta)} \sqrt{a_{p+n,1} \, a_{p+n,2}} \le 1. \tag{7.6}$$

Thus it is sufficient to show that

$$\frac{(n+p-\gamma)G^p(\alpha, n+1)}{(p-\gamma)} a_{p+n,1} \, a_{p+n,2} \le \frac{(n+p-\beta)G^p(\alpha, n+1)}{(p-\beta)} \sqrt{a_{p+n,1} \, a_{p+n,2}}$$

$$(n \ge 1), \tag{7.7}$$

that is , that

$$\sqrt{a_{p+n,1}\ a_{p+n,2}} \leq \frac{(n+p-\beta)(p-\gamma)}{(n+p-\gamma)(p-\beta)}. \tag{7.8}$$

Note that

$$\sqrt{a_{p+n,1}\ a_{p+n,2}} \leq \frac{(p-\beta)}{(n+p-\beta)G^p(\alpha,n+1)}\ (n \geq 1). \tag{7.9}$$

Consequently, we need only to prove that

$$\frac{(p-\beta)}{(n+p-\beta)G^p(\alpha,n+1)} \leq \frac{(n+p-\beta)(p-\gamma)}{(n+p-\gamma)(p-\beta)}\ (n \geq 1) \tag{7.10}$$

or, equivalently , that

$$\gamma \leq p - \frac{n(p-\beta)^2}{(n+p-\beta)^2G^p(\alpha,n+1)-(p-\beta)^2}\ (n \geq 1). \tag{7.11}$$

Since

$$A(n) = p - \frac{n(p-\beta)^2}{(n+p-\beta)^2G^p(\alpha,n+1)-(p-\beta)^2} \tag{7.12}$$

is an increasing function of $n(n \geq 1)$ for $0 \leq \alpha \leq \frac{2p-1}{2}$, $0 \leq \beta < p$ and $p \in N$, letting $n = 1$ in (7.12) , we obtain

$$\gamma \leq A(1) = p - \frac{(p-\beta)^2}{2(1+p-\beta)^2(p-\alpha)-(p-\beta)^2}, \tag{7.13}$$

which completes the proof of Theorem 12.

Finally , by taking the functions

$$f_j(z) = z^p - \frac{(p-\beta)}{2(1+p-\beta)(p-\alpha)}\ z^{p+1} (j = 1,2) \tag{7.14}$$

we can see that the result is sharp.

Corollary 21 . *For* $f_j(z)(j = 1,2)$ *as in Theorem 12, we have*

$$h(z) = z^p - \sum_{n=1}^{\infty} \sqrt{a_{p+n,1}\ a_{p+n,2}}\, z^{p+n} \tag{7.15}$$

belongs to the class $R^p[\alpha,\beta]$.

The result follows from the inequality (7.6). It is sharp for the same functions as in Theorem 12.

Corollary 22 .*Let the functions* $f_j(z)(j = 1,2)$ *defined by* (3.1) *be in the class* $C^p[\alpha,\beta]$. *Then* $(f_1 * f_2)(z) \in C^p[\alpha, \lambda(\alpha,\beta,p)]$, *where*

$$\lambda(\alpha,\beta,p) = p - \frac{(p-\beta)^2}{2(\frac{p+1}{p})(1+p-\beta)^2(p-\alpha) - (p-\beta)^2}. \tag{7.16}$$

The result is sharp for the functions

$$f_j(z) = z^p - \frac{(p-\beta)}{2(\frac{p+1}{p})(1+p-\beta)(p-\alpha)}\, z^{p+1}(j = 1,2). \tag{7.17}$$

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