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Non-Linear Analysis of Arbitrary Quadrilateral Plates by Use of Kirchhoff-Love Theory

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Abstract

Very large displacement but small strain of a very thin quadrilateral plate is studied using Kirchhoff-Love theory. The numerical investigation is based on the mapping of the quadrilateral plate onto the computational coordinates of a standard square and interpolation of the displacement field over the whole domain with no director assignment. The present investigation, which is basically a limiting analysis of the Cosserat's theory, enforces the well known Kirchhoff's hypothesis which denies the existence of shear strain in the direction of the plate's thickness. After forming the decoupled nonlinear equations, the material and geometric tangential stiffness matrices are derived through a linearization process and different stages of the problem solution are presented. Finally through certain numerical examples and comparison of the results with some existing researches the validity and the accuracy of the present method are verified.

Keywords: Kirchhoff-Love theory, Numerical solution, Quadrilateral plates

1 Introduction

Thin quadrilateral steel plates are widely used as the main structural components of box girders in bridges, plate girders, and controller gates in water and wastewater channels, platforms of offshore structures, shipbuilding and aircraft industries. Rectangular and skew plates are used for a variety of functions in aeronautical and aerospace constructions. It is well known that the famous Kirchhoff's hypothesis, that straight lines perpendicular to the midsurface remain perpendicular to the deformed midsurface, is satisfactory only when the thickness approaches zero and the deformation is not too large. The hypothesis, if carried over to cases where large deformations are experienced, can lead to certain numerical difficulties. However, in this research a procedure is presented through which the hypothesis is employed in the Cosserat's theory to numerically investigate large deformations in a thin rectangular plate. Large deformations in plates and shells are usually studied using two different approaches- three dimensional theory and what is known as direct method or Cosserat's theory in which a director is assigned to a non-Euclidean plane [1-4]. The accuracy of the direct method approaches that of three dimensional theory if more terms are included in the series expansion of directors.

Analogous to theoretical approaches, in numerical analysis of large deformations of plates and shells, the above mentioned two methods have also been employed extensively. As for three dimensional theory, a so called three dimensional modified element was first developed by Ahmad et al [6]. Other investigators like Hughes and Liu [7-8], Hughes and Carnoy [9] further developed this element in their works on nonlinear analysis of plates and shells. Presently, this method is recorded in standard finite element textbooks like Bathe [10], Hughes [11] and Belytschko [12]. Numerical formulation of the second approach was first presented by Wempner [13] whose article introduces co-rotational coordinates or co-rotational finite elements in nonlinear analysis of shells. The work of Argyris [4] can also be mentioned along these investigations. Finite element formulation of Cosserat's theory goes back to the first part of a series of papers written by Simo et al [15-21]. The works of others like Wangner [22] on central symmetrical problems can also be cited.

Co-rotational coordinates or co-rotational finite elements have also found their applications in problems in which large deformations exist along with small strains. Some examples are works of Parish [23], Beuchter *et. al* [24], Sansour *et. al* [25], Peng *et. al* [26], Jiang *et. al* [27], Moita *et. al* [28], and Liu *et. al* [29].

All the above works on large deformations in plates and shells, involve shear deformations in the direction of the thickness, and as the thickness approaches zero mostly experience shear locking or membrane locking in the process of their numerical analysis. In the present research an arbitrary quadrilateral plate is mapped onto the computational coordinates of a standard square and interpolation is merely done for the components of the displacement field over the entire domain. Consequently, a numerical procedure based on the Kirchhoff-Love theory for large deformations of the plate is presented without appealing to the director field. This is in fact a limiting analysis of the Cosserat's theory in which Kirchhoff's hypothesis which denies the presence of shear strains in the direction of the shell's thickness is enforced.

2 Kirchhoff's Theory

Figure 1 shows a thin quadrilateral plate with dimensions $a \times b$ before and after deformation while the Kirchhoff's hypothesis is assumed to hold. The stretch and bending effects are considered simultaneously in writing the equilibrium conditions [1]. The left superscripts denote the time of measurement. In this figure x^1, x^2, $x^3$are Cartesian reference coordinates and θ^1, $\theta^2$the convective coordinates which are assumed to be orthogonal at the reference configuration at time $t = 0$. ${}^0\mathbf{a}_\alpha$ and ${}^t\mathbf{a}_\alpha$ are the base vectors and ${}^0\mathbf{d}$ and ${}^t\mathbf{d}$ the directors at times $t = 0$ and $t = t$, respectively ($\alpha = 1, 2$). Given Kirchhoff's hypothesis the director is always perpendicular to the midsurface.

The position vector of a material point, being dependent on the variables θ^1, θ^2 is

$$ {}^t\mathbf{x} = \left\langle {}^tx^1,\, {}^tx^2,\, {}^tx^3 \right\rangle^T \tag{1} $$

The base vectors and the normal unit vector to the midsurface are

$$ \left.\begin{array}{l} {}^t\mathbf{a}_\alpha = {}^t\mathbf{x}_{,\alpha} = \frac{\partial {}^t\mathbf{x}}{\partial \theta^\alpha} \\ {}^t\mathbf{a}_3 = \frac{{}^t\mathbf{a}_1 \times {}^t\mathbf{a}_2}{\left\| {}^t\mathbf{a}_\alpha \times {}^t\mathbf{a}_\beta \right\|} \end{array}\right\} \tag{2} $$

The components of the first and second fundamental tensors of the surface and the components of the membrane and bending strains can be written as

$$ \left.\begin{array}{l} {}^ta_{\alpha\beta} = {}^t\mathbf{x}_{,\alpha}.{}^t\mathbf{x}_{,\beta} \\ {}^tb_{\alpha\beta} = -{}^t\mathbf{a}_{3,\alpha}.{}^t\mathbf{a}_\beta = {}^t\mathbf{a}_3.{}^t\mathbf{a}_{\alpha,\beta} \\ {}^t_0e_{\alpha\beta} = \frac{1}{2}({}^t\mathbf{x}_{,\alpha}.{}^t\mathbf{x}_{,\beta} - {}^0\mathbf{x}_{,\alpha}.{}^0\mathbf{x}_{,\beta}) \\ {}^t_0\kappa_{\alpha\beta} = {}^t\mathbf{a}_3.{}^t\mathbf{x}_{,\alpha\beta} = \frac{1}{\sqrt{{}^ta}}\left[{}^t\mathbf{x}_{,\alpha\beta}.{}^t\mathbf{x}_{,1} \times {}^t\mathbf{x}_{,2}\right] \end{array}\right\} \tag{3} $$

where the left subscripts denote the reference configuration in measurement of the strain components. ${}^ta_{\alpha\beta}$, ${}^tb_{\alpha\beta}$ are the first and second fundamental tensors of the surface, respectively. ${}^ta = \det({}^ta_{\alpha\beta})$, ${}^t_0e_{\alpha\beta}$: the membrane strain and ${}^t_0\kappa_{\alpha\beta}$: the bending strain. Now the stress components can also be defined. In Cosserat's theory, the membrane, shear and bending stresses are defined in terms of effective stress components in the direction of the shell's thickness [1]. Figure 2 shows the effective Cauchy stress at a material point at the deformed configuration. Also shown in this figure are the components of the effective symmetric Piola stress corresponding to Cauchy stress.

In this figure, ${}^t_tn^{\alpha\beta}$, ${}^t_tq^\alpha$, ${}^t_tm^{\alpha\beta}$are the effective membrane stress, shear stress and bending moment, respectively, measured per unit length of the deformed configuration. These are in fact the resultants of the Cauchy stresses and their invariant forms can be written as

$$ \left.\begin{array}{l} {}^t_t\mathbf{n} = {}^t_tn^{\alpha\beta}\,{}^t\mathbf{a}_\alpha \otimes {}^t\mathbf{a}_\beta \\ {}^t_t\mathbf{q} = {}^t_tq^\alpha\,{}^t\mathbf{a}_\alpha \\ {}^t_t\mathbf{m} = {}^t_tm^{\alpha\beta}\,{}^t\mathbf{a}_\alpha \otimes {}^t\mathbf{a}_\beta \end{array}\right\} \tag{4} $$

Similarly, ${}^t_0n^{\alpha\beta}$, ${}^t_0q^\alpha$, ${}^t_0m^{\alpha\beta}$ are the effective membrane stress, shear stress and bending moment, respectively, measured per unit length of the reference configuration. These are in fact the resultants of the Piola stresses and their invariant forms can be written as

$$\left.\begin{array}{l} {}^t_0\mathbf{n} = {}^t_0 n^{\alpha\beta}\, {}^0\mathbf{a}_\alpha \otimes {}^0\mathbf{a}_\beta \\ {}^t_0\mathbf{q} = {}^t_0 q^{\alpha}\, {}^0\mathbf{a}_\alpha \\ {}^t_0\mathbf{m} = {}^t_0 m^{\alpha\beta}\, {}^0\mathbf{a}_\alpha \otimes {}^0\mathbf{a}_\beta \end{array}\right\} \tag{5}$$

These stresses are related according to the following relations

$$\left.\begin{array}{l} {}^t_0 n^{\alpha\beta} = J\, {}^t_t n^{\alpha\beta} \\ {}^t_0 q^{\alpha} = J\, {}^t_t q^{\alpha} \\ {}^t_0 m^{\alpha\beta} = J\, {}^t_t m^{\alpha\beta} \end{array}\right\} \tag{6}$$

where $J = \frac{d^t\sigma}{d^0\sigma}$ is the transformation Jacobian, namely, the ratio between elements of area after and before the deformation. Using the principle of virtual work, the equilibrium condition at time t can be written as

$$\left.\begin{array}{l} \int_{{}^t\sigma} ({}^t_t n^{\alpha\beta} \delta^t_0 e_{\alpha\beta} + {}^t_t m^{\alpha\beta} \delta^t_0 \kappa_{\alpha\beta}) d^t\sigma = {}^t R_{ext} \\ \int_{{}^0\sigma} ({}^t_0 n^{\alpha\beta} \delta^t_0 e_{\alpha\beta} + {}^t_0 m^{\alpha\beta} \delta^t_0 \kappa_{\alpha\beta}) d^0\sigma = {}^t R_{ext} \end{array}\right\} \tag{7}$$

where ${}^t R_{ext}$ is the virtual work of the external forces and in terms of boundary tractions is

$${}^t R_{ext} = \int_{{}^t\sigma} ({}^t\bar{\mathbf{n}}.\delta^t\mathbf{x} + {}^t\bar{\mathbf{m}}.\delta^t\mathbf{d}) d^t\sigma + \int_{\partial^t\sigma} ({}^t\bar{\bar{\mathbf{n}}}.\delta^t\mathbf{x} + {}^t\bar{\bar{\mathbf{m}}}.\delta^t\mathbf{d}) d\partial^t\sigma \tag{8}$$

where ${}^t\bar{\mathbf{n}}$, ${}^t\bar{\mathbf{m}}$ are the distributed force and moment vectors at time t, respectively, measured per unit area of the deformed configuration. Similarly ${}^t\bar{\bar{\mathbf{n}}}$, ${}^t\bar{\bar{\mathbf{m}}}$ are the distributed force and moment vectors at time t, respectively, measured per unit length of the boundary $\partial^t\sigma$. Defining computational stress ${}^t_0\mathbf{r}$ and strain ${}^t_0\boldsymbol{\varepsilon}$ vectors according to

$$\left.\begin{array}{l} {}^t_0\mathbf{r} = \langle {}^t_0 n^{11},\, {}^t_0 n^{22},\, {}^t_0 n^{12},\, {}^t_0 m^{11},\, {}^t_0 m^{22},\, {}^t_0 m^{12} \rangle^T \\ {}^t_0\boldsymbol{\varepsilon} = \langle {}^t_0 e_{11},\, {}^t_0 e_{22},\, 2{}^t_0 e_{12},\, {}^t_0 \kappa_{11},\, {}^t_0 \kappa_{22},\, 2{}^t_0 \kappa_{12} \rangle^T \end{array}\right\} \tag{9}$$

the virtual work principle can be written in the concise form

$$\int_{{}^0\sigma} {}^t_0\mathbf{r}^T.\delta^t_0\boldsymbol{\varepsilon}\, d^0\sigma = {}^t R_{ext} \tag{10}$$

For a simple linear hyperelastic material we have

$${}^t_0\mathbf{r} = \mathbf{C}.{}^t_0\boldsymbol{\varepsilon} \tag{11}$$

where

$$\left.\begin{array}{l}\mathbf{C}=\begin{bmatrix}\mathbf{C}_m & \mathbf{0}\\ \mathbf{0} & \mathbf{C}_b\end{bmatrix}\\ \mathbf{C}_m=\frac{Eh}{(1-\nu^2)}\begin{bmatrix}({}^0a^{11})^2 & \nu({}^0a^{11})({}^0a^{22})+(1-\nu)({}^0a^{12})^2 & ({}^0a^{11})({}^0a^{12})\\ & ({}^0a^{11})^2 & ({}^0a^{22})({}^0a^{12})\\ sym & & \frac{1+\nu}{2}({}^0a^{12})^2+\frac{1-\nu}{2}({}^0a^{11})({}^0a^{22})\end{bmatrix}\\ \mathbf{C}_b=\frac{h^2}{12}\mathbf{C}_m\end{array}\right\} \tag{12}$$

In the above relations ${}^ta^{\alpha\beta}$ is the conjugate of the first fundamental tensor of the surface, E, Young's modulus of elasticity, h, plate's thickness and ν, the Poisson's ratio. Variations in strain components are as follow

$$\left.\begin{array}{l}\delta_0^t e_{\alpha\beta}=\frac{1}{2}(\delta^t\mathbf{x}_{,\alpha}.\,{}^t\mathbf{x}_{,\beta}+{}^t\mathbf{x}_{,\alpha}.\,\delta^t\mathbf{x}_{,\beta})\\ \delta_0^t\kappa_{\alpha\beta}=\frac{1}{\sqrt{{}^ta}}\delta\left[{}^t\mathbf{x}_{,\alpha\beta}.{}^t\mathbf{x}_{,1}\times{}^t\mathbf{x}_{,2}\right]-\frac{\delta^ta}{2\sqrt{({}^ta)^3}}\left[{}^t\mathbf{x}_{,\alpha\beta}.{}^t\mathbf{x}_{,1}\times{}^t\mathbf{x}_{,2}\right]\end{array}\right\} \tag{13}$$

where

$$\delta^ta=2^ta_{22}{}^t\mathbf{x}_{,1}.\delta^t\mathbf{x}_{,1}+2^ta_{11}{}^t\mathbf{x}_{,2}.\delta^t\mathbf{x}_{,2}-2^ta_{12}({}^t\mathbf{x}_{,1}.\delta^t\mathbf{x}_{,2}+{}^t\mathbf{x}_{,2}.\delta^t\mathbf{x}_{,1}) \tag{14}$$

By substituting Eq. (14) into Eq. (13), the variations of strain and curvature components are obtained. Then by substituting Eq. (13) into Eq. (10), the nonlinear form of virtual work is established. The step by step analysis of the nonlinear problem is explained in the next section.

3 Numerical Solution

The three stages mentioned below are followed in the development of the numerical solution:

- Mapping of the plate to the computational domain of a standard square and interpolating the displacement field.
- Linearization of the weak form of the equilibrium equations.
- Deriving appropriate expressions for the tangential, material and geometric stiffness matrices.

3.1 Mapping to Computational Domain and Interpolating Displacement Field

An arbitrary shaped quadrilateral plate with line supports in Cartesian coordinates may be expressed simply by the mapping of a square plate defined in its natural coordinates by the simple boundary Eqns. $\theta^\alpha = \pm 1$ as shown in Fig. 3. For a general quadrilateral plate with straight edges, the displacement transformation from the physical coordinates system to the computational coordinates system is achieved in exact form since the physical coordinates system itself is interpolated exactly.

Transformation equations for this linear mapping are

$$\left.\begin{array}{l} x^\alpha = \sum\limits_{I=1}^{4} x^\alpha_{(I)} N_{(I)}(\theta^1,\, \theta^2) \\ N_{(I)}(\theta^1,\, \theta^2) = \frac{1}{4}(1+\theta^1_{(I)}\,\theta^1)(1+\theta^2_{(I)}\,\theta^2) \end{array}\right\} \tag{15}$$

where $x^\alpha_{(I)}$ are the coordinates of four corners of the quadrilateral plate, $N^{(}_{(I)}\theta^\alpha)$ the interpolation functions and $\theta^\alpha_{(I)}$ the natural coordinates of the i^{th} corner. The first fundamental tensor of the surface at time $t = 0$ and its conjugate, to be used in the constitutive equations are

$$^0a_{\alpha\beta} = \sum_{\gamma=1}^{2} \frac{\partial x^\gamma}{\partial \theta^\alpha} \frac{\partial x^\gamma}{\partial \theta^\beta}, \quad \left[{}^0a^{\alpha\beta}\right] = \left[{}^0a_{\alpha\beta}\right]^{-1}. \tag{16}$$

Given the equilibrium state at time t, the equilibrium state at time $t + \Delta t$ is derived by interpolating the variation in the displacement field, i.e., $\mathbf{x} = {}^{t+\Delta t}\mathbf{x} - {}^t\mathbf{x}$ over the plate surface. As for the interpolation of the first two components of the displacement field, i.e., x^1, x^2 the following simple polynomials are used

$$\left.\begin{array}{l} P_1(\theta^\alpha) = (1-\theta^\alpha),\ P_2(\theta^\alpha) = (1+\theta^\alpha) \\ P_k(\theta^\alpha) = (\theta^\alpha)^{k-3}(1-(\theta^\alpha)^2),\ k = 3, 4, ... \end{array}\right\} \tag{17}$$

where the first two functions, namely, P_1, P_2 are chosen such that the in-plane boundary conditions are satisfied. To interpolate the third component of the displacement field, i.e., x^3, the following functions are used

$$\left.\begin{array}{ll} P_k(\theta^\alpha) = H_k(\theta^\alpha), & k = 1, 2, 3, 4 \\ P_k(\theta^\alpha) = (1-(\theta^\alpha)^2)^{k-3}, & k = 5, 6, ... \end{array}\right\} \tag{18}$$

where functions H_k which are four Hermite functions for $k = 1, 2, 3, 4$ are chosen such that the out-plane boundary conditions are satisfied. The two-dimensional interpolation functions can be formed through the products
$N_i(\theta^1,\, \theta^2) = P_j(\theta^1)P_k(\theta^2)$ and we have

$$\mathbf{x} = \left\{\begin{array}{c} x^1 \\ x^2 \\ x^3 \end{array}\right\} = \left[\begin{array}{ccc} \mathbf{N}^1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{N}^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{N}^3 \end{array}\right] \left\{\begin{array}{c} \hat{\mathbf{x}}^1 \\ \hat{\mathbf{x}}^2 \\ \hat{\mathbf{x}}^3 \end{array}\right\} = \mathbf{N}\hat{\mathbf{x}} \tag{19}$$

where $\mathbf{N}^i$ is the vector of shape functions and $\hat{\mathbf{x}}^i$ is the vector of generalized coordinate for interpolation ofx^i (i^{th} coordinate). Now by define of ${}^tQ_{\alpha\beta} \hat{=} [{}^t\mathbf{x}_{,\alpha\beta}.{}^t\mathbf{x}_{,1} \times {}^t\mathbf{x}_{,2}]$ we have

$$\left.\begin{array}{l} \delta_0^t e_{\alpha\beta} = \mathbf{E}_{\alpha\beta}.\delta\hat{\mathbf{x}} \\ \delta_0^t \kappa_{\alpha\beta} = \mathbf{K}_{\alpha\beta}.\delta\hat{\mathbf{x}} \\ \delta^t a = \mathbf{A}.\delta\hat{\mathbf{x}} \\ \delta^t Q_{\alpha\beta} = \mathbf{Q}_{\alpha\beta}.\delta\hat{\mathbf{x}} \end{array}\right\} \tag{20}$$

where

$$\left.\begin{array}{l} \mathbf{E}_{\alpha\beta} = \frac{1}{2}({}^t\mathbf{x}_{,\alpha}^T.\mathbf{N}_{,\beta} + {}^t\mathbf{x}_{,\beta}^T.\mathbf{N}_{,\alpha}) \\ \mathbf{K}_{\alpha\beta} = [\frac{1}{\sqrt{({}^ta)}}\mathbf{Q}_{\alpha\beta} - \frac{{}^tQ_{\alpha\beta}}{2\sqrt{({}^ta)^3}}\mathbf{A}] \\ \mathbf{Q}_{\alpha\beta} = ({}^t\mathbf{x}_{,1} \times {}^t\mathbf{x}_{,2})^T.\mathbf{N}_{,\alpha\beta} + ({}^t\mathbf{x}_{,2} \times {}^t\mathbf{x}_{,\alpha\beta})^T.\mathbf{N}_{,1} + ({}^t\mathbf{x}_{,\alpha\beta} \times {}^t\mathbf{x}_{,1})^T.\mathbf{N}_{,2} \\ \mathbf{A} = 2{}^ta_{11}{}^t\mathbf{x}_{,2}^T.\mathbf{N}_{,2} + 2{}^ta_{22}{}^t\mathbf{x}_{,1}^T.\mathbf{N}_{,1} - 2{}^ta_{12}{}^t\mathbf{x}_{,2}^T.\mathbf{N}_{,1} - 2{}^ta_{12}{}^t\mathbf{x}_{,1}^T.\mathbf{N}_{,2} \end{array}\right\} \tag{21}$$

3.2 Linearization

The appropriate linear forms of equations (10) and (13) are formed through differentiation. The results are

$$\int_{{}^0\sigma} \Delta_0^t\mathbf{r}^T.\delta_0^t\boldsymbol{\varepsilon}\, d^0\sigma + \int_{{}^0\sigma} {}_0^t\mathbf{r}^T.\Delta\delta_0^t\boldsymbol{\varepsilon}\, d^0\sigma = \Delta^t R_{ext} \tag{22}$$

and

$$\left.\begin{array}{l} \Delta\delta_0^t e_{\alpha\beta} = \delta\hat{\mathbf{x}}^T.\boldsymbol{\mathcal{E}}_{\alpha\beta}.\hat{\mathbf{x}} \\ \Delta\delta_0^t \kappa_{\alpha\beta} = \delta\hat{\mathbf{x}}^T.\boldsymbol{\mathcal{K}}_{\alpha\beta}.\hat{\mathbf{x}} \\ \Delta\delta^t a = \delta\hat{\mathbf{x}}^T.\boldsymbol{\mathcal{A}}.\hat{\mathbf{x}} \\ \Delta\delta^t Q_{\alpha\beta} = \delta\hat{\mathbf{x}}^T.\boldsymbol{\mathcal{Q}}_{\alpha\beta}.\hat{\mathbf{x}} \end{array}\right\} \tag{23}$$

where

$$\left.\begin{array}{l}
\boldsymbol{\mathcal{E}}_{\alpha\beta}=\frac{1}{2}(\mathbf{N}_{,\alpha}^{T}.\mathbf{N}_{,\beta}+\mathbf{N}_{,\beta}^{T}.\mathbf{N}_{,\alpha})\\
\boldsymbol{\mathcal{K}}_{\alpha\beta}=(\frac{3\,{}^{t}Q_{\alpha\beta}}{4\sqrt{({}^{t}a)^{5}}})\mathbf{A}^{T}.\mathbf{A}-(\frac{{}^{t}Q_{\alpha\beta}}{2\sqrt{({}^{t}a)^{3}}})\boldsymbol{\mathcal{A}}-(\frac{1}{2\sqrt{({}^{t}a)^{3}}})\mathbf{A}^{T}.\mathbf{Q}_{\alpha\beta}\\
\quad-(\frac{1}{2\sqrt{({}^{t}a)^{3}}})\mathbf{Q}_{\alpha\beta}^{T}.\mathbf{A}+\frac{1}{\sqrt{({}^{t}a)}}\boldsymbol{\mathcal{Q}}_{\alpha\beta}\\
\boldsymbol{\mathcal{A}}=4\mathbf{N}_{,2}^{T}.{}^{t}\mathbf{x}_{,2}^{T}.{}^{t}\mathbf{x}_{,1}.\mathbf{N}_{,1}+4\mathbf{N}_{,1}^{T}.{}^{t}\mathbf{x}_{,1}^{T}.{}^{t}\mathbf{x}_{,2}.\mathbf{N}_{,2}-2\mathbf{N}_{,1}^{T}.{}^{t}\mathbf{x}_{,2}^{T}.{}^{t}\mathbf{x}_{,1}.\mathbf{N}_{,2}\\
\quad-2\mathbf{N}_{,1}^{T}.{}^{t}\mathbf{x}_{,2}^{T}.{}^{t}\mathbf{x}_{,2}.\mathbf{N}_{,1}-2\mathbf{N}_{,2}^{T}.{}^{t}\mathbf{x}_{,1}^{T}.{}^{t}\mathbf{x}_{,1}.\mathbf{N}_{,2}-2\mathbf{N}_{,2}^{T}.{}^{t}\mathbf{x}_{,1}^{T}.{}^{t}\mathbf{x}_{,2}.\mathbf{N}_{,1}\\
\boldsymbol{\mathcal{Q}}_{\alpha\beta}=\boldsymbol{\mathcal{B}}_{12}^{T}.\mathbf{N}_{,\alpha\beta}+\boldsymbol{\mathcal{B}}_{2\alpha\beta}^{T}.\mathbf{N}_{,1}+\boldsymbol{\mathcal{B}}_{\alpha\beta 1}^{T}.\mathbf{N}_{,2}\\
\boldsymbol{\mathcal{B}}_{12}=\begin{bmatrix}\mathbf{0} & {}^{t}x_{,2}^{3}\mathbf{N}_{,1}^{2}-{}^{t}x_{,1}^{3}\mathbf{N}_{,2}^{2} & {}^{t}x_{,1}^{2}\mathbf{N}_{,2}^{3}-{}^{t}x_{,2}^{2}\mathbf{N}_{,1}^{3}\\ {}^{t}x_{,1}^{3}\mathbf{N}_{,2}^{1}-{}^{t}x_{,2}^{3}\mathbf{N}_{,1}^{1} & \mathbf{0} & {}^{t}x_{,2}^{1}\mathbf{N}_{,1}^{3}-{}^{t}x_{,1}^{1}\mathbf{N}_{,2}^{3}\\ {}^{t}x_{,2}^{2}\mathbf{N}_{,1}^{1}-{}^{t}x_{,1}^{2}\mathbf{N}_{,2}^{1} & {}^{t}x_{,1}^{1}\mathbf{N}_{,2}^{2}-{}^{t}x_{,2}^{1}\mathbf{N}_{,1}^{2} & \mathbf{0}\end{bmatrix}\\
\boldsymbol{\mathcal{B}}_{2\alpha\beta}=\begin{bmatrix}\mathbf{0} & {}^{t}x_{,\alpha\beta}^{3}\mathbf{N}_{,2}^{2}-{}^{t}x_{,2}^{3}\mathbf{N}_{,\alpha\beta}^{2} & {}^{t}x_{,2}^{2}\mathbf{N}_{,\alpha\beta}^{3}-{}^{t}x_{,\alpha\beta}^{2}\mathbf{N}_{,2}^{3}\\ {}^{t}x_{,2}^{3}\mathbf{N}_{,\alpha\beta}^{1}-{}^{t}x_{,\alpha\beta}^{3}\mathbf{N}_{,2}^{1} & \mathbf{0} & {}^{t}x_{,\alpha\beta}^{1}\mathbf{N}_{,2}^{3}-{}^{t}x_{,2}^{1}\mathbf{N}_{,\alpha\beta}^{3}\\ {}^{t}x_{,\alpha\beta}^{2}\mathbf{N}_{,2}^{1}-{}^{t}x_{,2}^{2}\mathbf{N}_{,\alpha\beta}^{1} & {}^{t}x_{,2}^{1}\mathbf{N}_{,\alpha\beta}^{2}-{}^{t}x_{,\alpha\beta}^{1}\mathbf{N}_{,2}^{2} & \mathbf{0}\end{bmatrix}\\
\boldsymbol{\mathcal{B}}_{\alpha\beta 1}=\begin{bmatrix}\mathbf{0} & {}^{t}x_{,1}^{3}\mathbf{N}_{,\alpha\beta}^{2}-{}^{t}x_{,\alpha\beta}^{3}\mathbf{N}_{,1}^{2} & {}^{t}x_{,\alpha\beta}^{2}\mathbf{N}_{,1}^{3}-{}^{t}x_{,1}^{2}\mathbf{N}_{,\alpha\beta}^{3}\\ {}^{t}x_{,\alpha\beta}^{3}\mathbf{N}_{,1}^{1}-{}^{t}x_{,1}^{3}\mathbf{N}_{,\alpha\beta}^{1} & \mathbf{0} & {}^{t}x_{,1}^{1}\mathbf{N}_{,\alpha\beta}^{3}-{}^{t}x_{,\alpha\beta}^{1}\mathbf{N}_{,1}^{3}\\ {}^{t}x_{,1}^{2}\mathbf{N}_{,\alpha\beta}^{1}-{}^{t}x_{,\alpha\beta}^{2}\mathbf{N}_{,1}^{1} & {}^{t}x_{,\alpha\beta}^{1}\mathbf{N}_{,1}^{2}-{}^{t}x_{,1}^{1}\mathbf{N}_{,\alpha\beta}^{2} & \mathbf{0}\end{bmatrix}
\end{array}\right\}\tag{24}$$

3.3 Stiffness Matrices

Substituting the interpolated displacement field in relations (22) and using the results in relation (23) and (24), the linear form of the decoupled equations is obtained as follows

$$({}^{t}\mathbf{K}_{M}+{}^{t}\mathbf{K}_{G})\hat{\mathbf{x}}=\mathbf{F}_{ext}={}^{t+\Delta t}\mathbf{F}_{ext}-{}^{t}\mathbf{F}_{ext}\tag{25}$$

where $\hat{\mathbf{x}}$is the vector for the degrees of freedom, ${}^{t}\mathbf{K}_{M}$, the material tangential stiffness matrix, ${}^{t}\mathbf{K}_{G}$, the geometric tangential stiffness matrix, and ${}^{t}\mathbf{F}_{ext}$, the nodal force vector at time t, where

$$\left.\begin{array}{l}
{}^{t}\mathbf{K}_{M}=\int_{{}^{0}\sigma}(\mathbf{B}^{T}.{}_{0}^{t}\mathbf{C}.\,\mathbf{B})d\,{}^{0}\sigma\\
{}^{t}\mathbf{K}_{G}=\int_{{}^{0}\sigma}({}_{0}^{t}n^{\alpha\beta}\boldsymbol{\mathcal{E}}_{\alpha\beta}+{}_{0}^{t}m^{\alpha\beta}\boldsymbol{\mathcal{K}}_{\alpha\beta})d\,{}^{0}\sigma\\
\mathbf{B}=\left\langle\mathbf{E}_{11}^{T},\,\mathbf{E}_{22}^{T},\,2\mathbf{E}_{12}^{T},\,\mathbf{K}_{11}^{T},\,\mathbf{K}_{22}^{T},\,2\mathbf{K}_{12}^{T}\right\rangle^{T}\\
{}^{t}\mathbf{F}_{ext}=\int_{{}^{0}\sigma}\mathbf{B}^{T}.{}_{0}^{t}\mathbf{r}d\,{}^{0}\sigma
\end{array}\right\}\tag{26}$$

Now, through an iteration process and using equation (25), the problem can be solved.

4 Numerical Results

4.1 Cantilevered Beam

A simple example is first chosen to verify the validity and accuracy of the proposed method. Figure 4 shows a cantilevered beam with rectangular cross section subjected to three different loading conditions at its free end.

The first case is for a concentrated moment M at the free end. Table 1 shows the convergence trend in lateral displacement as the number of interpolation polynomials is increased. Figure 5 shows moment versus displacement as compared to the exact solution. The exact solution of the problem is known to be $1/\rho = M/EI$, where ρ is the radius of curvature and EI , the bending rigidity and we have

$$\left.\begin{array}{l} \frac{u_o}{L} = 1 - \frac{\sin(ML/EI)}{(ML/EI)} \\ \frac{w_o}{L} = \frac{1-\cos(ML/EI)}{(ML/EI)} \end{array}\right\} \tag{27}$$

The second case is for a concentrated load P whose vertical direction remains unchanged during deformation, and finally the third case for a concentrated load P whose direction remains perpendicular to the deformed direction of the beam. Figures 6 and 7 shows deformed configurations of this beam for numerical values $h/b = 0.1$, $b/L = 0.1$, $\nu = 0.3$, and for the last two loading conditions.

4.2 Skew Plate Subjected to Distributed Load

This example presents a skew plate subjected to a constant distributed load P. The geometric boundary conditions are: simple supports for the out-plane motion and two different cases, clamped and free, as for the in-plane motion. Figure 8 shows the geometric conditions of the plate for the last two cases.

Most researches on this problem use von-Karman theory. In his *"Non-Linear Analysis of Plates"*, Chia [30] has collected researches and analysis on large deformations of plates. In the previous investigation Shahidi et. al. [31] studied this problem by use of Cosserat theory and interpolation of director field on the whole domain of plate. Some of his results, which are based on von-Karman theory and previous investigation, are compared with the results of the present research in Table 2.

Figure 9 also shows load versus displacement at the center of the skew plate when the boundary conditions are clamped for the in-plane motion. For this case, since in-plane motions are prevented, tensile stresses are produced at the boundary when the lateral load is applied.

4.3 Trapezoidal Plate Subjected to Distributed Load

This example presents a trapezoidal plate, subjected to two different load cases, a constant distributed load P and a concentrated load P at the centre. The geometric boundary conditions are: simple supports for the out-plane motion and clamped for the in-plane motion. Figure 10 shows the geometric conditions of the plate for the last two cases.

Figure 11 and 12 also shows load versus displacement at the center of the trapezoidal plate when the boundary conditions are clamped for the in-plane motion. For this case, since in-plane motions are prevented, tensile stresses are produced at the boundary when the lateral load is applied.

5 Conclusion

Using Cosserat's theory and imposing Kirchhoff'-Love assumption a procedure for the numerical analysis of the theory was developed. Numerical results for a cantilevered beam with rectangular cross section subjected to concentrated loads at its free end, and also for quadrilateral plates, were derived and compared to existing researches. The present methodology can be regarded as a limiting analysis of Cosserat's theory as thickness approaches zero. By imposing Kirchhoff-Love hypothesis in the proposed numerical procedure, problems like shear locking and membrane locking, were circumvented.

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