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On the Involute and Evolute Curves of the Spacelike Curve with a Spacelike Binormal in Minkowski $3-$Space

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Abstract. In this study, we have generalized the involute and evolute curves of the spacelike curve α with a spacelike binormal in Minkowski 3-Space. Firstly, we have shown that, the length between the spacelike curve α and the timelike curve β is constant. Furthermore, the Frenet frame of the involute curve β has been found as depend on curvatures of the curve α. We have determined the curve α is planar in which conditions. Secondly, we have found transformation matrix between the evolute curve β and the curve α. Finally, we have computed the curvatures of the evolute curve β.

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1. Preliminaries

Let $IR^3 = \{(x_1, x_2, x_3) | x_1, x_2, x_3 \in IR\}$ be a 3-dimensional vector space, and let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be two vectors in IR^3. The Lorentz scalar product of x and y is defined by

$$\langle x, y\rangle_L = -x_1 y_1 + x_2 y_2 + x_3 y_3,$$

$IE_1^3 = (R^3, \langle x, y\rangle_L)$ is called 3-dimensional Lorentzian space, Minkowski 3-Space or 3-dimensional semi-euclidean space. The vector x in IE_1^3 is called a spacelike vector, null vector or a timelike vector if $\langle x, x\rangle_L > 0$ or $x = 0$, $\langle x, x\rangle_L = 0$ or $\langle x, x\rangle_L < 0$, respectively. For $x \in IE_1^3$, the norm of the vector x defined by $\|x\|_L = \sqrt{|\langle x, x\rangle_L|}$, and x is called a unit vector if $\|x\|_L = 1$. For any x, $y \in IE_1^3$, Lorentzian vectoral product of x and y is defined by

$$x \wedge_L y = (x_3 y_2 - x_2 y_3, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1).$$

We denote by $\{T(s), N(s), B(s)\}$ the moving Frenet frame along the curve $\alpha(s)$. Then $T(s), N(s)$ and $B(s)$ are tangent, the principal normal and the binormal vector of the curve $\alpha(s)$, respectively. Depending on the causal character of the curve α , we have the following Frenet-Serret formulas :

If α is a spacelike curve with a spacelike principal normal N;

$$T' = \kappa N\,,\ N = -\kappa T + \tau B\,,\ B' = \tau N \tag{1.1}$$

$$\langle T, T\rangle_L = \langle N, N\rangle_L = 1, \langle B, B\rangle_L = -1, \langle T, N\rangle_L = \langle N, B\rangle_L = \langle T, B\rangle_L = 0.$$

If α is a spacelike curve with a spacelike binormal B;

$$T' = \kappa N,\ N' = \kappa T + \tau B,\ B' = \tau N \tag{1.2}$$

$$\langle T, T\rangle_L = \langle B, B\rangle_L = 1, \langle N, N\rangle_L = -1, \langle T, N\rangle_L = \langle N, B\rangle_L = \langle T, B\rangle_L = 0.$$

If α is a timelike curve and finally;

$$T' = \kappa N,\ N = \kappa T + \tau B,\ B' = -\tau N \tag{1.3}$$

$$\langle T, T\rangle_L = -1, \langle B, B\rangle_L = \langle N, N\rangle_L = 1, \langle T, N\rangle_L = \langle N, B\rangle_L = \langle T, B\rangle_L = 0.$$

[2].If the curve α is non-unit speed, then

$$\kappa(t) = \frac{\left\|\alpha'(t) \wedge_L \alpha''(t)\right\|_L}{\left\|\alpha'(t)\right\|_L^3},\ \ \tau(t) = \frac{\det\left(\alpha'(t), \alpha''(t), \alpha'''(t)\right)}{\left\|\alpha'(t) \wedge_L \alpha''(t)\right\|_L^2}. \tag{1.4}$$

If the curve α is unit speed, then

$$\kappa(s) = \|\alpha''(s)\|_L\,,\ \tau(s) = \|\,B'(s)\|_L\,. \tag{1.5}$$

2. The Involute of The Spacelike Curve with a Spacelike Binormal

Definition 1. *Let unit speed spacelike curve* $\alpha : I \longrightarrow E_1^3$ *and the timelike curve* $\beta : I \longrightarrow E_1^3$ *be given. For* $\forall s \in I$*, then the curve* β *is called the involute of the curve* α*, if the tangent at the point* $\alpha(s)$ *to the curve* α *passes through the tangent at the point* $\beta(s)$ *to the curve* β *and*

$$\langle T^*(s), T(s)\rangle_L = 0. \tag{2.1}$$

Let the Frenet-Serret frames of the curves α and β be $\{T, N, B\}$ and $\{T^*, N^*, B^*\}$, respectively. In this case, the causal characteristics of the Frenet-Serret frames of the curves α and β must be of the form.

$$\{T \text{ spacelike}, N \text{ timelike}, B \text{ spacelike}\}$$

and

$$\{T^* \text{ timelike}, N^* \text{ spacelike}, B^* \text{ spacelike}\}.$$

Theorem 1. *Let the curve* β *be involute of the the curve* α *and let* k *be a constant real number. Then*

$$\beta(s) = \alpha(s) + (k - s)T(s). \tag{2.2}$$

Proof. The curve $\beta(s)$ may be given as

$$\beta(s) = \alpha(s) + u(s)T(s) \tag{2.3}$$

If we take the derivative Eq. (2.3), then we have

$$\beta'(s) = \left(1 + u'(s)\right) T(s) + u(s)\kappa(s)N(s).$$

Since the curve β is involute of the curve α, $\langle T^*(s), T(s)\rangle_L = 0$. Then, we get

$$1 + u'(s) = 0 \text{ or } u(s) = k - s. \tag{2.4}$$

Thus we get

$$\beta(s) - \alpha(s) = (k - s)\, T(s) \tag{2.5}$$

∎

Corollary 2. *The distance between the curves* β *and* α *is* $|k - s|$.

Proof. If we take the norm in Eq. (2.5), then we get

$$\|\beta(s) - \alpha(s)\|_L = |k - s|. \tag{2.6}$$

∎

Theorem 3. *Let the curve* β *be involute of the the curve* α*, then*

$$\begin{bmatrix} T^* \\ N^* \\ B^* \end{bmatrix} = \left(\sqrt{\kappa^2 + \tau^2}\right)^{-1} \begin{bmatrix} 0 & 1 & 0 \\ \kappa & 0 & -\tau \\ -\tau & 0 & -\kappa \end{bmatrix} . \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

Proof. If we take the derivative Eq.(2.5), we can write

$$\beta^{'}(s) = (k-s)\kappa(s)N(s)$$

and

$$\left\|\beta^{'}(s)\right\|_L = \left|(k-s)\kappa(s)\right|.$$

Furthermore, we get

$$T^{*}(s) = \frac{\beta^{'}(s)}{\left\|\beta^{'}(s)\right\|_L} = \frac{(k-s)\kappa(s)}{\left|(k-s)\kappa(s)\right|}N(s).$$

From the last equation, we must have

$$T^{*}(s) = N(s) \text{ or } T^{*}(s) = -N(s).$$

We assume that $T^{*}(s) = N(s)$. Let's denote the coordinate function on IR by x. Then, for$\forall s \in IR$, $x(s) = s$, we get

$$\begin{aligned} \beta^{'}(s) &= (k-s)\kappa(s)N(s), \\ \beta^{'} &= (k-x)\kappa N \\ \beta^{'} &= (k-x)\kappa\,(0,1,0) \end{aligned}$$

Thus, we have

$$\begin{aligned} \beta^{''} &= -\kappa N + (k-x)\kappa^{'}N + (k-x)\kappa(\kappa T + \tau B) \\ \beta^{''} &= (k-x)\kappa^{2}T + \left((k-x)\kappa^{'} - \kappa\right)N + (k-x)\kappa\tau B \\ \beta^{''} &= \kappa(k-x)\,(\kappa, *, \tau) \end{aligned}$$

Hence, we have

$$\beta^{'} \wedge_L \beta^{''} = (k-x)^{2}\kappa^{2}\left(-\tau T - \kappa B\right)$$

and

$$\left\|\beta^{'} \wedge_L \beta^{''}\right\|_L = \left|k-x\right|^{2}\kappa^{2}\sqrt{\left|\kappa^{2}+\tau^{2}\right|}.$$

Furthermore, we get

$$B^{*} = \frac{\beta^{'} \wedge_L \beta^{''}}{\left\|\beta^{'} \wedge_L \beta^{''}\right\|} = \frac{(k-x)^{2}.\kappa^{2}.\left(-\tau T - \kappa B\right)}{(k-x)^{2}.\kappa^{2}.\sqrt{\kappa^{2}+\tau^{2}}} = \frac{-\tau T - \kappa B}{\sqrt{\kappa^{2}+\tau^{2}}}.$$

Since $N^{*} = B^{*} \wedge_L T^{*}$, then we obtain

$$\begin{aligned} N^{*} &= \left(\frac{-\tau T - \kappa B}{\sqrt{\kappa^{2}+\tau^{2}}}\right) \wedge_L N, \\ N^{*} &= \frac{\kappa T - \tau B}{\sqrt{\kappa^{2}+\tau^{2}}}. \end{aligned}$$

∎

Theorem 4. *Let the curve* β *be involute of the the curve* α*. Let the curvature and torsion of the curve* β *be* κ^* *and* τ^**, respectively. Then*

$$\kappa^*(s) = \frac{\sqrt{\kappa^2(s)+\tau^2(s)}}{|k-s|\,.\kappa(s)}\ ,\ \tau^*(s) = \frac{\kappa(s)\tau^{'}(s)-\kappa^{'}(s)\tau(s)}{|k-s|\,.\kappa(s).\sqrt{\kappa^2(s)+\tau^2(s)}}.$$

Proof. From Eq. (1.3) and Eq. (1.4), we have

$$\kappa^*(s) = \frac{|k-s|^2\,\kappa^2(s)}{|k-s|^3\,\kappa^3(s)} = \frac{\sqrt{\kappa^2(s)+\tau^2(s)}}{\kappa(s)\,|k-s|}$$

and

$$\begin{aligned}
\beta^{'''} &= \left[-\kappa^2 T + (k-x)2\kappa\kappa^{'}T + (k-x)\kappa^2(\kappa N)\right] \\
&\quad + \left[-\kappa^{'} - \kappa^{'} + (k-x)\kappa^{''}\right] N \\
&\quad + \left[-\kappa + (k-x)\kappa^{'})\right](\kappa T + \tau B) \\
&\quad + \left[-\kappa\tau + (k-x)\kappa^{'}\tau + (k-x)\kappa\tau^{'}\right] B \\
&\quad + \left[(k-x)\kappa\tau\right]\tau N \\
&= \left(-2\kappa^2 + 3\,(k-x)\,\kappa\kappa^{'}\right) T \\
&\quad + \left((k-x)\,\kappa^3 - 2\kappa^{'} + (k-x)\,\kappa^{''} - (k-x)\,\kappa\tau^2\right) N \\
&\quad + \left(-2\kappa\tau + 2\,(k-x)\,\kappa^{'}\tau + (k-x)\,\kappa\tau^{'}\right) B. \\
\beta^{'''} &= \left(-2\kappa^2 + 3\,(k-x)\,\kappa\kappa^{'}, \star, -2\kappa\tau + 2\,(k-x)\,\kappa^{'}\tau + (k-x)\,\kappa\tau^{'}\right)
\end{aligned}$$

Furthermore, since

$$\tau^*(s) = \frac{\det\left(\beta^{'}(s), \beta^{''}(s), \beta^{'''}(s)\right)}{\left\|\beta^{'}(s) \wedge_L \beta^{''}(s)\right\|_L^2},$$

we have

$$\begin{aligned}
\Delta &= -(k-x)^2\kappa^2 \begin{bmatrix} 0 & 1 & 0 \\ \kappa & * & \tau \\ -2\kappa^2 + 3\,(k-x)\,\kappa\kappa^{'} & \star & -2\kappa\tau + 2\,(k-x)\,\kappa^{'}\tau + (k-x)\,\kappa\tau^{'} \end{bmatrix} \\
&= -(k-x)^2\kappa^2\left[-2\kappa^2\tau + 2\,(k-x)\,\kappa^{'}\kappa\tau + (k-x)\,\kappa^2\tau^{'} + 2\kappa^2\tau - 3\,(k-x)\,\kappa\kappa^{'}\tau\right] \\
&= -(k-x)^2\kappa^3\left[(k-x)\,\kappa\tau^{'} - (k-x)\,\kappa^{'}\tau\right] \\
&= (k-x)^3\kappa^3\left(\kappa^{'}\tau - \kappa\tau^{'}\right), \\
\Delta &= \det\left(\beta^{'}, \beta^{''}, \beta^{'''}\right).
\end{aligned}$$

Hence, we get

$$\begin{aligned}
\tau^*(s) &= \frac{\kappa^3.(k-s)^3.\left[\kappa^{'}(s)\tau(s)-\kappa(s)\tau^{'}(s)\right]}{\kappa^4\left|k-s\right|^4(\kappa^2(s)+\tau^2(s))}, \\
\tau^*(s) &= \frac{\kappa^{'}(s)\tau(s)-\kappa(s)\tau^{'}(s)}{\kappa(s).\left|k-s\right|(\kappa^2(s)+\tau^2(s))}.
\end{aligned}$$

∎

From the last equation, we have the following corollaries:

Corollary 5. *If the curve α is planar, then its involute curve β is also planar.*

Corollary 6. *If the curvature $\kappa \neq 0$ and the torsion $\tau \neq 0$ of the curve α are constant ,then the involute curve β is planar, i.e., if the curve α is a ordinary helix, then its the involute curve β is planar.*

Corollary 7. *If the curvature $\kappa \neq 0$ and the torsion $\tau \neq 0$ of the curve α are not constant but $\frac{\tau}{\kappa}$ is constant, then the involute curve β is planar, i.e. if the curve α is a general helix, then their the involute curve β is planar.*

Theorem 8. *Suppose that the spacelike curve $\alpha : I \longrightarrow E_1^3$ with arc-length parameter are given. Then, the locus of the centre of the curvature of the curve α is the unique involute of the curve α which lies on the plane of the curve α.*

Proof. The locus of the centre of the curvature of the curve α is

$$C(s) = \alpha(s) - \frac{1}{\kappa(s)}N(s), \ \ \forall\ s \in R,\ \kappa(s) \neq 0$$

If we take the derivative in the above equation, then we have

$$\begin{aligned}
\frac{dC}{ds} &= T + \left(\frac{1}{\kappa}\right)^{'} N - \frac{1}{\kappa}(\kappa T) \\
&= T + \left(\frac{1}{\kappa}\right)^{'} N - \frac{1}{\kappa}\kappa T \\
C^{'} &= \left(\frac{1}{\kappa}\right)^{'} N \\
\left\langle C^{'}, T\right\rangle_L &= \left(\frac{1}{\kappa}\right)^{'} \langle N, T\rangle_L \\
\left\langle C^{'}(s), T(s)\right\rangle_L &= 0.
\end{aligned}$$

Therefore, the involute C of the spacelike curve α is the locus of the centre of the curvature. Is the curve C planar ? If the torsion of the curve C is denoted by τ^*, then

$$\tau^*(s) = \frac{\left(\kappa^{'}\tau - \kappa\tau^{'}\right)(s)}{\kappa(s)\left|k-s\right|(\kappa^2(s)+\tau^2(s))}.$$

If we take $\tau = 0$, then we have

$$\tau^*(s) = 0.$$

Thus, the curve C is planar. ∎

3. The Evolute of The Spacelike Curve with a Spacelike Binormal

Definition 2. *Let the unit speed curve α and the curve β with the same interval be given. For $\forall s \in I$, the tangent at the point $\beta(s)$ to the curve β passes through the point $\alpha(s)$ and*

$$\langle T^*(s), T(s)\rangle_L = 0.$$

Then, β is called the evolute of the curve α. Let the Frenet-Serret frames of the curves α and β be (T, N, B) and (T^*, N^*, B^*), respectively.

Theorem 9. *Let the curve β be the evolute of the unit speed spacelike curve α , Then*

$$\beta(s) = \alpha(s) - \frac{1}{\kappa(s)}N(s) + \frac{1}{\kappa(s)}\left[\tanh\left(\varphi(s) + c\right)\right]B(s), \tag{3.1}$$

where $c \in IR$ and $\varphi(s) + c = \int \tau(s)ds$. Furthermore, in the normal plane of the point $\alpha(s)$ the measure of directed angle between $\beta(s) - \alpha(s)$ and $N(s)$ is

$$\varphi(s) + c.$$

Proof. The tangent of the curve β at the point $\beta(s)$ is the line constructed by the vector $T^*(s)$. Since this line passes through the point $\alpha(s)$, the vector $\beta(s) - \alpha(s)$ is perpendicular to the vector $T(s)$. Then

$$\beta(s) - \alpha(s) = \lambda N(s) + \mu B(s). \tag{3.2}$$

If we take the derivative of Eq. (3.2), then we have

$$\beta'(s) = \alpha'(s) + \lambda' N + \lambda(\kappa T + \tau B) + \mu' B(s) + \mu(\tau N)$$

$$\beta'(s) = (1 + \lambda\kappa)\, T + \left(\lambda' + \mu\tau\right) N + \left(\lambda\tau + \mu'\right) B. \tag{3.3}$$

According to the definition of the evolute, since $\langle T^*(s), T(s)\rangle = 0$, from Eq. (3.3), we get

$$\lambda = -\frac{1}{\kappa}\ , \tag{3.4}$$

and

$$\beta' = \left(\lambda' + \mu\tau\right) N + \left(\lambda\tau + \mu'\right) B. \tag{3.5}$$

From the Eq. (3.2) and Eq. (3.5), the vector field $\beta'$is parallel to the vector field $\beta - \alpha$. Then we have

$$\frac{\lambda' + \mu\tau}{\lambda} = \frac{\lambda\tau + \mu'}{\mu}.$$

After that, we have

$$\begin{aligned} \tau &= \frac{\lambda'\mu - \lambda\mu'}{\lambda^2 - \mu^2} \\ &= -\frac{\left(\frac{\mu}{\lambda}\right)'}{1 - \left(\frac{\mu}{\lambda}\right)^2}. \end{aligned}$$

If we take the integral the last equation, we get

$$\varphi(s) + c = -\arg\tanh\left(\frac{\mu(s)}{\lambda(s)}\right).$$

Hence, we find

$$\mu(s) = -\lambda(s)\tanh\left[\varphi(s) + c\right]. \tag{3.6}$$

If we substitute Eq. (3.4) and Eq. (3.6) into Eq. (3.2), we have

$$\begin{aligned} \beta(s) &= \alpha(s) - \frac{1}{\kappa(s)}N(s) + \frac{1}{\kappa(s)}\left[\tanh\left(\varphi(s) + c\right)\right]B(s) \\ \beta(s) &= M(s) + \frac{1}{\kappa(s)}\left[\tanh\left(\varphi(s) + c\right)\right]B(s). \end{aligned}$$

Then, we obtain an evolute curve for each $c \in IR$. Since

$$\left\langle \overrightarrow{M(s)\beta(s)}, \overrightarrow{M(s)\ \alpha(s)} \right\rangle_L = 0,$$

in the Lorentzian triangle which have corners $\beta(s)$, $M(s)$ and $\alpha(s)$, the angle M is right angle in the Lorentzian mean. In the same triangle, the tangent of the angle $\alpha(s)$ is

$$\frac{\frac{1}{\kappa(s)}\tanh\left[\varphi(s) + c\right]}{\frac{1}{\kappa(s)}} = \tanh\left[\varphi(s) + c\right]. \tag{3.7}$$

Then, the measure of the angle between the vectors $\beta(s) - \alpha(s)$ and N is $\varphi(s) + c$. ∎

Theorem 10. *Let the timelike curve $\beta : I \longrightarrow E_1^3$ be evolute of the unit speed spacelike curve $\alpha : I \longrightarrow E_1^3$. If the Frenet-Serret vector fields of the curve β are T^* (timelike), N^* (spacelike), B^* (spacelike), then*

$$\begin{bmatrix} T^* \\ N^* \\ B^* \end{bmatrix} = \begin{bmatrix} 0 & \cosh(\varphi + c) & -\sinh(\varphi + c) \\ -1 & 0 & 0 \\ 0 & -\sinh(\varphi + c) & \cosh(\varphi + c) \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{3.8}$$

Proof. Since the Frenet-Serret vector fields of the curve β are T^*, N^*, B^* and

$$\beta = \alpha - \rho N + \rho \tanh(\varphi + c) B,$$

we have

$$\begin{aligned}
\beta'(s) &= \alpha' - \rho' N - \rho(\kappa T + \tau B) + \rho' \tanh(\varphi + c) B \\
&\quad + \rho\varphi' \left[1 - \tanh^2(\varphi + c)\right] B + \rho \tanh(\varphi + c)\tau N \\
&= (1 - \rho\kappa) T + \left(-\rho' + \rho\tau \tanh(\varphi + c)\right) N \\
&\quad + \left[\left(\rho\varphi' - \rho\tau\right) + \rho' \tanh(\varphi + c) - \rho\varphi' \tanh^2(\varphi + c)\right] B \\
&= \left(-\rho' + \rho\tau \tanh(\varphi + c)\right) N - \left[-\rho' + \rho\tau \tanh(\varphi + c)\right] . \tanh(\varphi + c) B \\
&= \left[-\rho' + \rho\tau \tanh(\varphi + c)\right] \left[N - \tanh(\varphi + c) B\right]
\end{aligned}$$

(3.9)

$$\beta'(s) = \left[\frac{-\rho' + \rho\tau \tanh(\varphi + c)}{\cosh(\varphi + c)}\right] \left[\cosh(\varphi + c) N - \sinh(\varphi + c) B\right].$$

If we take the norm in the Eq. (3.9), then we obtain

$$\begin{aligned}
\left\|\beta'(s)\right\|_L &= \frac{\left|-\rho' + \rho\tau \tanh(\varphi + c)\right|}{\cosh(\varphi + c)} \\
&\quad \frac{\left|-\left(\frac{1}{\kappa}\right)' + \frac{1}{\kappa}\tau \frac{\sinh(\varphi+c)}{\cosh(\varphi+c)}(\varphi + c)\right|}{\cosh(\varphi + c)} \\
&= \frac{\left|\cosh(\varphi + c)\kappa' + \kappa\tau \sinh(\varphi + c)\right|}{\kappa^2 \cosh^2(\varphi + c)}.
\end{aligned}$$

Since $T^* = \frac{\beta'}{\left\|\beta'\right\|_L}$, then we get

$$T^* = \cosh(\varphi + c) N - \sinh(\varphi + c) B. \tag{3.10}$$

Therefore, we have obtained Eq. (3.9). The curve β is not a unit speed curve. If we take the derivative of Eq. (3.10) with respect to s, we find

$$\begin{aligned}
(T^*)' &= \varphi' \left[N \sinh(\varphi + c) - B \cosh(\varphi + c)\right] \\
&\quad + \cosh(\varphi + c)(\kappa T + \tau B) - \tau N \sinh(\varphi + c) \\
&= \left(\varphi' - \tau\right) \left[N \sin(\varphi + c) - B \cosh(\varphi + c)\right] + \kappa T \cosh(\varphi + c) \\
&= \kappa T \cosh(\varphi + c).
\end{aligned}$$

Since $T' = \left\|\alpha'\right\|_L \kappa N$, for curve β we have

$$(T^*)' = \left\|\beta'\right\|_L \kappa^* N^*.$$

Thus

$$\left\|\beta'\right\|_L \kappa^* N^* = \kappa T \cosh(\varphi + c).$$

Since the vectors N^* and T have the unit length, we get $N^* = -T$ or $N^* = T$. We assume that $N^* = -T$. Since $B^* = N^* \wedge_L (-T^*)$, we have

$$B^* = -\sinh(\varphi + c)N + \cosh(\varphi + c)B. \tag{3.11}$$

Thus, the proof is completed. ∎

Theorem 11. *Let the timelike curve $\beta : I \longrightarrow E_1^3$ be the evolute of the unit speed spacelike curve $\alpha : I \longrightarrow E_1^3$. Let the Frenet vector fields, curvature and torsion of the curve β be T^*, N^*, B^*, κ^* and τ^*, respectively. Then*

$$\begin{aligned} |\kappa^*| &= \frac{\kappa^3 \cosh^3(\varphi + c)}{|\kappa\tau \sinh(\varphi + c) + \kappa' \cosh(\varphi + c)|},\ \kappa > 0 \\ |\tau^*| &= \frac{\kappa^3 |\sinh(\varphi + c)| \cosh^2(\varphi + c)}{|\kappa\tau \sinh(\varphi + c) + \kappa' \cosh(\varphi + c)|}. \end{aligned}$$

Proof. Since N^* and T have unit length, then taking norm from equility $\left\|\beta'\right\|_L \kappa^* N^* = \kappa T \cosh(\varphi + c)$ we can write have

$$\left\|\beta'\right\|_L |\kappa^*| = \kappa \left|\cosh(\varphi + c)\right|.$$

Therefore, we have

$$|\kappa^*| = \frac{\kappa \cosh(\varphi + c)}{\|\beta'\|_L} \tag{3.12}$$

$$|\kappa^*| = \kappa \cosh(\varphi + c) : \frac{\left|\cosh(\varphi + c)\kappa' + \kappa\tau \sinh(\varphi + c)\right|}{\kappa^2 \cosh^2(\varphi + c)},$$

$$|\kappa^*| = \frac{\kappa^3 \cosh(\varphi + c)^3}{|\kappa\tau \sinh(\varphi + c) + \kappa' \cosh(\varphi + c)|}$$

If we take the derivative Eq. (3.11) with respect to s, then we have

$$\begin{aligned} (B^*)' &= \left(\varphi' - \tau\right) [B \sinh(\varphi + c) - N \cosh(\varphi + c)] - \kappa T \sinh(\varphi + c) \\ &= -\kappa T \sinh(\varphi + c). \end{aligned}$$

Since $(B^*)' = -\left\|\beta'\right\|_L \tau^* N^*$, we get

$$-\left\|\beta'\right\|_L \tau^* N^* = -\kappa T \sinh(\varphi + c).$$

and so we find that

$$|\tau^*| = \frac{\kappa \left|\sinh(\varphi + c)\right|}{\|\beta'\|_L},\ \kappa > 0 \tag{3.13}$$

$$|\tau^*| = \frac{\kappa^3 |\sinh(\varphi + c)| \cosh^2(\varphi + c)}{|\kappa\tau \sinh(\varphi + c) + \kappa' \cosh(\varphi + c)|}.$$

∎

Theorem 12. *Let $\beta : I \longrightarrow E_1^3$ be the evolute of the unit speed spacelike curve $\alpha : I \longrightarrow E_1^3$. Let the curvature and torsion of the curve β be κ^* and τ^*, respectively. Then*

$$\left|\frac{\tau^*}{\kappa^*}\right| = |\tanh(\varphi + c)|. \tag{3.14}$$

Furthermore, we denote by $\beta^{(1)}$ and $\beta^{(2)}$, the evolute curves obtained by using c_1 and c_2 instead of c, respectively. The tangents of the curves $\beta^{(1)}$ and $\beta^{(2)}$ at the points $\beta^{(1)}(s)$ and $\beta^{(2)}(s)$ intersect at the point $\alpha(s)$. The measure of the angle between the tangents is $c_1 - c_2$.

Proof. The Eq. (3.14) is obtained easily by using Eq. (3.12) and Eq. (3.13), i.e.,

$$\left|\frac{\tau^*}{\kappa^*}\right| = \frac{\kappa |\sinh(\varphi + c)|}{\|\beta'\|_L} : \frac{\kappa \cosh(\varphi + c)}{\|\beta'\|_L} = |\tanh(\varphi + c)|.$$

The measure of the angle between the vectors $\overrightarrow{\alpha(s)\beta^{(1)}(s)}$ and $V_2(s)$, and between the vectors $\overrightarrow{\alpha(s)\beta^{(2)}(s)}$ and $N(s)$ are $\varphi(s) + c_1$ and $\varphi(s) + c_2$, respectively. The vector $\overrightarrow{\alpha(s)\beta^{(1)}(s)}$ is parallel to the tangent of the curve $\beta^{(1)}$ at the point $\beta^{(1)}(s)$. The vector $\overrightarrow{\alpha(s)\beta^{(2)}(s)}$ is parallel to the tangent of the curve $\beta^{(2)}$ at the point $\beta^{(2)}(s)$. Furthermore, since $\overrightarrow{\alpha(s)\beta^{(1)}(s)}$, $\overrightarrow{\alpha(s)\beta^{(1)}(s)}$ and $\overrightarrow{N}$ are perpendicular to the vector $T(s)$, these three vectors are planar. Then, the measure of the angle between the tangents of the curves $\beta^{(1)}$ and $\beta^{(2)}$ at the points $\beta^{(1)}(s)$ and $\beta^{(2)}(s)$ is

$$\varphi(s) + c_1 - (\varphi(s) + c_2) = c_1 - c_2.$$

So, the proof is completed. ∎

Theorem 13. *Suppose that, two different evolutes of the spacelike curve α are given. Let the points on the evolutes of the curve α corresponding to the point P be P_1 and P_2. Then the angle $\widehat{P_1 P P_2}$ is constant.*

Proof. Let the evolutes of the curve α be β and γ. Let the arc-length parameters of the α, β and γ be s, $s^*$and $\widehat{s}$, respectively. Let the curvatures of the curves α, β and γ be k, k^* and $\widehat{k}$, respectively. And let the Frenet vectors of the curves α, β and γ be $\{T, N, B\}, \{T^*, N^*, B^*\}$ and $\left\{\widehat{T}, \widehat{N}, \widehat{B}\right\}$. Then

$$T = N^*, T = \widehat{N}. \tag{3.15}$$

Since the curves β and γ are evolute, then

$$\langle T, T^* \rangle_L = \left\langle T, \widehat{T} \right\rangle_L = 0 \tag{3.16}$$

Therefore, if $f(s) = \left\langle T^*, \widehat{T} \right\rangle_L$, then we have

$$\begin{aligned}
(f)'(s) &= \left\langle (T^*)', \widehat{T} \right\rangle_L + \left\langle T^*, \left(\widehat{T}\right)' \right\rangle_L \\
&= \left\langle \kappa^* N^* \frac{ds^*}{ds}, \widehat{T} \right\rangle_L + \left\langle T^*, \widehat{\kappa}\widehat{N}\frac{d\widehat{s}}{ds} \right\rangle_L \\
&= \kappa^* \frac{ds^*}{ds} \left\langle N^*, \widehat{T} \right\rangle_L + \widehat{\kappa}\frac{d\widehat{s}}{ds} \left\langle T^*, \widehat{N} \right\rangle_L \\
&= \kappa^* \frac{ds^*}{ds} \left\langle T, \widehat{T} \right\rangle_L + \widehat{\kappa}\frac{d\widehat{s}}{ds} \langle T^*, N^* \rangle_L \\
&= \kappa^* \frac{ds^*}{ds}.0 + \widehat{\kappa}\frac{d\widehat{s}}{ds}.0 \\
(f)'(s) &= 0.
\end{aligned}$$

Therefore, we have $f(s) = \theta$ =constant. Hence, $m\left(\widehat{P_1 P P_2}\right) = m\left(T^*, \widehat{T}\right) = \theta$ =constant. ∎

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