

A Numerical Solution for Nonlinear PDEs

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Abstract

In this paper, we solve system of time dependent partial differential equations (PDEs) by using pseudospectral method. Firstly, theory of application of spectral collocation method on system of time dependent partial differential equations presented. This method yields a system of ordinary differential equations (ODEs). Secondly, we use forth-order Runge-Kutta formula for the numerical integration of the system of ODE. we consider some examples to illustrate the performance of the method described.

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1 Introduction

We consider system of n_{npe} partial differential equations (see[5]) of the form:

$$\mathbf{u}_t = f(x, t, \mathbf{u}, (M(x, t, \mathbf{u})\mathbf{u}_x)_x), \quad (1)$$

$$\mathbf{u}(x, t) = \mathbf{u}_0(x), \quad (2)$$

$$\alpha(x)\mathbf{u} + \beta(x, t, \mathbf{u})\mathbf{u}_x = \gamma(x, t, \mathbf{u}), \quad (3)$$

$$x \in [a, b], \quad t \geq 0. \quad (4)$$

In order to solve (1)-(4), many researchers have used various numerical and analytical methods to solve the system of time dependent partial differential equations. Darvishi and Javidi [5] studied method of lines for solving system of time dependent partial differential equations. Abdou and Soliman [1] presented variational iteration method for solving Burger's and coupled Burger's equations. Nowak [8] presented a fully adaptive MOL-treatment of parabolic 1 - D problems with extrapolation techniques.

In this paper, system of partial differential equations was solved numerically by combination of pseudospectral method and fourth order Runge-Kutta method. Here, we consider one test problem of the reference [1] and one test problem of [5]. The numerical results are compared with the exact solutions. It was seen that the absolute of errors are very small.

Problem(a). The first problem is selected from [1]. The problem is

$$\begin{cases} u_t = u_{xx} + 2uu_x - (uv)_x, \\ v_t = v_{xx} + 2vv_x - (uv)_x, \end{cases} \quad x \in [0, 1], \quad t \geq 0, \quad (5)$$

with the initial conditions

$$u(x, 0) = v(x, 0) = \sin x = \alpha(x) \quad (6)$$

and boundary conditions

$$u(0, t) = v(0, t) = 0, \quad u(1, t) = v(1, t) = e^{-t} \sin 1 = \beta(t). \quad (7)$$

The exact solution of the Eq. (5) with the initial conditions (6) and boundary conditions (7) is obtained as

$$u(x, t) = v(x, t) = e^{-t} \sin x. \quad (8)$$

Problem(b). (see [5]) Consider the nonlinear parabolic problem

$$\begin{cases} u_{tt} = u_{xx} + u^2 + G(x, t), & (x, t) \in B, \\ u(x, t) = u_t(x, 0) = g(x), & 0 \leq x \leq 1, \\ u(0, t) = u(1, t) = h(t), & 0 \leq t \leq 1, \end{cases} \quad (9)$$

where

$$G(x, t) = \exp(t)(1 + \pi^2 - \exp(t) \sin \pi x) \sin \pi x$$

and

$$\begin{aligned} g(x) &= \sin \pi x, \quad h(t) = 0, \\ B &= \{(x, t) \mid 0 \leq x, t \leq 1\}. \end{aligned}$$

The analytic solution of the problem is

$$u(x, t) = e^t \sin \pi x. \quad (10)$$

By setting $u_t = v$ the following system of PDE is obtained from (9)

$$\begin{cases} u_t = v, \\ v_t = u_{xx} + u^2 + G(x, t), \end{cases} \quad 0 \leq x \leq 1, \quad 0 \leq t \leq 1, \quad (11)$$

with the following initial and boundary conditions

$$\begin{aligned} u(x, 0) &= v(x, 0) = g(x), \quad x \in [0, 1], \\ u(0, t) &= u(1, t) = h(t), \quad t \in [0, 1]. \end{aligned} \quad (12)$$

2 Pseudospectral Chebyshev method

One of the methods to solve partial differential equations in the spectral collocation method or the pseudospectral method (see [3, 6]). Pseudospectral methods have become increasingly popular for solving differential equations and also very useful in providing highly accurate solutions to differential equations.

In this method, such an approximation of $f_N(x)$ to $f(x)$ is presented that $f_N(x_i) = f(x_i)$ for some collocation point x_i . After setting f_N in the differential equation, we have to use derivative(s) of f_N at the collocation point. A straightforward implementation of the spectral collocation methods involves the use of spectral differentiation matrices to compute derivatives at the collocation points, in which if $\vec{F} = \{f(x_i)\}$ is the vector consisting of values of $f(x)$ at the $N + 1$ collocation points and $\vec{F}' = \{f'(x_i)\}$ consists of the values of the derivatives at the collocation points, then the collocation derivative matrix D is the matrix mapping $\vec{F} \rightarrow \vec{F}'$. The entries of derivative matrix D can be computed analytically. To obtain optimal accuracy this matrix must be computed carefully. In [2, 3, 9] the authors describe the subject very well. Let $f(x)$ be a function on $[-1, 1]$. We interpolate $f(x)$ by the polynomial $f_N(x)$ of degree at most N of the form

$$f_N(x) = \sum_{j=0}^N g_j(x)f(x_j). \quad (13)$$

In the Chebyshev-Gauss-Lobatto points

$$x_j = \cos\left(\frac{j\pi}{N}\right), \quad j = 0(1)N,$$

with $g_j(x)$, $j = 0(1)N$ are polynomial of degree at most N such that:

$$g_j(x_k) = \delta_{jk}, \quad j, k = 0(1)N.$$

It can be shown that [2]

$$g_j(x) = \frac{(-1)^{j+1}(1-x^2)T'_N(x)}{c_j N^2(x-x_j)}, \quad j = 0(1)N, \quad (14)$$

where

$$c_0 = c_N = 2, \quad c_j = 1, \quad j = 1(1)N - 1$$

and $T_N(x)$ the Chebyshev polynomial, i.e.,

$$T_N(x) = \cos(N \arccos x).$$

The derivatives of the approximate solution $f_N(x)$ are then estimated at the collocation points by differentiating (13) and evaluating the resulting expression [2]. This yields

$$f_N^{(n)}(x) = \sum_{j=0}^N g_j^{(n)}(x) f(x_j), \quad n = 1, 2, \dots, \quad (15)$$

or in matrix notation

$$F^{(n)} = D^{(n)} F, \quad n = 1, 2, \dots,$$

where

$$F^{(n)} = [f_N^{(n)}(x_0), f_N^{(n)}(x_1), \dots, f_N^{(n)}(x_N)]^T$$

and

$$F = [f(x_0), f(x_1), \dots, f(x_N)]^T$$

and where $D^{(n)}$ is the $(N+1) \times (N+1)$ matrix whose entries are given by

$$d_{kj}^{(n)} = g_j^{(n)}(x_k), \quad j, k = 0(1)N.$$

The first-order Chebyshev differentiation matrix $D^{(1)} = D = (d_{kj})$, is given by (see [4, 6, 7]):

$$d_{kj} = \begin{cases} -\frac{c_k}{2c_j} \frac{(-1)^{j+k}}{\sin((k+j)\frac{\pi}{2N}) \sin((k-j)\frac{\pi}{2N})}, & k \neq j, \\ -\frac{1}{2} \cos(\frac{k\pi}{N}) (1 + \cot^2(\frac{k\pi}{N})), & k = j, k \neq 0, N \\ d_{00} = -d_{NN} = \frac{2N^2+1}{6}. \end{cases} \quad (16)$$

3 Semi-discrete models for system of PDEs

We will describe the pseudospectral Chebyshev method for problem (a) and (b). Let N be a nonnegative integer and denote by $z_i = \cos(\frac{i\pi}{N})$, $i = 0(1)N$, the Chebyshev-Gauss-Lobatto points in the interval $[-1, 1]$ and put

$$\begin{aligned} x &= \frac{b-a}{2}z + \frac{b+a}{2}, \\ x_i &= \frac{b-a}{2}z_i + \frac{b+a}{2}, \quad i = 0(1)N, \\ u(x, t) &= U(z, t), \\ v(x, t) &= V(z, t), \\ c &= \frac{b-a}{2}. \end{aligned}$$

Then from (5), we have

$$\begin{cases} U_t = \frac{1}{c^2}U_{zz} + \frac{2}{c}UU_z - \frac{1}{c}(UV)_z, \\ V_t = \frac{1}{c^2}V_{zz} + \frac{2}{c}VV_z - \frac{1}{c}(UV)_z, \end{cases} \quad z \in [-1, 1] \quad , \quad t \geq 0, \quad (17)$$

with the following initial and boundary conditions

$$U(z, 0) = V(z, 0) = \alpha\left(\frac{b-a}{2}z + \frac{b+a}{2}\right), \quad z \in [-1, 1], \quad (18)$$

$$U(-1, t) = U(1, t) = V(-1, t) = V(1, t) = \beta(t). \quad (19)$$

We discretize (17) in space by the method of lines replacing $\frac{\partial^k U}{\partial z^k}(z_i, t)$ and $\frac{\partial^k V}{\partial z^k}(z_i, t)$, $k = 1, 2$, by pseudospectral approximations given by

$$\frac{\partial^k U}{\partial z^k}(z_i, t) \approx \sum_{j=0}^N d_{ij}^{(k)} U(z_j, t), \quad i = 1(1)N - 1, \quad k = 1, 2 \quad (20)$$

and

$$\frac{\partial^k V}{\partial z^k}(z_i, t) \approx \sum_{j=0}^N d_{ij}^{(k)} V(z_j, t), \quad i = 1(1)N - 1, \quad k = 1, 2. \quad (21)$$

Here

$$D^{(k)} = [d_{ij}^{(k)}]_{i,j=0}^N \quad k = 1, 2$$

are differentiation matrices of order k . Put $Y_i(t) = U(z_i, t)$, $i = 0(1)N$ and $Y_{i+N+1}(t) = V(z_i, t)$, $i = 0(1)N$. Substituting (20) and (21) into (17) and taking into account that $Y_0(t) = Y_N(t) = Y_{N+1}(t) = Y_{2N+1}(t) = \beta(t)$, we obtain

$$\begin{aligned} Y'_i(t) &= \frac{1}{c^2}((d_{i0}^{(2)} + d_{iN}^{(2)})\beta(t) + \sum_{j=1}^{N-1} d_{ij}^{(2)} Y_j(t)) \\ &\quad + \frac{2}{c} Y_i(t)((d_{i0}^{(1)} + d_{iN}^{(1)})\beta(t) + \sum_{j=1}^{N-1} d_{ij}^{(1)} Y_j(t)) \\ &\quad - \frac{1}{c}((d_{i0}^{(1)} + d_{iN}^{(1)})\beta^2(t) + \sum_{j=1}^{N-1} d_{ij}^{(1)} Y_j(t) Y_{j+N+1}(t)) \\ &\quad i = 1(1)N - 1 \\ Y'_{i+N+1}(t) &= \frac{1}{c^2}((d_{i0}^{(2)} + d_{iN}^{(2)})\beta(t) + \sum_{j=1}^{N-1} d_{ij}^{(2)} Y_{j+N+1}(t)) \\ &\quad + \frac{2}{c} Y_{i+N+1}(t)((d_{i0}^{(1)} + d_{iN}^{(1)})\beta(t) + \sum_{j=1}^{N-1} d_{ij}^{(1)} Y_{j+N+1}(t)) \\ &\quad - \frac{1}{c}((d_{i0}^{(1)} + d_{iN}^{(1)})\beta^2(t) + \sum_{j=1}^{N-1} d_{ij}^{(1)} Y_j(t) Y_{j+N+1}(t)) \\ &\quad i = 1(1)N - 1 \end{aligned} \quad (22)$$

with the following initial conditions

$$Y_i(0) = Y_{i+N+1}(0) = \alpha\left(\frac{b-a}{2}z_i + \frac{b+a}{2}\right), \quad i = 1(1)N - 1. \quad (23)$$

Then the system (22) can be rewritten in the following form

$$\begin{cases} Y'(t) = F(t, Y(t)), \\ Y(0) = Y_0, \end{cases} \quad (24)$$

where

$$\begin{aligned} Y(t) &= [Y_1(t), \dots, Y_{N-1}(t), Y_{N+2}(t), \dots, Y_{2N}(t)]^T, \\ Y_0 &= [\alpha(x_1), \dots, \alpha(x_{N-1}), \alpha(x_1), \dots, \alpha(x_{N-1})]^T, \\ Y'(t) &= [Y'_1(t), \dots, Y'_{N-1}(t), Y'_{N+2}(t), \dots, Y'_{2N}(t)]^T, \end{aligned}$$

$$F(t, Y(t)) = [F_1(t, Y(t)), \dots, F_{N-1}(t, Y(t)), F_{N+2}(t, Y(t)), \dots, F_{2N}(t, Y(t))]^T$$

and

$$\begin{aligned} F_i(t, Y(t)) &= \frac{1}{c^2}((d_{i0}^{(2)} + d_{iN}^{(2)})\beta(t) + \sum_{j=1}^{N-1} d_{ij}^{(2)} Y_j(t)) \\ &\quad + \frac{2}{c} Y_i(t)((d_{i0}^{(1)} + d_{iN}^{(1)})\beta(t) + \sum_{j=1}^{N-1} d_{ij}^{(1)} Y_j(t)) \\ &\quad - \frac{1}{c}((d_{i0}^{(1)} + d_{iN}^{(1)})\beta^2(t) + \sum_{j=1}^{N-1} d_{ij}^{(1)} Y_j(t) Y_{j+N+1}(t)) \\ &\quad i = 1(1)N - 1 \\ F_{i+N+1}(t, Y(t)) &= \frac{1}{c^2}((d_{i0}^{(2)} + d_{iN}^{(2)})\beta(t) + \sum_{j=1}^{N-1} d_{ij}^{(2)} Y_{j+N+1}(t)) \\ &\quad + \frac{2}{c} Y_{i+N+1}(t)((d_{i0}^{(1)} + d_{iN}^{(1)})\beta(t) + \sum_{j=1}^{N-1} d_{ij}^{(1)} Y_{j+N+1}(t)) \\ &\quad - \frac{1}{c}((d_{i0}^{(1)} + d_{iN}^{(1)})\beta^2(t) + \sum_{j=1}^{N-1} d_{ij}^{(1)} Y_j(t) Y_{j+N+1}(t)) \\ &\quad i = 1(1)N - 1 \end{aligned} \quad (25)$$

Equations (24) form a system of ordinary differential equations (ODEs) in time. Therefore, to advance the solution in time, we use ODE solver such as the Runge-Kutta methods of order fourth.

4 Numerical results

We now obtain numerical solutions of problem (a) and (b). To show the efficiency of the present method for our problem in comparison with the exact solution, we report absolute errors which defined by

$$|Eu|_{ij} = |u_{approx}(x_i, t_j) - u_{exact}(x_i, t_j)|$$

and

$$|Ev|_{ij} = |v_{approx}(x_i, t_j) - v_{exact}(x_i, t_j)|,$$

Where $u_{approx}(x_i, t_j)$ and $v_{approx}(x_i, t_j)$ are the solution obtained by Eq. (24) solved by forth-order Runge-Kutta method and $u_{exact}(x_i, t_j)$ and $v_{exact}(x_i, t_j)$ are the exact solutions. As seen in Table 1-3 , the absolute errors are very small.

Problem(a). In Table 1, we show absolute error for various values of x and t with $\Delta t = 0.0001$ and $N = 16$.

Table1. Absolute error $|Eu|_{ij} = |Ev|_{ij}$ for various values of x and t with $\Delta t = 0.0001$ and $N = 16$ of problem(a) .

| $x[i] \setminus t_j$ | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|----------------------|----------------|----------------|----------------|----------------|----------------|
| $x[1]$ | $1.2690E - 10$ | $1.0390E - 10$ | $8.5063E - 11$ | $6.9644E - 11$ | $5.7019E - 11$ |
| $x[3]$ | $4.2830E - 12$ | $3.5062E - 12$ | $2.8703E - 12$ | $2.3500E - 12$ | $1.9241E - 12$ |
| $x[5]$ | $5.0293E - 13$ | $4.1012E - 13$ | $3.3556E - 13$ | $2.7456E - 13$ | $2.2493E - 13$ |
| $x[7]$ | $1.5360E - 13$ | $1.2335E - 13$ | $1.0059E - 13$ | $8.1257E - 14$ | $6.7557E - 14$ |
| $x[9]$ | $8.1990E - 14$ | $6.4726E - 14$ | $5.2569E - 14$ | $4.2993E - 14$ | $3.5305E - 14$ |
| $x[11]$ | $7.4191E - 14$ | $5.9119E - 14$ | $4.8184E - 14$ | $3.9427E - 14$ | $3.2307E - 14$ |
| $x[13]$ | $1.4855E - 13$ | $1.2091E - 13$ | $9.8928E - 14$ | $8.0963E - 14$ | $6.6312E - 14$ |
| $x[15]$ | $1.5609E - 12$ | $1.2770E - 12$ | $1.0463E - 12$ | $8.5660E - 13$ | $7.0133E - 13$ |

Problem(b). In Table 1-2, we show absolute error $|Eu|_{ij}$ and $|Ev|_{ij}$ for various values of x and t with $N = 16$. In Table1, we take $\Delta t = 0.0001$ and in Table 2 , we take $\Delta t = 0.001$.

Table2. Absolute error $|Eu|_{ij}$ and $|Ev|_{ij}$ for various values of x and t with $\Delta t = 0.0001$ and $N = 16$ of problem(b).

| $x[i] \setminus t_j$ | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|----------------------|----------------|----------------|----------------|----------------|----------------|
| $ Eu _{ij}$ | | | | | |
| $x[3]$ | $4.1633E - 16$ | $1.9984E - 15$ | $5.3291E - 15$ | $9.2704E - 15$ | $1.2101E - 14$ |
| $x[7]$ | $5.3291E - 15$ | $2.7311E - 14$ | $5.1958E - 14$ | $7.7049E - 14$ | $9.6367E - 14$ |
| $x[11]$ | $5.3219E - 15$ | $2.7311E - 14$ | $5.2847E - 14$ | $7.6605E - 14$ | $9.7256E - 14$ |
| $x[14]$ | $1.6653E - 16$ | $9.4924E - 15$ | $1.3156E - 14$ | $2.5868E - 14$ | $2.8311E - 14$ |
| $ Ev _{ij}$ | | | | | |
| $x[3]$ | $3.2807E - 14$ | $1.2962E - 14$ | $1.7789E - 13$ | $7.2886E - 14$ | $4.3243E - 14$ |
| $x[7]$ | $5.1292E - 14$ | $9.7700E - 15$ | $5.8176E - 14$ | $1.1480E - 13$ | $1.8963E - 13$ |
| $x[11]$ | $5.1736E - 14$ | $8.8818E - 15$ | $4.7962E - 14$ | $1.1391E - 13$ | $2.1982E - 13$ |
| $x[14]$ | $1.4044E - 12$ | $7.7716E - 16$ | $7.6050E - 14$ | $7.4052E - 14$ | $4.6074E - 14$ |

Table3. Absolute error $|Eu|_{ij}$ and $|Ev|_{ij}$ for various values of x and t with $\Delta t = 0.001$ and $N = 16$ of problem(b).

| $x[i] \setminus t_j$ | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|----------------------|----------------|----------------|----------------|----------------|----------------|
| $ Eu _{ij}$ | | | | | |
| $x[3]$ | $3.3307E - 15$ | $9.0206E - 15$ | $1.5210E - 14$ | $2.4298E - 14$ | $3.7637E - 14$ |
| $x[7]$ | $2.8866E - 15$ | $3.1086E - 14$ | $7.3497E - 14$ | $1.2790E - 13$ | $1.7976E - 13$ |
| $x[11]$ | $3.1086E - 15$ | $3.0864E - 14$ | $7.3719E - 14$ | $1.2834E - 13$ | $1.7879E - 13$ |
| $x[14]$ | $6.1062E - 15$ | $1.8485E - 14$ | $3.2696E - 14$ | $5.1736E - 14$ | $8.1823E - 14$ |
| $x[i] \setminus t_j$ | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
| $ Ev _{ij}$ | | | | | |
| $x[3]$ | $3.4556E - 14$ | $4.1717E - 14$ | $4.1050E - 14$ | $5.5622E - 14$ | $1.0897E - 13$ |
| $x[7]$ | $1.2723E - 13$ | $2.5402E - 13$ | $3.2374E - 13$ | $3.6326E - 13$ | $3.3529E - 13$ |
| $x[11]$ | $1.1968E - 13$ | $2.5668E - 13$ | $3.1708E - 13$ | $3.4728E - 13$ | $3.1930E - 13$ |
| $x[14]$ | $7.6716E - 14$ | $8.7597E - 14$ | $1.3844E - 13$ | $1.8308E - 13$ | $1.5332E - 13$ |

5 Conclusions

In this paper, we have proposed an efficient Spectral collocation for solve system of time dependent partial differential equations (PDEs), with highly convergence and very small error. As seen in Table 1-3, errors are very small and they are very better than the results of another papers cited in this article.

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