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Ideal amenability of module extension Banach algebras

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Abstract

Let $\mathcal{A}$ be a Banach algebra and let I be a closed two-sided ideal in $\mathcal{A}$, $\mathcal{A}$ is $\mathcal{A}$ is I-weakly amenable if the first cohomology group of $\mathcal{A}$ with coefficients in the dual space I^* is zero; i.e., $H^1(\mathcal{A}, I^*) = \{0\}$. Further, $\mathcal{A}$ is ideally amenable if $\mathcal{A}$ is I-weakly amenable for every closed two-sided ideal I in $\mathcal{A}$. In this paper we find the necessary and sufficient conditions for a module extension Banach algebra to be ideally amenable.

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1 Introduction

Let $\mathcal{A}$ be a Banach algebra and let X be a Banach $\mathcal{A}$-bimodule, then X^* is a Banach $\mathcal{A}$-bimodule if for each $a \in \mathcal{A}$ and $x \in X$ and $x^* \in X^*$ we define

$$\langle x, ax^* \rangle = \langle xa, x^* \rangle, \qquad \langle x, x^*a \rangle = \langle ax, x^* \rangle.$$

In particular I is a Banach $\mathcal{A}$-bimodule and I^* is a dual $\mathcal{A}$-bimodule for every closed two-sided ideal I in $\mathcal{A}$. If X is a Banach $\mathcal{A}$-module then a derivation from $\mathcal{A}$ into X is a continuous linear operator D if for every $a, b \in \mathcal{A}$,

$$D(ab) = D(a).b + a.D(b).$$

If $x \in X$ and we define δ_x from $\mathcal{A}$ into X as follows

$$\delta_x(a) = a.x - x.a \qquad (a \in \mathcal{A}),$$

δ_x is a derivation. Derivations of this form are called inner derivations. A Banach algebra $\mathcal{A}$ is amenable if every derivation from $\mathcal{A}$ into every dual

$\mathcal{A}$-module X is inner; i.e. $H^1(\mathcal{A}, X^*) = \{0\}$, where $H^1(\mathcal{A}, X^*)$ is the first cohomology group from $\mathcal{A}$ with coefficients in X^*. This definition was introduced by B. E. Johnson in [Jo1] (see [Ru] and [He]). $\mathcal{A}$ is weakly amenable if, every derivation from $\mathcal{A}$ into $\mathcal{A}^*$ is inner (see [Jo3], [D-Gh], [Gr1], [Gr2] and [Gr3]). Bade, Curtis and Dales [B-C-D] have introduced the concept of weak amenability for commutative Banach algebras. Let $n \in \mathbb{N}$. A Banach algebra $\mathcal{A}$ is called n-weakly amenable if, $H^1(\mathcal{A}, \mathcal{A}^{(n)}) = \{0\}$. Dales, Ghahramani and Gronbaek started the concept of n-weak amenability of Banach algebras in [D-Gh-G]. A Banach algebra $\mathcal{A}$ is ideally amenable if $H^1(\mathcal{A}, I^*) = \{0\}$, for every closed two-sided ideal I in $\mathcal{A}$. Eshaghi-Gordji and Yazdanpanah have introduced the concept of ideal amenability of Banach algebras in [G-Y] (see [E-H] and [E-M]). Let $\mathcal{A}$ be a Banach algebra and X be a Banach $\mathcal{A}$-bimodule, and let $B_1 = \mathcal{A} \oplus X$ as a Banach space, so that

$$\|(a,x)\| = \|a\| + \|x\| \qquad (a \in \mathcal{A}\ ,\ x \in X).$$

Then B_1 is a Banach algebra with the product

$$(a_1, x_1)(a_2, x_2) = (a_1a_2, x_1 \cdot a_2 + a_1 \cdot x_2)\ .$$

B_1 is called a module extension Banach algebra. It is easy to show that ${B_1}^* = \mathcal{A}^* \oplus X^*$, where this sum is $\mathcal{A}$-bimodule l_∞-sum. Yong Zhang in [Zh] found the necessary and sufficient conditions for a module extension Banach algebra to be n-weakly amenable ($n \in \mathbb{N}$). We prove Zhang's Theorems for n-ideal amenability when n=1.

2 Module extension Banach algebras

Let $\mathcal{A}$ be a Banach algebra and let X be a Banach $\mathcal{A}$-bimodule. It is easy to show that J is a closed ideal in $\mathcal{A} \oplus X$, if and only if there exist closed ideal I in $\mathcal{A}$ and a closed $\mathcal{A}$-submodule Y of X such that $J = I \oplus Y$ and that $IX \cup XI \subseteq Y$. We will find the necessary and sufficient conditions for a module extension Banach algebra to be ideally amenable. First we give the following Lemmas.

Lemma 2.1. Suppose that $\Gamma : X \longrightarrow I^*$ is a continuous $\mathcal{A}$-bimodule morphism. Then $\bar{\Gamma} : A \oplus X \longrightarrow (I \oplus Y)^*$ defined by $\bar{\Gamma}((a,x)) = (\Gamma(x), 0)$ is a continuous derivation. The derivation $\bar{\Gamma}$ is inner if and only if there exists $F \in Y^*$such that $aF - Fa = 0$ and $\Gamma(x) = xF - Fx$ for $a \in A$ and $x \in X$.

proof It is straightforward to check that $\bar{\Gamma}$ is continuous derivation. Noting that $(\Gamma(x), 0) = \bar{\Gamma}((0,x))$, $\bar{\Gamma}((a,0)) = (0,0)$. Now let $\bar{\Gamma}$ is inner then

$\bar{\Gamma}((a,x)) = (a,x).\xi - \xi.(a,x)$ where $\xi \in (I \oplus Y)^*$. We have $(\Gamma(x),0) = \bar{\Gamma}((0,x)) = (0,x).\xi - \xi.(0,x)$ and

$$(0,0) = \bar{\Gamma}((a,0)) = (a,0).\xi - \xi.(a,0).$$

We define $F : Y \longrightarrow \mathbb{C}$ by $F(y) = \xi(0,y)$. Since Y is a submodule of X, for $y \in Y$ and $a \in A$ we have $ya, ay \in Y$ and $(aF)(y) = F(ya), (Fa)(y) = F(ay)$ and hence aF, Fa are well-defined. Furthermore

$$\begin{aligned}(aF - Fa)(y) &= F(ya - ay) = \xi(0, ya - ay)\\ &= ((a,0)\xi)(0,y) - (\xi(a,0))(0,y)\\ &= 0,\end{aligned}$$

then $aF - Fa = 0$ where $a \in A$. For every $x \in X$ we have

$$(\Gamma(x),0) = \bar{\Gamma}((0,x)) = (0,x).\xi - \xi.(0,x).$$

But $\xi \in (I \oplus Y)^*$ and hence there exists $u \in I^*$ and $F \in Y^*$ such that $\xi = (u,F)$. Then we have

$$(\Gamma(x),0) = \bar{\Gamma}((0,x)) = (0,x).(u,F) - (u,F).(0,x) = (xF - Fx, aF - Fa).$$

Therefore $xF - Fx = \Gamma(x)$ for each $x \in X$. For converse, if such an element F exists, then

$$\bar{\Gamma}((a,x)) = (\Gamma(x),0) \ = (xF - Fx, aF - Fa) = (a,x).(0,F) - (0,F).(a,x),$$

where $\xi \in (I \oplus Y)^*$ and then $\bar{\Gamma}$ is inner. ■

Lemma 2.2. Let $D : A \longrightarrow Y^*$ be a continuous derivation. Then $\bar{D} : (A \oplus X) \longrightarrow (I \oplus Y)^*$, defined by $\bar{D}((a,x)) = (0, D(a))$ is also a continuous derivation. Also we have
1. if $\bar{D}$ is inner, then so is D;
2. if D is inner, then there exists a continuous derivation $\tilde{D} : (A \oplus X) \longrightarrow (I \oplus Y)^*$ satisfying $\tilde{D}((a,0)) = 0$ $a \in A$ and for which $\bar{D} - \tilde{D}$ is inner.

proof Clearly $\bar{D}$ is a continuous derivation. Let $\bar{D}$ is inner then for some $u \in I^*$ and $F \in Y^*$ we have $\bar{D}((a,x)) = (a,x).(u,F) - (u,F).(a,x)$ but

$$\begin{aligned}(0, D(a)) &= \bar{D}((a,0)) = (a,0).(u,F) - (u,F).(a,0)\\ &= (au, aF) - (ua, Fa) = (au - ua, aF - Fa),\end{aligned}$$

and then $D(a) = aF - Fa$ for $F \in Y^*$ and D is inner. Now let D is inner. We have, there exists $F \in Y^*$ such that $D(a) = aF - Fa$ for $a \in A$. Let

$T : X \longrightarrow I^*$ be defined by $T(x) = Fx - xF$ $x \in X$ (Fx and xF are well-defined because $IX, XI \subseteq Y$) and $\bar{T} : (A \oplus X) \longrightarrow (I \oplus Y)^*$ defined by $\bar{T}((a,x)) = (T(x), 0)$ $(a,x) \in (A \oplus X)$. Then

$$\begin{aligned}(\bar{D} - \bar{T})((a,x)) &= \bar{D}((a,x)) - \bar{T}((a,x)) \\ &= (0, D(a)) - (T(x), 0) \\ &= (xF - Fx, aF - Fa) \\ &= (a,x).(0,F) - (0,F).(a,x).\end{aligned}$$

Now let $\tilde{D} = \bar{T}$, then we have $\tilde{D}((a,0)) = \bar{T}((a,0)) = (T(0),0) = 0$, and $\bar{D} - \tilde{D}$ is inner. ■

If $D : A \longrightarrow I^*$ is a continuous derivation, we define $\bar{D} : (A \oplus X) \longrightarrow (I \oplus Y)^*$ by $\bar{D}((a,x)) = (D(a), 0)$.

If $T : X \longrightarrow Y^*$ is a continuous $\mathcal{A}$-bimodule morphism, we define $\bar{T} : (A \oplus X) \longrightarrow (I \oplus Y)^*$ by $\bar{T}((a,x)) = (0, T(x))$.

If $D : A \longrightarrow Y^{**}$ is a continuous derivation, we define $\bar{D} : (A \oplus X) \longrightarrow (I \oplus Y)^{**}$ by $\bar{D}((a,x)) = (0, D(a))$.

If $T : X \longrightarrow I^{**}$ is a continuous $\mathcal{A}$-bimodule morphism, we define $\bar{T} : (A \oplus X) \longrightarrow (I \oplus Y)^{**}$ by $\bar{T}((a,x)) = (T(x), 0)$.

Lemma 2.3. The operators $\bar{D}$ and $\bar{T}$ defined above are continuous derivations. Furthermore, the derivation $\bar{D}$ is inner if and only if D is inner, and $\bar{T}$ is inner if and only if $T = 0$.

proof Let $(a,x), (b,y) \in (A \oplus X)$, we have

$$\begin{aligned}\bar{D}((a,x),(b,y)) &= \bar{D}((ab, ay + xb)) \\ &= (D(ab), 0) = (D(a)b + aD(b), 0) \\ &= (D(a), 0).(b,y) + (a,x).(D(b), 0) \\ &= \bar{D}((a,x)).(b,y) + (a,x).\bar{D}((b,y)).\end{aligned}$$

Then $\bar{D}$ is a (continuous) derivation, and similarly for $\bar{T}$. Now let $\bar{D}$ be inner, then there exists $u \in I^*$ and $F \in Y^*$ such that $\bar{D}((a,x)) = (a,x).(u,F) - (u,F).(a,x)$. In particular

$$\begin{aligned}(D(a), 0) &= \bar{D}((a,0)) \\ &= (a,0).(u,F) - (u,F).(a,0) \\ &= (au - ua, aF - Fa),\end{aligned}$$

then $D(a) = au - ua$ for $u \in I^*$ i.e D is inner. For converse, let D is inner, then there exists $u \in I^*$ such that $D(a) = au - ua$ for $a \in A$. We have

$$\begin{aligned}\bar{D}((a,x)) &= (D(a), 0) = (au - ua, 0) \\ &= (a,x).(u,0) - (u,0).(a,x).\end{aligned}$$

By putting $\xi = (u, 0) \in (I \oplus Y)^*$, $\bar{D}((a, x)) = (a, x).\xi - \xi.(a, x)$ where $\xi \in (I \oplus Y)^*$. Thus $\bar{D}$ is inner. Let $\bar{T}$ is inner, then there exists $u \in I^*$ and $F \in Y^*$ such that

$$\bar{D}((a, x)) = (a, x).(u, F) - (u, F).(a, x).$$

In particular,

$$(0, T(x)) = \bar{T}((0, x)) = (0, x).(u, F) - (u, F).(0, x) = (xF - Fx, 0),$$

thus $T = 0$. The converse is evident, and a similar proof gives the rest. ∎

Theorem 2.4. $\mathcal{A} \oplus X$ is ideally amenable if and only if for arbitrary ideal $I \oplus Y$ of $\mathcal{A} \oplus X$ the following conditions hold:
1. $\mathcal{A}$ is I-weakly amenable.
2. $H^1(A, Y^*) = \{0\}$.
3. For every continuous $\mathcal{A}$-bimodule morphism $\Gamma : X \longrightarrow I^*$, there exists $F \in Y^*$ such that $aF - Fa = 0$ for $a \in A$ and $\Gamma(x) = xF - Fx$ for $x \in X$.
4. The only continuous $\mathcal{A}$-bimodule morphism $T : X \longrightarrow Y^*$ for which $xT(y) + T(x)y = 0$ $(x, y \in X)$ in I^* is $T = 0$.

proof Let $I \oplus Y$ be an arbitrary ideal of $\mathcal{A} \oplus X$. Denote by Δ_1 the projection from $(I \oplus Y)^*$ onto I^* with kernel Y^*. Let Δ_2 be the projection $id - \Delta_1 : (I \oplus Y)^* \longrightarrow Y^*$ and let $\tau_1 : A \longrightarrow (A \oplus X)$ be the inclusion mapping (i.e $\tau_1(a) = (a, 0)$). Then Δ_1, Δ_2 are $\mathcal{A}$-bimodule morphisms, and τ_1 is an algebra homomorphism. We now prove the sufficiency. Suppose that conditions 1-4 hold. Suppose also that $D : (A \oplus X) \longrightarrow (I \oplus Y)^*$ is a continuous derivation. Then $Do\tau_1 : A \longrightarrow (I \oplus Y)^*$ is a continuous derivation. This implies that $\Delta_1 oDo\tau_1 : A \longrightarrow I^*$ and $\Delta_2 oDo\tau_1 : A \longrightarrow Y^*$ are continuous derivations. By conditions 1 and 2, they are inner. Therefore, $Do\tau_1$ is inner. From Lemmas 2.2 and 2.3, the mapping

$$\overline{Do\tau_1} = \overline{\Delta_1 oDo\tau_1} + \overline{\Delta_2 oDo\tau_1} : (A \oplus X) \longrightarrow (I \oplus Y)^*,$$

is a continuous derivation, and there is a continuous derivation $\tilde{D} : (A \oplus X) \longrightarrow (I \oplus Y)^*$ satisfying $\tilde{D}((a, 0)) = 0$ for $a \in A$ and such that $\overline{Do\tau_1} - \tilde{D}$ is inner. On the other hand,

$$(D - \overline{Do\tau_1})((a, 0)) = D((a, 0)) - \overline{Do\tau_1}((a, 0)) = Do\tau_1(a) - Do\tau_1(a) = 0.$$

Let $\hat{D} = D - \overline{Do\tau_1} + \tilde{D}$. Then $\hat{D}$ is a continuous derivation from $\mathcal{A} \oplus X$ into $(I \oplus Y)^*$, satisfying $\hat{D}((a, 0)) = 0$ for $a \in A$. So,

$$\begin{aligned} \hat{D}((0, ax)) &= \hat{D}((a, 0).(0, x)) \\ &= (a, 0).\hat{D}((0, x)) \\ &= a.\hat{D}((0, x)) \quad (a \in A, x \in X), \end{aligned}$$

and $\hat{D}((0, xa)) = \hat{D}((0, x)).a \quad (a \in A, x \in X)$. Denote by $\tau_2 : X \longrightarrow A \oplus X$ the inclusion mapping given by $\tau_2(x) = (0, x)$ $(x \in X)$. Then $\hat{D}o\tau_2 : X \longrightarrow (I \oplus Y)^*$ is a continuous $\mathcal{A}$-bimodule morphism. From condition 3 there exists $F \in Y^*$ for which $\Delta_1 o\hat{D}o\tau_2(x) = xF - Fx$ and $aF - Fa = 0$ for $x \in X, a \in A$. Since

$$\begin{aligned} (0,0) &= \hat{D}((0,0)) = \hat{D}((0,x).(0,y)) \\ &= \hat{D}((0,x)).(0,y) + (0,x).\hat{D}((0,y)) \\ &= ([\Delta_2 o\hat{D}o\tau_2(x)]y + x[\Delta_2 o\hat{D}o\tau_2(y)], 0), \end{aligned}$$

we have $(\Delta_2 o\hat{D}o\tau_2(x))y + x(\Delta_2 o\hat{D}o\tau_2(y)) = 0$ for every $x, y \in X$. From 4, $\Delta_2 o\hat{D}o\tau_2 = 0$. Thus

$$\begin{aligned} \hat{D}((a,x)) &= \hat{D}((0,x)) = \hat{D}o\tau_2(x) \\ &= (\Delta_1 o\hat{D}o\tau_2(x), \Delta_2 o\hat{D}o\tau_2(x)) \\ &= (xF - Fx, 0) \\ &= (a,x).(0,F) - (0,F).(a,x). \end{aligned}$$

We have that $\hat{D}$ is inner. Thus $D = \hat{D} + (\overline{Do\tau_1} - \tilde{D})$ is inner. This proves that $\mathcal{A} \oplus X$ is ideally amenable. For the converse, let $\mathcal{A} \oplus X$ is ideally amenable and let $I \oplus Y$ is an arbitrary ideal of $\mathcal{A} \oplus X$. Then every continuous derivation from $\mathcal{A} \oplus X$ into $(I \oplus Y)^*$ is inner. Let $D : A \longrightarrow I^*$ be a continuous derivation, then By lemmas 2.3, $\bar{D} : A \oplus X \longrightarrow (I \oplus Y)^*$ defined by $\bar{D}((a, x)) = (D(a), 0)$ is inner, and condition 1 hold. Let $T : X \longrightarrow Y^*$ be a continuous A-bimodule morphism, then by lemma 2.3, $\bar{T} : A \oplus X \longrightarrow (I \oplus Y)^*$ defined by $\bar{T}((a, x)) = (0, T(x))$ is inner, and condition 2 hold. Now if $\Gamma : X \longrightarrow I^*$ be an arbitrary continuous $\mathcal{A}$-bimodule morphism, by Lemma 2.1, there exists $F \in Y^*$ such that $aF - Fa = 0$ $a \in A$ and $\Gamma(x) = xF - Fx$ $x \in X$. This gives condition 3. If $T : X \longrightarrow Y^*$ is a continuous $\mathcal{A}$-bimodule morphism for which $xT(y) + T(x)y = 0$ $x, y \in X$ in I^*, then by lemma 2.3, $T = 0$. This gives condition 4 and the proof is complete. ■

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