On the Degree of Theta Pairs
of Finite Groups\textsuperscript{1}

A. Erfanian and R. Rezaei

Department of Mathematics
Faculty of Mathematical Sciences
Ferdowsi University of Mashhad, Mashhad, Iran
erfanian@math.um.ac.ir, ras_rezaei@yahoo.com

Abstract

Let $G$ be a finite group and $M$ a maximal subgroup of $G$. A $\theta$-pair of $M$ is any pair of subgroups $(C, D)$ of $G$ such that (i) $D < G$, $D < C$, (ii) $< M, C > = G$, $< M, D > = M$ and (iii) $\frac{G}{D}$ has no proper normal subgroup of $\frac{C}{D}$. A $\theta^*$-pair of $M$ is a pair of subgroups $(C, D)$ satisfying conditions (i) and (iii) and a property that $D \leq M$ and $C^g \not\subseteq M$ for every $g \in G$.

In this paper, we introduce the degree of $\theta$-pairs, denoted by $d\theta(G)$ as the ratio $|\theta(G)|/m(G)$, where $\theta(G)$ is the union of all $\theta$-pairs of the maximal subgroups of $G$ and $m(G)$ is the total number of distinct maximal subgroups of $G$. Similarly, we define the degrees of maximal $\theta$-pairs, $\theta^*$-pairs and maximal $\theta^*$-pairs of a finite group $G$ and give some evaluations on the above degrees for some simple groups, nilpotent groups and solvable groups. Moreover, we prove that if $G$ is nilpotent then the degree of maximal $\theta$-pairs of $G$ is exactly 1.

\textbf{Keywords:} $\theta$-pair, $\theta^*$-pair, maximal $\theta$-pair, maximal $\theta^*$-pair, degree of $\theta$-pair, degree of $\theta^*$-pair

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1 Introduction

In 1990, Mukhrejee and Bhattacharya [4] introduced the notion of $\theta$-pair for every maximal subgroup of a finite group. They have proved the existence of

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a \( \theta \)-pair for any maximal subgroup \( M \) of a finite group \( G \). Two years later, Beidleman and Smith [2], generalized the concept of \( \theta \)-pair for infinite groups. In [8], Yaoqing improved the definition of \( \theta \)-pair for a maximal subgroup of a finite group \( G \) to a maximal \( \theta \)-pair and normal \( \theta \)-pair. He proved that for any maximal subgroup \( M \) of a finite group \( G \), there exists a normal maximal \( \theta \)-pair related to \( M \). The investigation on \( \theta \)-pairs are continued by others in [1, 7-9]. Most of this research deals with the structure of finite groups. Later, in 2000, Shirong and Yaoqing in [6], defined the new concept \( \theta^* \)-pair which is a special case of \( \theta \)-pairs by some additional conditions. Using the concept of \( \theta^* \)-pair, they obtained several results on maximal \( \theta^* \)-pairs which imply a finite group \( G \) to be solvable or supersolvable.

In this paper, we introduced a degree on the \( \theta \)-pairs of a finite group \( G \) which is called \( \theta \)-pairs degree of \( G \) and will be stated later. In this article, we prove that if \( G \) is a finite nilpotent group, then the \( \theta \)-pairs and \( \theta^* \)-pairs degrees are exactly 1. Moreover, we will compare the above degrees of a group \( G \) with the quotient group \( {\frac{G}{N}} \), where \( N \) is a normal subgroup of \( G \) and contained in the Frattini subgroup \( \Phi(G) \). Now we give some definitions and basic results.

2 Some definitions and basic results

All groups considered in this paper, are assumed to be finite. We also denote \( M_{\text{max}} \leq G \) as a convenience that \( M \) is a maximal subgroup of \( G \). Now, we start with the definition of \( \theta \)-pair given by Mukhrejee and Bhattocharya in [4].

**Definition 2.1** Let \( G \) be a finite group and \( M \) a maximal subgroup of \( G \). A \( \theta \)-pair of \( M \) is any pair \( (C, D) \) of subgroups satisfying the following conditions:

(a) \( D \triangleleft G \), \( D < C \).
(b) \( < M, C > = G \) and \( < M, D > = M \).
(c) \( \frac{C}{D} \) has no proper normal subgroup of \( \frac{G}{D} \).

**Definition 2.2** Let \( (C, D) \) be a \( \theta \)-pair of a maximal subgroup \( M \) of \( G \). Then \( (C, D) \) is called a normal \( \theta \)-pair if \( C \triangleleft G \) and is said to be a maximal \( \theta \)-pair if there is no \( \theta \)-pair \( (C', D') \) such that \( C < C' \).

The set of all \( \theta \)-pairs of a maximal subgroup \( M \) of \( G \) is denoted by \( \theta(M) \). We also denote \( \theta(G) \) as the union of \( \theta(M) \) for all maximal subgroups \( M \) of \( G \). Similarly, \( \theta_m(M) \) and \( \theta_m(G) \) will be denoted for those of \( \theta \)-pairs which are maximal.

It is clear that for every maximal subgroup \( M \) of \( G \), \( \theta(M) \) and \( \theta_m(M) \) are not empty. Since \( (C, M_G) \) is a \( \theta \)-pair of \( M \), where \( M_G \) is the core of \( M \) in \( G \) and \( C \) is a subgroup of \( G \) such that \( \frac{C}{M_G} \) is a chief factor of \( G \). It is easy to see that \( (C, M_G) \) is a maximal \( \theta \)-pair of \( M \) as well.
Definition 2.3 given a maximal subgroup $M$ of a finite group $G$, a $\theta^*$-pair of $M$ is any pair $(C, D)$ of subgroup of $G$ such that
(a) $D < C$ and $D \leq G$,
(b) $D \leq M$ but $C^g \not\subseteq M$ for every $g \in G$ and
(c) $\frac{C}{D}$ has no proper normal subgroup of $\frac{G}{D}$.

Similar to those of $\theta$-pairs, we can define the concepts : normal, maximal $\theta^*$-pair and the notations $\theta^*(M)$, $\theta^*_m(M)$ and $\theta^*(G)$.

By the above two definitions, we can see that every $\theta^*$-pair of $M$ is a $\theta$-pair of $M$ and so $\theta^*(M) \subseteq \theta(M)$. Thus, we have $\theta^*(G) \subseteq \theta(G)$. It is also clear that a normal $\theta^*$-pair is certainly maximal and $\theta^*$-pair$(M^g)=\theta^*$-pair$(M)$ for every maximal subgroup $M$ of $G$ and every $g \in G$.

In the rest of this section, we state some known results on the $\theta$-pairs and $\theta^*$-pairs. We omitt the proofs and one may find them in [4].

Theorem 2.4 If $(C, D)$ is a maximal $\theta$-pair in $\theta(M)$ and $N < G$, $N < D$, then $(\frac{C}{N}, \frac{D}{N})$ is a maximal $\theta$-pair in $\theta(\frac{M}{N})$. Conversely, if $(\frac{C}{N}, \frac{D}{N})$ is a maximal $\theta$-pair in $\theta(\frac{M}{N})$, then $(C, D)$ is a maximal $\theta$-pair in $\theta(M)$.

Theorem 2.5 Let $G$ be a finite group. Then $G$ is nilpotent if and only if $\theta(M)$, for every $M \leq G$, contains a maximal pair $(C, D)$ such that $\frac{G}{D}$ is nilpotent.

Theorem 2.6 A solvable group $G$ is nilpotent if and only if for all $M \leq G$, $\theta(M)$ contains exactly one maximal pair.

There are some similar results on $\theta^*$-pair that we referred to [5] and [6].

3 Degree of $\theta$-pairs and $\theta^*$-pairs

In this section, we introduce a concept of degree for $\theta$-pairs and $\theta^*$-pairs of a finite group $G$. We state the following definition.

Definition 3.1 Let $G$ be a finite group and $m(G)$ be the total number of distinct maximal subgroup of $G$. Then, $d\theta(G)$ is said to be the degree of $\theta$-pairs of $G$ and is

$$d\theta(G) = \frac{|\theta(G)|}{m(G)}.$$  

Similarly, the degree of $\theta^*$-pair of $G$ is denoted by

$$d\theta^*(G) = \frac{|\theta^*(G)|}{m(G)}.$$
We may also define the above degree for maximal $\theta$-pairs and $\theta^*$-pairs of $G$. We denote them by $d\theta_m(G)$ and $d\theta_m^*(G)$, respectively. In the following lemma we compare the above degrees.

**Lemma 3.2** Let $G$ be a finite group $G$. then

(i) $d\theta_m(G) \leq d\theta(G)$;
(ii) $d\theta_m^*(G) \leq d\theta^*(G)$;
(iii) $d\theta_m^*(G) \leq d\theta_m(G)$.

*Proof* It is clear by definitions 2.2 and 2.3.

**Lemma 3.3** Let $G$ be a finite non-abelian simple group. Then $|\theta_m(G)| = 1$.

*Proof* It is clear that $(G, 1)$ is a $\theta$-pair for every maximal subgroup $M$ of $G$. So we have to show that $(G, 1)$ is a unique maximal $\theta$-pair. If $(C, D)$ is a $\theta$-pair of a maximal subgroup of $G$, then there are two possibilities for $D$. If $D = G$, then by condition (b) of Definition 2.1, we should have $<M, D> = M$ which is a contradiction. Thus, $D$ must be the identity subgroup. So, $(C, D) = (C, 1)$ and therefore

$$\theta(M) = \{(C, 1) : <C, M> = G \text{ and } C \text{ has no proper normal subgroup of } G\}.$$ 

Since $(G, 1) \in \theta(M)$ and is the maximal element of $\theta(M)$, we have $\theta_m(G) = \{(G, 1)\}$ and the proof is completed.

**Lemma 3.4** Let $G$ be a finite non-abelian simple group. Then $m(G) > 1$.

*Proof* Assume that $m(G) = 1$ and $M$ is a unique maximal subgroup of $G$. Then we should have $M^g = M$ for every $g \in G$ and so $M < G$. If $M = 1$, then $G$ is a cyclic group of prime order which is a contradiction. Hence $M \neq 1$ and it implies that $m(G) > 1$.

**Theorem 3.5** Let $G$ be a finite non-abelian simple group. Then

$$d\theta_m(G) = d\theta_m^*(G) = \frac{1}{m(G)} < 1.$$ 

*Proof* By lemmas 3.2 and 3.3, the proof is clear.

The degree of $\theta$-pairs or $\theta^*$-pairs may have the values equal or bigger than 1. The following example proves this fact.

**Example 3.6**

(i) Let $G = S_3$, then we have 4 maximal subgroups

$M_1 = <(12)>, M_2 = <(13)>, M_3 = <(23)>.$
and $M_4 = A_3$. We can easily see that

$$\theta(G) = \theta^*(G) = \theta_m(G) = \theta^*_m(G) = \{(G, A_3), (A_3, 1), (M_1, 1), (M_2, 1), (M_3, 1)\}.$$  

Thus

$$d\theta(G) = d\theta^*(G) = d\theta_m(G) = d\theta^*_m(G) = \frac{5}{4} > 1.$$  

(ii) Let $G = \mathbb{Z}_3$, then by the similar method as the previous example we can find that $d\theta_m(G) = d\theta^*_m(G) = 1$.

**Theorem 3.7** Let $G$ be a cyclic group of prime order. Then

$$d\theta(G) = d\theta^*(G) = d\theta_m(G) = d\theta^*_m(G) = 1$$

*Proof* It is clear that the only proper subgroup of $G$ is the identity subgroup. Thus $m(G) = 1$ and if $M = 1$ is a maximal subgroup of $G$, then $(G, 1)$ is the only $\theta$-pair and, of course is a $\theta^*$-pair and maximal. Therefore $\theta(M) = \theta_m(M) = \theta^*_m(M) = \{(G, 1)\}$ and it completes the proof.

The following theorem is a comparison of the maximal $\theta$-pair degree of group $G$ and quotient group $G/N$.

**Theorem 3.8** Let $G$ be a nilpotent group and $N$ be a normal subgroup of $G$ such that $N \subseteq \Phi(G)$. Then $d\theta_m(G) = d\theta_m(G/N)$.

*Proof* First, we prove that $m(G) = m(G/N)$. If $M$ is a maximal subgroup of $G$, then $M/N$ is a maximal subgroup of $G/N$, because if there exists a subgroup $M$ of $G$ such that $M/N \nsubseteq H \nsubseteq G$, then $M \nsubseteq H \nsubseteq G$ which is a contradiction. Thus $M/N$ is a maximal subgroup of $G/N$. Conversely, if $M/N$ is a maximal subgroup of $G/N$, then by the similar method, $M$ is a maximal subgroup of $G$. Therefore, we have $m(G) = m(G/N)$. Now, assume that $M$ is a maximal subgroup of $G$. Then $M \lhd G$ and so $(G, M)$ is a $\theta$-pair of $M$ in $G$, because $G/N$ is simple. We can also see that $(G, M)$ is a maximal $\theta$-pair of $M$. Since, if $(C, D)$ is another maximal $\theta$-pair of $M$, then $C = G$ and so $(G, D)$ is a normal maximal $\theta$-pair. Hence, by (Theorem 2.3, [5]), $D = M_G$ and therefore $(C, D) = (G, M)$. It implies that $\theta_m(M) = \{(G, M)\}$. Now, assume that $m(G) = k$ and $M_1, M_2, \ldots, M_k$ are distinct maximal subgroups of $G$, then

$$\theta_m(G) = \{(G, M_1), (G, M_2), \ldots, (G, M_k)\}.$$  

By the similar way we can show that $\theta_m(G/N) = \{(G/N, M_1/N), (G/N, M_2/N), \ldots, (G/N, M_k/N)\}$. 


Hence
\[ d\theta_m(G) = \frac{|\theta_m(G)|}{m(G)} = \frac{|\theta_m(G)|}{m\left(\frac{G}{N}\right)} = d\theta_m\left(\frac{G}{N}\right). \]

**Proposition 3.9** Let \( G \) be a nilpotent group. Then \( d\theta_m(G) = 1 \).

**Proof** It is clear by Theorem 3.8.

We can establish the above result for the degree of \( \theta^* \)-pairs and the proofs are very similar.

Finally, we state the following conjectures that we may find some evidence on it.

**Conjecture 1** If \( d\theta_m(G) = 1 \), then \( G \) is nilpotent.

**Conjecture 2** Let \( G_1 \) and \( G_2 \) be two finite groups. Then
\[ d\theta_m(G_1 \times G_2) \leq d\theta_m(G_1)d\theta_m(G_2). \]

**References**


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