Int. J. Contemp. Math. Sciences, Vol. 2, 2007, no. 7, 343 - 348

Shape Operator for Real Space Forms in Hessian Manifolds of Constant Hessian Sectional Curvature

Münevver YILDIRIM YILMAZ and Mehmet BEKTAŞ

Fırat University, Faculty of Science and Arts
Department of Mathematics, 23119 Elazığ, Turkey
myildirim@firat.edu.tr, mbektas@firat.edu.tr

Abstract

Chen [1] establishes a relationship between sectional curvature and the shape operator for submanifold in a real space form. A similar inequality for slant submanifolds in a complex space form is proved in [2]. As a natural generalization to the above two kinds of this result, an inequality is established in [3]. In this paper, we find a similar inequality between the shape operator and sectional curvature for Riemannian submanifolds in a Hessian manifolds of constant Hessian sectional curvature.

Mathematics Subject Classification: 53C55

Keywords: Hessian Metric, Hessian Manifolds, Shape Operator

1. PRELIMINARIES: Let M^m be a flat affine manifold with flat affine connection D. Among Riemannian metrics on M^m there exists an important class of Riemannian metrics compatible with the flat affine connection D. A Riemannian metric g on M is said to be Hessian metric if g is locally expressed by $g = D^2u$ where u is a local smooth function.We call such a pair $(D\,,\,g)$ a Hessian structure on M and a triple $(M\,,\,D\,,\,g)$ a Hessian manifold.Geometry of Hessian manifold is deeply related to Kählerian geometry and affine differential geometry [4] .

We use the same notations and terminology as in [4] unless otherwise stated.Let $M^{\ m}$ be a Hessian manifold with Hessian structure $(D\,,\,g\,)$.We express various geometric concepts for the Hessian structure $(D\,,\,g)$ in terms of affine coordinate system $\{\,x^1\,,\,\ldots\,,\,x^m\}$ with respect to D , i.e $D\,dx^i = 0$

i) The Hessian metric ;

$$g_{ij} = \frac{\partial^2 u}{\partial x^i \partial x^j}$$

ii) Let γ be a tensor field of type $(1, 2)$ defined by

$$\gamma\left(X, Y\right) = \triangledown_X Y - D_X Y$$

where ∇ is the Riemannian connection for g . Then we have

$$\gamma^i_{jk} = \Gamma^i_{jk} = \frac{1}{2} g^{ir} \frac{\partial g_{rj}}{\partial x^k}, \quad \gamma_{ijk} = \frac{1}{2} \frac{\partial g_{ij}}{\partial x^k} = \frac{1}{2} \frac{\partial^3 u}{\partial x^i \partial x^j \partial x^k}, \quad \gamma_{ijk} = \gamma_{jik} = \gamma_{kji}$$

where Γ^i_{jk} are the Christoffel ' s symbols of ∇ .

iii)Define a tensor field S of type $(1, 3)$ by

$$S = D_\gamma$$

and call it the Hessian curvature tensor for (D, g) .Then we have

$$S^i_{jkl} = \frac{\partial \gamma^i_{jl}}{\partial x^k}$$

$$S_{ijkl} = \frac{1}{2} \frac{\partial^4 u}{\partial x^i \partial x^j \partial x^k \partial x^l} - \frac{1}{2} g^{rs} \frac{\partial^3 u}{\partial x^i \partial x^k \partial x^r} - \frac{\partial^3 u}{\partial x^j \partial x^l \partial x^s}$$

$$S_{ijkl} = S_{ilkj} = S_{kjil} = S_{jilk} = S_{klij}.$$

iv) The Riemannian curvature tensor for ∇ ;

$$R_{ijkl} = \frac{1}{2} \left(S_{jikl} - S_{ijkl}\right). \tag{1.1}$$

Definition 1. For a non-zero contravariant symmetric tensor ξ_x of degree at x we set

$$h\left(\xi_x\right) = \frac{\left\langle \varsigma\left(\xi_x\right), \xi_x \right\rangle}{\left\langle \xi_x, \xi_x \right\rangle}$$

and call it the Hessian sectional curvature in the direction ξ_x .

Theorem 1.1. Let (M, D, g) be a Hessian manifold of dimension ≥ 2. If the Hessian sectional curvature $h\left(\xi_x\right)$ depends only x then (M, D, g)

is of constant Hessian sectional curvature .(M , D , g) is of constant Hessian sectional curvature c if and only if

$$S_{ijkl} = \frac{c}{2}\left(g_{ij}g_{kl} + g_{il}g_{kj}\right) . \tag{1.2}$$

Corollary 1.1. If a Hessian manifold (M , D , g) is a space of constant Hessian sectional curvature c , then the Riemannian manifold (M , g) is a space of constant sectional curvature $-\frac{c}{4}$.

2. SHAPE OPERATOR AND SECTIONAL CURVATURE

Theorem 2.1. Let $x : M^n \to M^m\left(-\frac{c}{4}\right)$ be an isometric immersion of a Riemannian n-manifold in to an m-dimensional Hessian manifold $M^m\left(-\frac{c}{4}\right)$ of constant sectional curvature $-\frac{c}{4}$. If there exist a point $p \in M^n$ such that $c \equiv \ \inf \ K \neq -\frac{c}{4}$ at p, then the shape operator at the mean curvature H satisfies

$$A_H > \frac{n-1}{n}\left(c + \frac{c}{4}\right) I_n \qquad \text{at} \;\; p \tag{2.1}$$

where I_n is the identity map.

Proof: We assume that M^n is a submanifold in $M^m\left(-\frac{c}{4}\right)$. Choose orthonormal basis $e_1, ..., e_n, e_{n+1,...,}e_m$ at p such that e_{n+1} is parallel to the mean curvature vector H and $e_1, ..., e_n$ diagonolize the shape operator A_{n+1}. Then we have

$$A_{n+1} = \begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ 0 & 0 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{pmatrix} \tag{2.2}$$

$$A_r = \sum_{i,j=1}^{n}\left(h_{ij}^r\right) \quad , \sum_{i=1}^{n}\left(h_{ii}^r\right) = 0 \quad , \quad r = n+2, \cdots , m$$

we put $u_{ij} = u_{ji} = a_i a_j$.From Gauss equation , we get

$$u_{ij} \geq c + \frac{c}{4} + \sum_{r=n+2}^{m}\left(h_{ij}^r\right)^2 - \sum_{r=n+2}^{n}\left(h_{ii}^r\right)\left(h_{jj}^r\right) \qquad 1 \leq i \neq j \leq n. \tag{2.3}$$

We need following lemmas.

Lemma 1. The following statements hold

(1) For any fixed $i \in \{1, 2, ..., n\}$, we have

$$\sum_{i \neq j}^{n} u_{ij} \geq (n-1)\left(c + \frac{c}{4}\right).$$

(2) $u_{ij} \neq 0$ for $i \neq j$.
(3) For distinct $i, j, k \in \{1, 2, ..., n\}$ it follows that $a_i^2 = u_{ij}u_{ik}/u_{jk}$.
(4) For a fixed $k, 1 \leq k \leq [\frac{n}{2}]$, and each $B \in S_k$, we have

$$\sum_{,j \in B} \sum_{t \in \overline{B}} u_{jt} \geq (n-k)\, k \left(c + \frac{c}{4}\right).$$

where $\overline{B}$ is the complement of B in $\{1, 2, ..., n\}$.

Proof: From (2.2) and (2.3), we get

$$\sum_{i \neq j} u_{ij} \geq (n-1)\left(c + \frac{c}{4}\right) + \sum_{r=n+2}^{m} \left(\sum_{i \neq j} \left(h_{ij}^r\right)^2 - \left(h_{ii}^r\right) \sum_{i \neq j} \left(h_{jj}^r\right) \right)$$

$$= (n-1)\left(c + \frac{c}{4}\right) + \sum_{r=n+2}^{m} \sum_{i \neq j} \left(h_{ij}^r\right)^2$$

$$\geq (n-1)\left(c + \frac{c}{4}\right)$$

which yields statement (1). For statement (2), let us assume $u_{ij} = a_i a_j = 0$ if $a_i = 0$ then $u_{it} = 0$ for any $t \neq i$. Hence $\sum_{t \neq i} u_{it} = 0$ which contradicts statement (1).

(3) We have $u_{ij}u_{ik}/u_{jk} = a_i a_j a_i a_k / a_j a_k = a_i^2$

(4) Without loss of generality, we may assume $B = \{1, 2, ..., k\}$. From (2.3), we find

$$\sum_{j \in B} \sum_{t \in \overline{B}}^{n} u_{jt} \geq (n-k)\, k \left(c + \frac{c}{4}\right) + \sum_{j=1}^{k} \sum_{t=k+1}^{n} \sum_{r=n+2}^{m} [\left(h_{jt}^r\right)^2 - \left(h_{jj}^r\right)\left(h_{tt}^r\right)]$$

$$= (n-k)\, k \left(c + \frac{c}{4}\right) + \sum_{r=n+2}^{m} [\sum_{j=1}^{k} \sum_{t=k+1}^{n} \left(h_{jt}^r\right)^2 + \sum_{j=1}^{k} \left(h_{jj}^r\right)^2]$$

which implies the statement.

Lemma 2. For any $1 \leq i \neq j \leq n,$ we have $u_{ij} > 0$.

Proof. Assume $u_{1n} < 0,$ then by statement (3) of Lemma 1, we get $u_{1i}u_{in} < 0$ for $1 < i < n$. Without loss of generality we may assume

$$u_{12}, ..., u_{1l}, u_{(l+1)n}, ..., u_{(n-1)n} > 0$$

$$u_{1(l+1)}, ..., u_{1n}, u_{2n}, ..., u_{\text{ln}} < 0 \tag{2.4}$$

For some $[\frac{n+1}{2}] \leq l \leq n-1$.If $l = n-1,$ then $u_{1n} + u_{2n} + \cdots + u_{(n-1)n} < 0$ that contradicts statement (1) of Lemma 1. Thus $l < n-1$. For statement (3) of Lemma 1., we get

$$a_n^2 = \frac{u_{in}u_{tn}}{u_{it}} > 0, \tag{2.5}$$

where $2 \leq i \leq l$ and $l+1 \leq t \leq n-1$. Using (2.4) and (2.5) we obtain $u_{it} < 0$ which implies

$$\sum_{i=1}^{l}\sum_{t=l+1}^{n} u_{it} = \sum_{i=2}^{l}\sum_{t=l+1}^{n-1} u_{it} + \sum_{i=1}^{l} u_{in} + \sum_{t=l+1}^{n} u_{lt} < 0.$$

This is a contradiction.

If we return to proof of Theorem 2.1, from Lemma 2., it follows that $a_1, \cdots, a_n$
are of the same sign. Therefore the shape operator A_H is positive definite. From statement (1) Lemma 1, we get

$$na_i\left|H\right| - a_i^2 = a_i \sum_{j \neq i} a_j \geq (n-1)\left(c + \frac{c}{4}\right).$$

This completes the proof.

References :

[1] Chen, B.Y., Mean curvature and shape operator of isometric immersions in real space forms, Glasgow Math. J. , 38(1996) 87 – 97.

[2] Matsumoto, K., Mihai, I and Oiaga, Shape operator for slant manifolds in generalized Complex space forms, Bull. Yamagata Univ. 14, (2000) , 169 – 177.

[3] Kim, J.S. Song, Y.M. and Tripathi, M.M., Shape operator for slant manifolds in generalized complex space forms, Indian J. Pure Appl. Math. 34,(2003) , 1153 – 1163.

[4] Shima, H., Hessian manifolds of constant Hessian sectional curvature, J. Math. Soc. Japan, 47(4) (1995) , 735 – 753.

[5] Yıldırım, M., Geometrical structures of Hesian manifolds, Master Thesis, (2004).

Received: July 18, 2006