Curves in Projective Spaces through a Given Set and Their Number of Moduli

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Abstract. Fix integers \( r, d, g, s \) such that \( r \geq 3, \ d \geq r + 2, \ g \geq 2, \ s \geq 0, \) and \( (r + 1)d - rg - r^2 \geq 0. \) There is a nice family \( W(d, g, r)' \) of smooth curves of degree \( d \) and genus \( g \) of \( \mathbb{P}^r \) such that the natural map \( W(d, g, r)' \to M_g \) is dominant. Fix a general \( S \subset \mathbb{P}^r \) such that \( \#(S) = s \) and set \( W(d, g, r, s; S)' := \{ C \in W(d, g, r)' : S \subset C \}. \) Here we give upper bounds on \( s \) to get that the induced map \( W(d, g, r, s; S)' \to M_g \) is dominant.

Mathematics Subject Classification: 14H50

Keywords: pointed curve, pointed projective curves, Hilbert scheme of curves

For all non-negative integers \( g, s \) such that either \( g \geq 2 \) and \( s \geq 0 \) or \( g = 1 \) and \( s \geq 1 \) or \( g = 0 \) and \( s \geq 3 \) let \( M_{g, s} \) denote the moduli space of all pairs \( (C, (P_1, \ldots, P_s)) \), where \( C \) is a smooth and connected projective curve with genus \( g \) and \( (P_1, \ldots, P_s) \) is an ordered \( s \)-ple of distinct point of \( C \). For all integers \( d, g, r \) such that \( r \geq 3, \ g \geq 0 \) and either \( d \geq g + r \) or \( d \geq r + 2 \) and \( d - r < g \leq d - r + \lfloor (d - r - 2)/(r - 2) \rfloor \) let \( W(d, g, r) \) denote the irreducible component of \( \text{Hilb}(\mathbb{P}^r) \) described in [1], §1, and \( W(d, g, r)' \) the open part of \( W(d, g, r) \) parametrizing the smooth non degenerate curves of \( \mathbb{P}^r \) with degree \( d \) and genus \( g \) belonging to \( W(d, g, r) \). \( W(d, g, r) \neq \emptyset, \ h^1(C, \mathcal{O}_C(2)) = 0 \) and \( h^1(C, N_C) = 0 \) for a general \( C \in W(d, g, r)' \); \( \dim(W(D, g, r)) = (r + 1)d + (3 - r)(g - 1) \) and \( W(d, g, r) \) is generically smooth; if \( d \geq g + 3, \) then \( h^1(C, \mathcal{O}_C(1)) = 0 \) for a general \( C \in W(d, g, r)' \); if \( g \geq 2 \) and \( \rho(g, r, d) := g - (r + 1)(g + r - d) \geq 0, \) then the natural map \( u_{d, g, r} : W(d, g, r)' \to M_g \) is dominant ([1], §2 and §3).

Now fix a finite \( S \subset \mathbb{P}^r \) and order the points of it. Let \( W(d, g, r; S) \) denote

\(^1\) The author was partially supported by MIUR and GNSAGA of INdAM (Italy).
the set of all \( C \in W(d, g, r) \) such that \( S \subset C_{\text{reg}} \). Let \( W(d, g, r; S)' \) denote the set of all \( C \in W(d, g, r) \) such that \( S \subset C \). The ordering of the points of \( S \) induces a morphism \( u_{d,g,r,S} : W(d, g, r; S)' \to M_{g,s} \), \( s := \sharp(S) \); we need to assume \( s \geq 3 \) if \( g = 0 \) and \( s \geq 1 \) if \( g = 1 \), otherwise \( M_{g,s} \) is not defined. If \( S, S' \) are general and \( \sharp(S) = \sharp(S') = s \), then the maps \( u_{d,g,r,S} \) and \( u_{d,g,r,S'} \) have the same numerical properties and we will write \( u_{d,g,r,S} \) for any of them. For all integers \( g \geq 2 \) and \( s \geq 0 \) there is a natural morphism \( \phi_{g,s} : M_{g,s} \to M_g \).

Set \( v_{d,g,r,S} := \phi_{g,s} \circ u_{d,g,r,S} \) and \( v_{d,g,r,S} := \phi_{g,s} \circ u_{d,g,r,S} \). The map \( v_{d,g,r,S} \) does not depend upon the choice of an ordering of \( S \). We will prove the following results.

**Theorem 1.** Fix integers \( d, g, r, s \) such that \( r \geq 3, g \geq 2, d \geq r + 2, s \geq 0 \) and \((r+1)d - rg \geq r(r+1)\). Fix a general \( S \subset P^r \) such that \( \sharp(S) = s \).

(i) If \( s \leq r + 2 \), then \( v_{d,g,r,S} : W(d, g, r; S)' \to M_g \) is dominant.

(ii) Assume \( s \geq r+3 \) and write \( g = 2+ar+br \) and \( d = 2+r+ar+b(r-1)+c \) for some non-negative integers \( a, b, c \). If \( s \leq c + a \), then \( v_{d,g,r,S} \) is dominant.

**Proposition 1.** Fix non-negative integers \( d, g, r, s \) such that \( r \geq 3, g \geq 2, s \geq 3, \) and \( \rho(g, r, d-s+2) \geq 0 \). Fix \( S \subset P^r \) such that \( \sharp(S) = s \) and there is a line \( D \) such that \( S \subset D \). Then the natural map \( v_{d,g,r,S} : W(d, g, r; S)' \to M_g \) is dominant.

**Proposition 2.** Fix integers \( d, g, r, s \) such that \( r \geq 3, g \geq 2, s \geq 4, \) and \( \rho(g, r, d-s+3) \geq 0 \). Fix \( S \subset P^r \) such that \( \sharp(S) = s \) and there is a line \( D \) such that \( \sharp(S \cap D) = s - 1 \). Then the natural map \( v_{d,g,r,s} : W(d, g, r; S)' \to M_g \) is dominant.

We work over an algebraically closed field \( K \) such that \( \text{char}(K) = 0 \).

**Remark 1.** Here we explain why in the statements of Theorem 1 and Propositions 1 we distinguished the cases:

(i) \( s \leq r + 2 \);

(ii) \( s \geq r + 3 \).

Let \( S \subset P^r \) be a subset in linearly general position such that \( \sharp(S) = s \). If \( s \geq r + 2 \) then no automorphism of \( P^r \) fix pointwise \( S \). If \( s \leq r + 1 \) the set of all projectivities fixing pointwise \( S \) has dimension \( r(r + 2 - s) \) and hence we need to take into account the integer \( r(r + 2 - s) \) in each dimensional computation. However, if \( s \leq r + 2 \) and \( u_{g,r,d,0} \) is dominant then \( u_{g,r,d,s} \) and hence \( u_{g,r,d,s} \) are dominant. Thus if \( s \leq r + 2 \), then \( u_{d,g,r,s} \) and/or \( v_{d,g,r,s} \) are dominant if and only if \( \rho(g, r, d) \geq 0 \) ([1], §3), proving the case \( s \leq r + 2 \) of Theorem 1. Furthermore, all linearly normal subsets of \( P^r \) with at most \( r + 2 \) points are projectively equivalent. Hence if \( s \leq r + 2 \) in this range \( u_{d,g,r,S} \) and/or are dominant if and only \( \rho(g, r, d) \geq 0 \), just assuming that \( S \) is in linearly general position. This is the reason why for \( s \leq r + 2 \) to get something new in Propositions 1 and 2 we look at points not in linearly general position. Hence we need to assume \( s \geq 3 \). The extremal case is when the \( s \) points are
collinear (Proposition 1). The next extremal case is when \( s \geq 4 \) and exactly \( s - 1 \) points are collinear (Proposition 2).

For all integers \( d, g, r, s \) such that \( r \geq 3 \) and \( s \geq r + 2 \) set \( \rho(g, r, d, s) := (r + 1)d - rg + r - s(r - 1) \).

**Remark 2.** Let \( C \subset \mathbb{P}^r \), \( r \geq 3 \), be a rational normal curve. Then \( N_C \) is isomorphic to the direct sum of \( r - 1 \) line bundles of degree \( r + 2 \) ([2] or [3], proof of Th. 5.2). Hence \( h^1(C, N_C(-S)) = 0 \) for every \( S \subset C \) such that \( \sharp(S) \leq r + 2 \). Now fix an integer \( m \) such that \( 2 \leq m < r \) and let \( M \subset \mathbb{P}^r \) any \( m \)-dimensional linear subspace. Let \( T \subset M \) be a rational normal curve of \( M \). The first part of this Remark gives that \( N_{T,M} \) is the direct sum of \( m - 1 \) line bundles of degree \( m + 2 \). Hence \( N_T \) is the direct sum of \( m - 1 \) line bundles of degree \( m + 2 \) and \( m - m \) line bundles of degree \( m \).

**Lemma 1.** Fix integers \( d, g, r, s \) such that \( W(d, g, r) \) is defined, \( d \geq r + 1 \), and \( s \geq r + 2 \). Fix a general \( S \subset \mathbb{P}^r \) such that \( \sharp(S) = s \). Assume that \( v_{d,g,r,s} \) is dominant and the existence of \( C \in W(d, g, r; S) \) such that \( h^1(C, N_C(-S)) = 0 \). Then \( v_{d+r-1,g+r,r,s} \) is dominant and \( h^1(X, N_X(-S)) = 0 \) for a general \( X \in W(d + r - 1, g + r, S) \).

**Proof.** By semicontinuity we may assume that \( C \) is general in \( W(d, g, r; S) \). Hence \( v_{d,g,r,s}^{-1}(v_{d,g,r,s}(C)) \) has dimension \( \rho(g, r, d, s) \) at \( C \). Take a general hyperplane \( H \subset \mathbb{P}^r \). Thus \( H \) is transversal to \( C \) and we may find a set \( A \subset C \cap H \) which are in linear general position in \( H \) and \( \sharp(A) = r + 1 \). Let \( T \subset H \) the general linearly normal curve of \( H \) containing \( A \). Thus \( C \cap T = A \) and \( T \) intersects \( C \) quasi-transversally. Notice that \( Y := C \cup T \in W(d, g, r; S) \) ([1], Lemmas 1.2 and 1.3) and that the proof of the vanishing of \( h^1(Y, N_Y) \) given in [1], Lemma 1.2, gives \( h^1(Y, N_Y(-S)) = 0 \) for both assertions we use Remark 2 with \( m = r - 1 \). Thus \( h^1(X, N_X(-S)) = 0 \) for a general \( X \in W(d + r - 1, g + r, S) \). The curve \( Y \) is stable and \( W(d + r - 1, g + r, r; S) \) is smooth at \( Y \). Hence check that \( v_{d+r-1,g+r,r,s} \) is dominant, it is sufficient to check that the fiber of \( v_{d+r-1,g+r,r,s} \) over \( Y \) has dimension \( \rho(g + r, r, d + r - 1, s) \). Notice that \( \rho(g + r, r, d + r - 1, s) = \rho(g, r, d, s) \) and copy the part “ \( \rho(g, r, d) \geq 0 \) ” of [1], Prop. 3.1.

**Lemma 2.** Fix integers \( d, g, r, s \) such that \( W(d, g, r) \) is defined, \( d \geq r + 1 \), and \( s \geq r + 2 \). Fix a general \( S \subset \mathbb{P}^r \) such that \( \sharp(S) = s \). Assume that \( v_{d,g,r,s} \) is dominant and the existence of \( C \in W(d, g, r; S) \) such that \( h^1(C, N_C(-S)) = 0 \). Fix a general \( P \in \mathbb{P}^r \) and set \( S' := S \cup \{P\} \). Then \( v_{d+r,g+r,r,s}^{S'} \) is dominant and \( h^1(X, N_X(-S')) = 0 \) for a general \( X \in W(d + r, g + r, S') \).

**Proof.** We follow the proof of Lemma 2. Now we take as \( T \) a general rational normal curve of \( \mathbb{P}^r \) containing \( P \) and \( r + 1 \) general points of \( C \). Hence \( Y := C \cup T \in W(d + r, g + r, r; S') \). We have \( h^1(Y, N_Y(-S')) = 0 \) by Remark 2 and the proof of [1], Lemma 1.2. Notice that \( \rho(g + r, r, d + r, s + 1) = \rho(g, r, d, s) + 1 \). By Brill-Noether theory for \( C \) and the generality of \( C \cap T \) there are \( \infty^1 \) embeddings of \( Y \) in \( \mathbb{P}^r \) near the given one and with \( S' \) in there images.
Proof of Theorem 1. The case \(s \leq g + 2\) follows from Remark 1 and the dominance of the map \(v_{d,g,r,0}\) proved in [1], Prop. 3.1. Then use \(a\) times Lemma 1 and \(b\) times Lemma 2. If \(c > 0\) use that for any fixed curve \(C\), any \(P \in C_{\text{reg}}\) and for a general \(A \subset \mathbb{P}^r\) such that \(\sharp(A) = c\) there is a smooth degree \(c\) rational curve \(T\) containing \(A\), such that \(T \cap C = \{P\}\) and \(T\) intersects quasi-transversally \(C\).

Proof of Proposition 1. Fix \(S' \subset S\) such that \(\sharp(S') = 2\). Since \(\sharp(S') \leq r + 2\) and \(\rho(g, r, d - s + 2) \geq 0\), Remark 1 and [1], §3, implies that \(v_{d-s+2,g,r;S'}\) is dominant. Take a general \(C \in W(d-s+2,g,r;S')\). By the generality of \(C\) we may assume \(S \cap C = S'\). Call \(C'\) the union of \(C\) and \(s-2\) general lines, each of them intersecting quasi-transversally \(C\) at exactly one point and containing a different point of \(S \setminus S'\). \(C'\) is the stable curve associated to this union.

Proof of Proposition 2. Take \(S' \subset S\) such that \(\sharp(S') = 3\) and \(\sharp(S' \cap D) = 2\). Since \(\sharp(S') \leq r + 2\) we may repeat verbatim the proof of Proposition 1.

References


Received: July 24, 2006