Int. J. Contemp. Math. Sciences, Vol. 2, 2007, no. 5, 241 - 244

Plane Curves with Ordinary Multiple Points and Variables Ordinary Nodes

E. Ballico[1]

Dept. of Mathematics
University of Trento
38050 Povo (TN), Italy
ballico@science.unitn.it

Abstract. Fix integers $s > 0$, $m_i > 0$, $1 \le i \le s$, $d > 0$, $z \ge 0$, and s distinct points of $\mathbf{P}^2$. Let $W(d; m_1P_1, \dots, m_sP_s; z)$ denote the set of all integral degree d plane curves with an ordinary point with multiplicity m_i at each P_i and z ordinary nodes in $\mathbf{P}^2\backslash\{P_1, \dots, P_s\}$ as its only singularities. Here we show under certain assumptions (mainly if either $s \le 5$ or $z \gg 0$) that $W(d; m_1P_1, \dots, m_sP_s; z)$ and with the expected dimension.

Mathematics Subject Classification: 14H50

Keywords: plane curve, singular plane curve, ordinary multiple point

Fix integers $s > 0$, $m_i > 0$, $1 \le i \le s$, $d > 0$, $z \ge 0$, and s distinct points of $\mathbf{P}^2$. Let $W(d; m_1P_1, \dots, m_sP_s; z)$ denote the set of all integral degree d plane curves with an ordinary point with multiplicity m_i at each P_i and z ordinary nodes in $\mathbf{P}^2\backslash\{P_1, \dots, P_s\}$ as its only singularities. Thus each $C \in W(d; m_1P_1, \dots, m_sP_s; z)$ has geometric genus $(d-1)(d-2)/2 - \sum_{i=1}^{s} \binom{m_i}{2} - z$. Let $u : M \to \mathbf{P}^2$ be the blowing up of $\{P_1, \dots, P_s\}$. Set $E_i := \pi^{-1}(P_i)$ and $L_d := u^*(\mathcal{O}_{\mathbf{P}^2}(d))(-m_1E_1 - \cdots - m_sE_s)$. Let $V(M, d, z)$ denote the set of all $C \in |L_d|$ which are integral and with z ordinary points (none of them on $E_1 \cup \cdots \cup E_s$) as its only singularities. The strict transform induces an insomorphim between $W(d; m_1P_1, \dots, m_sP_s; z)$ and $V(M, d, z)$.

Notice that either $W(d; m_1P_1, \dots, m_sP_s; 0) = \emptyset$ or $W(d; m_1P_1, \dots, m_sP_s; 0)$ is irreducible and rational. We will say that an irreducible component Γ of the

[1]The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

scheme $W(d; m_1P_1, \dots, m_sP_s; z)$ has the *expected dimension* if $\Gamma \neq \emptyset$ and

$$\dim(\Gamma) = (d^2+3d)/2 - \sum_{i=1}^{s} m_i(m_i+1)/2 - z \tag{1}$$

Here we state our main results.

Theorem 1. *Assume $2 \le s \le 5$, $m_1 \ge \cdots \ge m - s > 0$ and that no 3 of the points $P_1, \dots, P_s$ are collinear. If $s = 2, 3, 4$ assume $d \ge m_1 + m_2$. If $s = 5$ assume $d \ge m_1 + m_2$ and $m_2 \ge m_4 + m + 5$. Fix any integer z such that $0 \le z \le (d^2+3d)/2 - \sum_{i=1}^{s} m_i(m_i+1)/2$. Then $W(d; m_1P_1, \dots, m_sP_s; z)$ is non-empty and it has an irreducible component with the expected dimension.*

Theorem 2. *Assume $d \ge \sum_{i=1} m_i$ and fix any integer z such that $0 \le z \le (d^2+3d)/2 - \sum_{i=1}^{s} m_i(m_i+1)/2$. Then $W(d; m_1P_1, \dots, m_sP_s; z)$ is non-empty and it has an irreducible component with the expected dimension.*

Remark 1. Here we assume $1 \le s \le 8$ and that $(P_1, \dots, P_s)$ is general in $(\mathbf{P}^2)^s$. Hence K_M^* is ample. Thus $K_M \cdot T < 0$ for all integral curves $T \subset M$. We will always assume $m_1 \ge \cdots \ge m_s > 0$. Here we will only do the case $z = 0$. For the case $z > 0$, see Remark 2. Since the case $s = 1$ was done in [1], we will assume $2 \le s \le 8$. In all cases (under suitable assumpitions of the integers d and m_i, $1lei \le s$, we will show that $W(d; m_1P_1, \dots, m_sP_s; 0)$ is non-empty and with the expected dimension given by (1).

(a) Here we assume $s = 2$. We also assume $d \ge m_1 + m_2$. Let $E \subset \mathbf{P}^2$ a general union of m_2 smooth conics through P_1 and P_2, $m_1 - m_2$ lines through P_1 and (if $d > m_1 + m_2$) a general union of $d - m_1 - m_2$ lines. Let $F \subset M$ the strict transform of E in M. Notice that F is reduced and connected. A general $X \in |F|$ is smooth and connected, and $W(d; m_1P_1, m_2P_2; 0)$ has the expected dimension ([1], Prop. 3.7, [4], Lemma 2 and Prop. 2.11, and [2], Lemma 3).

(b) Here we assume $s = 3$. We also assume $d \ge m_1 + m_2$. Let E be the general union of m_3 smooth conics containing P_1, P_2, P_3, $m_2 - m_3$ smooth conics through P_1 and P_2, $m_1 - m_2$ lines through P_1 and (if $d > m_1 + m_2$) a union of $d - m_1 - m_2$ lines. Let F be the strict stransform of E in M. F is reduced and connected. A general $X \in |F|$ is smooth and connected, and $W(d; m_1P_1, m_2P_2; 0)$ has the expected dimension ([1], Prop. 3.7, [4], Lemma 2 and Prop. 2.11, and [2], Lemma 3).

(c) Here we assume $s = 4$. We also assume $d \ge m_1 + m_2$. Let E be the general union of m_4 smooth conics containing P_1, P_2, P_3, P_4, $m_3 - m_4$ smooth conics containing P_1, P_2, P_3, $m_1 - m_3$ lines through P_1, $m_2 - m_3$ lines through P_2 and (if $d > m_1 + m_2$) a general union of $d - m_1 - m_2$ lines. Let F be the strict transform of E in M. F is reduced and connected. A general $X \in |F|$ is smooth and connected, and $W(d; m_1P_1, m_2P_2, m_3P_3; 0)$ has the expected dimension ([1], Prop. 3.7, [4], Lemma 2 and Prop. 2.11, and [2], Lemma 3). From no on, we omit to mention that $F \subset M$ is the strict transform of E and to quote [1], Prop. 3.7, [4], Lemma 2 and Prop. 2.11, and [2], Lemma 3.

(d) Here we assume $s = 5$. First assume $m_3 \geq m_4 + m_5$ and $d \geq m_1 + m_2$. Let E be the general union of m_4 smooth conics through P_1, P_2, P_3, P_4, m_5 smooth conics through P_1, P_2, P_3, P_5, $m_3 - m_4 - m_5$ smooth conics through P_1, P_2, P_3, $m_1 - m_3$ lines through P_1, m_2 lines through P_2 and (if $d > m_1 + m_2$) a general union of $d - m_1 - m_2$ lines. Now assume $m_2 \geq m_4 + m_5$, $m_3 < m_4 + m_5$, and $d \geq m_1 + m_2$. Let E be a general union of m_5 smooth conics through P_1, P_2, P_3, P_5, $m_4 + m_5 - m_3$ smooth conics through P_1, P_2, P_3, P_4, $m_3 - m_5$ smooth conics through P_1, P_2, P_4, $m_1 - m_4 - m_5$ lines through P_1, $m_2 - m_4 - m_5$ lines through P_2 and (if $d > m_1 + m_2$) a general plane curve of degree $d - m_1 - m_2$.

(e) Here we assume $s = 6$. Assume $d \geq 2(m_1 + m_2 + m_3)$. By [3], p. 303, $W(d; m_1P_1, \dots, m_sP_s; 0) \neq \emptyset$ and with the expected dimension.

(f) Here we assume $s = 7$. Assume $d \geq 2(m_1 + m_2 + m_3) + m_7$. Take a general solution C' of the case $s' := 6$, $d' := d - m_7$, $m_i' := m_i$ for $1 \leq i \leq 6$ (part (e)). We may assume $P_7 \notin C'$. Let E be the union of C' and m_7 general lines through P_7. Call $C'' \subset M$ the strict transform of C'. Since $K_M \cdot C'' < 0$, we get that $|F|$ contains a smooth and connected element. Hence $W(d; m_1P_1, m_2P_2; 0)$ is not empty and it has the expected dimension.

(g) Here we assume $s = 8$. Assume $d \geq 2(m_1 + m_2 + m_3) + m_7 + m_8$. Take a general solution C' of the case $s' := 6$, $d' := d - m_7 - m_8$, $m_i' := m_i$ for $1 \leq i \leq 6$ (part (e)). We may assume $P_i \notin C'$ for $i = 7, 8$. Let E be the union of C', m_8 general smooth conics containing P_7 and P_8 and (if $m_7 > m_8$) $m_7 - m_8$ general lines through P_7. Call $C'' \subset M$ the strict transform of C'. Since $K_M \cdot C'' < 0$, we get that $|F|$ contains a smooth and connected element. Hence $W(d; m_1P_1, m_2P_2; 0)$ is not empty and it has the expected dimension.

Remark 2. Here we assume $1 \leq s \leq 8$ and that $(P_1, \dots, P_s)$ is general in $(\mathbf{P}^2)^s$. Hence K_M^* is ample. Thus $K_M \cdot T < 0$ for all integral curves $T \subset M$. We will always assume $m_1 \geq \cdots \geq m_s > 0$. Here we will assume $z > 0$, see Remark 2. Since the case $s = 1$ was done in [1], we will assume $2 \leq s \leq 8$. In all cases (under suitable assumpitions of the integers d and m_i, $1lei \leq s$, we will show the existemce of an irreducible component Γ of $W(d; m_1P_1, \dots, m_sP_s; z)$ is non-empty and with the expected dimension given by (1). We will take the set-up of Remark 1. In each case we have a reducible degree d plane curve E and its transform F of E in M. We fix z suitable singular points of F, say $O_1, \dots, O_z$ and we call M' the blowing-up of M at these points. Notice that M' depends from the choice of E and then from the choice of $A := \{O_1, \dots, O_z\}$. Let G be the strict transform of F in M'. Hence G is reduced. The set A is said to be a non-disconnecting set if G is connected ([4]). We will always choose A so that G is connected. This is the only condition we need to apply [4], Lemma 2 and Prop. 2.11, and get the existence of an irreducible component of $W(d; m_1P_1, \dots, m_sP_s; z)$ with the expected dimension. In each case below let c_F denote the number of the irreducible components of F. In parts (a) each irreducible component of F is a smooth rational curve and

$p_a(F) = (d^2+3d)/2 - \sum_{i=1}^{s} m_i(m_i+1)/2$. Hence in these cases F has $(d^2 + 3d)/2 - \sum_{i=1}^{s} m_i(m_i+1)/2 + c_F - 1$ singular points. Notice that we may find a non-disconnecting subset of Sing(F) with cardinality $(d^2+3d)/2 - \sum_{i=1}^{s} m_i(m_i+1)/2$. Hence in these cases we may tale as z any integer such that $0 \le z \le (d^2+3d)/2 - \sum_{i=1}^{s} m_i(m_i+1)/2$.

(a) Here we asume $s = 2$. Use the construction of part (a) of Remark 1. We win if $d \ge m_1 + m_2$ and $0 \le z \le (d^2+3d)/2 - \sum_{i=1}^{2} m_i(m_i+1)/2$.

(b) Here we assume $s = 3$. Use the construction of part (b) of Remark 1. We win if $d \ge m_1 + m_2$ and $0 \le z \le (d^2+3d)/2 - \sum_{i=1}^{3} m_i(m_i+1)/2$.

(c) Here we assume $s = 4$. Use the construction of part (c) of Remark 1. We win if $d \ge m_1 + m_2$ and $0 \le z \le (d^2+3d)/2 - \sum_{i=1}^{4} m_i(m_i+1)/2$.

(d) Here we assume $s = 5$. Use the construction of part (a) of Remark 1. We win if $d \ge m_1 + m_2$ and $m_2 \ge m_3 + m_5$ and $0 \le z \le (d^2+3d)/2 - \sum_{i=1}^{2} m_i(m_i+1)/2$.

Proof of Theorem 1. All the statements were proved in Remark 2. □

Proof of Theorem 2. Let E be the general union of m_i lines through P_i for all $1 \le i \le s$ and (if $d > \sum_{i=1} m_i$) of $d - \sum_{i=1} m_i$ lines. Then use the first part of Remark 2. □

We work over an algebraically closed field $\mathbb{K}$ such that char$(\mathbb{K}) = 0$.

References

[1] E. Arbarello and M. Cornalba, Footnotes to a paper of Beniamino Segre, Math. Ann. 256 (1981), 341–362.

[2] M. Coppens, The Weierstrass gap sequence of the ordinary ramification points of trigonal coverings of $\mathbf{P}^1$, J. Pure Appl. Algebra 43 (1986), no. 1, 11–25.

[3] R. Hartshorne, Genre des courbes algébriques dans l'espace projectif, Sem. Bourbaki 1981/82, no. 592, Astérisque 92–93 (1992), 301–313.

[4] A. Tannenbaum, Families of algebraic curves with nodes, Compositio Math. 41 (1980), no. 1, 107–126.

Received: August 23, 2006