On Generalized Derivations in Semiprime Rings

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Abstract

The purpose of this note is to prove the following result. Let \( R \) be a semiprime ring of characteristic not 2 and \( G: R \rightarrow R \) be an additive mapping such that \( G(x^2) = G(x)x + xD(x) \) holds for all \( x \in R \) and some derivations \( D \) of \( R \). Then \( G \) is generalized derivation.

Mathematics Subject Classification: 16W10, 16W25, 16W20

Keywords: ring, prime ring, semiprime ring, centralizer, Jordan centralizer, derivation, Jordan derivation, generalized derivation, generalized derivation

1 Introduction

This note is motivated by the work of Zalar [6]. Throughout, \( R \) will represent an associative ring with center \( Z(R) \). A ring \( R \) is \( n \)-torsion free, if \( nx = 0 \), \( x \in R \) implies \( x = 0 \), where \( n \) is a positive integer. Recall that \( R \) is prime if \( aRb = (0) \) implies \( a = 0 \) or \( b = 0 \), and semiprime if \( aRa = (0) \) implies \( a = 0 \). An additive mapping \( T: R \rightarrow R \) is called a left (right) centralizer in case \( T(xy) = T(x)y \ (T(xy) = xT(y)) \) holds for all \( x, y \in R \) and is called a Jordan left (right) centralizer in case \( T(x^2) = T(x)x \ (T(x^2) = xT(x)) \) holds for all \( x \in R \). A result of Zalar [6] asserts that any Jordan centralizer on a semiprime ring of characteristic not 2 is a centralizer. An additive mapping \( D: R \rightarrow R \) is called a derivation if \( D(xy) = D(x)y + xD(y) \) holds for all pairs \( x, y \in R \).
and is called a Jordan derivation in case $D(x^2) = D(x)x + xD(x)$ holds for all $x \in R$. A derivation $D$ is inner if there exists $a \in R$ such that $D(x) = ax - xa$ holds for all $x \in R$. Every derivation is a Jordan derivation. The converse is in general not true. A classical result of Herstein \cite{4} asserts that any Jordan derivation on 2-torsion free prime ring is a derivation. Cusack \cite{2} generalized Herstein’s theorem to 2-torsion free semiprime ring.

In \cite{3}, Hvala has defined the notion of a generalized derivation as follows: An additive mapping $G : R \to R$ is said to be a generalized derivation if there exists a derivation $D : R \to R$ such that $G(xy) = G(x)y + xD(y)$ for all $x, y \in R$. Also, he called the maps of the form $x \to ax + xb$ where $a, b$ are fixed elements in $R$ by the inner generalized derivations. Ashraf and Nadeem-Ur-Rehman, in \cite{5}, have defined the concept of a Jordan generalized derivation as follows: An additive mapping $G : R \to R$ is said to be a Jordan generalized derivation if there exists a derivation $D : R \to R$ such that $G(x^2) = G(x)x + xD(x)$ for all $x \in R$. Hence the concept of a generalized derivation covers both the concepts of a derivation and a left centralizers and the concept of a Jordan generalized derivation covers both the concepts of a Jordan derivation and a left Jordan centralizers. In [1, Remark 1] Brešar proved that for a semiprime ring $R$, if $G$ is a function from $R$ to $R$ and $D : R \to R$ is an additive mapping such that $G(xy) = G(x)y + xD(y)$ for all $x, y \in R$, then $D$ is uniquely determined by $G$ and moreover $G$ must be a derivation. Ashraf and Nadeem-Ur-Rehman, in \cite{5}, proved the following result: Let $R$ be a 2-torsion free ring such that $R$ has a commutator which is not a zero divisor, then every Jordan generalized derivation on $R$ is a generalized derivation.

In this note, using Zalar’s method, we study the same result of Ashraf and Nadeem-Ur-Rehman but for a semiprime ring, and without the condition of zero divisor, i.e., if $R$ is a semiprime ring of characteristic not 2 and $G$ is an additive mapping which satisfies

\[ G(x^2) = G(x)x + xD(x) \]

holds for all $x \in R$ and some derivation $D$ of $R$, then $G$ is generalized derivation. This result will be a generalization of the result of Zalar \cite{6}. In order to prove our result we will need the following lemmas which are due to Zalar.

**Lemma 1.1** (\cite{6} Lemma 1.1). Let $R$ be a semiprime ring. If $a, b \in R$ are such that $axb = 0$ for all $x \in R$, then $ab = ba = 0$.

**Lemma 1.2** (\cite{6} Lemma 1.2). Let $R$ be a semiprime ring and $\theta, \phi : R \times R \to R$ biadditive mappings. If $\theta(x, y)w\phi(x, y) = 0$ for all $x, y, w \in R$, then $\theta(x, y)w\phi(u, v) = 0$ for all $x, y, u, v, w \in R$.

**Lemma 1.3** (\cite{6} Lemma 1.3). Let $R$ be a semiprime ring and $a \in R$ be some fixed element. If $a[\cdot, \cdot] = 0$ for all $x, y \in R$, then there exists an ideal $U$ of $R$ such that $a \in U \subseteq Z(R)$ holds.
2 The Main Result

Theorem 2.1. Let $R$ be a semiprime ring of characteristic not 2 and $G: R \rightarrow R$ be an additive mapping satisfying the relation

$$ G(x^2) = G(x)x + xD(x), $$

for all $x \in R$ and some derivation $D$ of $R$. Then $G$ is generalized derivation.

Proof. Replacing $x$ by $x + y$ in (1) we get

$$ G(xy + yx) = G(x)y + G(y)x + xD(y) + yD(x), \quad x, y \in R. $$

Replacing $y$ by $xy + yx$ in (2) and using (2) we obtain

$$ G(x^2y +yx^2) + 2G(xy) = G(x)xy + G(x)yx + G(y)x^2 + xD(y)x + yD(x)x + xD(xy + yx) + (xy + yx)D(x), \quad x, y \in R. $$

On the other hand, replacing $x$ by $x^2$ in (2) and adding $2G(xy)$ to both sides we get

$$ G(x^2y + yx^2) + 2G(xy) = G(x)xy + xD(x)y + G(y)x^2 + x^2D(y) + yxD(x) + yD(x)x + 2G(xy), \quad x, y \in R. $$

Comparing (3) and (4) we obtain

$$ G(xy) = G(x)yx + xD(yx), \quad x, y \in R. $$

Putting $x = x + z$ in (5), we get

$$ G(xyz + zyx) = G(x)yz + G(z)yx + xD(yz) + zD(yx), \quad x, y, z \in R. $$

Let $F = G(xzyx + yxzxy)$, we shall compute it in two different ways. Using (5) we have

$$ F = G(x)zyx + G(y)xzxy + xD(yzx) + yD(xzy), \quad x, y, z \in R. $$

Using (6) we have

$$ F = G(xy)zyx + G(y)xzxy + xyD(zyx) + yxD(zxy), \quad x, y, z \in R. $$

Comparing (7) and (8) we get

$$ \theta(x, y)zyx + \theta(y, x)zxy = 0, \quad x, y, z \in R, $$
where $\theta(x, y)$ stands for $G(xy) - G(x)y - xD(y)$. In the concept of the definition of $\theta$, equation (2) can be rewritten in the form $\theta(x, y) = -\theta(y, x)$. Using this notation in equation (9) we get

$$\theta(x, y)z[x, y] = 0, \quad x, y, z \in R. \quad (10)$$

Using Lemma 1.2 we get

$$\theta(x, y)z[u, v] = 0, \quad x, y, z, u, v \in R. \quad (11)$$

Using Lemma 1.1 we obtain

$$\theta(x, y)[u, v] = 0, \quad x, y, u, v \in R. \quad (12)$$

Now fix $x, y \in R$ and write $\theta$ instead of $\theta(x, y)$ to simplify further writing. Using Lemma 1.3 we get the existence of an ideal $U$ such that $\theta \in U \subset Z(R)$ holds. In particular, $b\theta$, $\theta b \in Z(R)$ for all $b \in R$. This gives us

$$x.\theta^2 y = \theta^2 y.x = y\theta^2 x = y.\theta^2 x.$$

This gives us $4G(x.\theta^2 y) = 4G(y.\theta^2 x)$. Now we will compute each side of this equality by using (2) and the above notation.

$$4G(x.\theta^2 y) = 2G(x\theta^2 y + \theta^2 yx) =$$

$$= 2G(x)\theta^2 y + 2xD(\theta^2 y) + 2G(\theta^2 y)x + 2\theta^2 yD(x) =$$

$$= 2G(x)\theta^2 y + G(\theta^2 y + y\theta^2)x + 2x D(\theta^2 y) + 2\theta^2 yD(x) =$$

$$= 2G(x)\theta^2 y + G(\theta)\theta y x + \theta D(\theta)y x + G(y)\theta^2 x + \theta^2 D(y)x + yD(\theta^2)x +$$

$$2x D(\theta^2 y) + 2\theta^2 yD(x).$$

So we get

$$4G(x.\theta^2 y) = 2G(x)\theta^2 y + G(\theta)\theta y x + \theta D(\theta)y x + G(y)\theta^2 x + \theta^2 D(y)x +$$

$$yD(\theta^2)x + 2x D(\theta^2 y) + 2\theta^2 yD(x), \quad x, y \in R. \quad (13)$$

Moreover,

$$4G(y.\theta^2 x) = 2G(y\theta^2 x + \theta^2 xy) =$$

$$= 2G(y)\theta^2 x + 2y D(\theta^2 x) + 2G(\theta^2 x)y + 2\theta^2 xD(y) =$$

$$= 2G(y)\theta^2 x + G(\theta^2 x + x\theta^2)y + 2y D(\theta^2 x) + 2\theta^2 xD(y) =$$

$$= 2G(y)\theta^2 x + G(\theta)\theta y x + \theta D(\theta)x y + G(x)\theta^2 y + \theta^2 D(x)y + xD(\theta^2)y +$$

$$2y D(\theta^2 x) + 2\theta^2 xD(y).$$
So we get
\[ 4G(y, \theta^2 x) = 2G(y)\theta^2 x + G(\theta)\theta xy + \theta D(\theta)xy + G(x)\theta^2 y + \theta^2 D(x)y + xD(\theta^2)y + 2yD(\theta^2)x + 2\theta^2 xD(y), \quad x, y \in R. \] (14)

Comparing (13) and (14) and using the following notations
\[ \theta yx = \theta y, x = x\theta y = \theta xy, \]
\[ \theta D(\theta)yx = D(\theta)\theta yx = D(\theta)\theta xy = \theta D(\theta)xy, \]
\[ x\theta D(\theta)y = D(\theta)x\theta y = D(\theta)\theta xy = D(\theta)\theta yx = \theta yD(\theta)x = \theta yD(\theta)x, \]
we obtain
\[ G(x)\theta^2 y + x\theta^2 D(y) = G(y)\theta^2 x + y\theta^2 D(x) \]
which gives
\[ \phi(x, y)\theta^2 = \phi(y, x)\theta^2, \]
where \( \phi(x, y) \) stands for \( G(x)y + xD(y) \). On the other hand, we also have
\[ 4G(xy\theta^2) = 4G(x\theta, y\theta). \]
We will compute each side of this equality by using (2) and the properties of \( \theta \), so we get
\[ 4G(xy\theta^2) = 2G(xy\theta^2 + \theta^2 xy) = 2G(xy)\theta^2 + 2xyD(\theta^2) + 2G(\theta^2)xy + 2\theta^2 D(xy), \]
which gives
\[ 4G(xy\theta^2) = 2G(xy)\theta^2 + 2xyD(\theta^2) + 2G(\theta^2)xy + 2\theta^2 D(xy), \quad x, y \in R. \] (16)
Moreover,
\[ 4G(x\theta, y\theta) = 2G(x\theta y\theta + y\theta x\theta) = \]
\[ = 2G(\theta x)\theta y + 2\theta xD(\theta y) + 2G(\theta y)\theta x + 2\theta yD(\theta x) = \]
\[ = G(\theta x + \theta x)\theta y + 2\theta xD(\theta y) + G(\theta y + \theta y)\theta x + 2\theta yD(\theta x) = \]
\[ = G(x)\theta^2 y + G(\theta)\theta xy + xD(\theta)\theta y + \theta D(x)\theta y + 2\theta xD(\theta y) + G(y)\theta^2 x + \]
\[ G(\theta)\theta yx + yD(\theta)\theta x + \theta D(y)\theta x + 2\theta yD(\theta x). \]
So we obtain
\[ 4G(x\theta, y\theta) = G(x)\theta^2 y + G(\theta)\theta xy + xD(\theta)\theta y \]
\[ + \theta D(x)\theta y + 2\theta xD(\theta y) + G(y)\theta^2 x + G(\theta)\theta yx + yD(\theta)\theta x \]
\[ + \theta D(y)\theta x + 2\theta yD(\theta x), \quad x, y \in R. \] (17)
Comparing (16) and (17), we obtain
\[ 2G(xy)\theta^2 = \phi(x, y)\theta^2 + \phi(y, x)\theta^2, \quad x, y \in R. \] (18)
Using (15), finally we get \( G(xy)\theta^2 = \phi(x, y)\theta^2 \). But \( \theta(x, y) = G(xy) - \phi(x, y) \) and this means \( \theta^3 = 0 \) so that
\[ \theta^2 R \theta^2 = \theta^4 R = (0), \]
\[ \theta R \theta = \theta^2 R = (0), \]

which implies \( \theta = 0 \), and the proof is complete.

It is clear that if we let the derivation \( D \) to be the zero derivation in the above theorem, we get the following result.

**Corollary 2.2 ([6] Proposition 1.4).** Let \( R \) be a semiprime ring of characteristic not 2 and \( T : R \rightarrow R \) an additive mapping which satisfies \( T(x^2) = T(x)x \) for all \( x \in R \). Then \( T \) is a left centralizer.

**References**


Received: June 16, 2007