Int. J. Contemp. Math. Sciences, Vol. 2, 2007, no. 29, 1441 - 1449

MC-Hypercentral Groups

Francesco Russo

Department of Mathematics
Office n.17, campus of Monte S.Angelo, via Cinthia 80126
University of Naples, Naples, Italy
E-mail: francesco.russo@dma.unina.it

Abstract

This paper is devoted to the imposition of some chain conditions on groups having a generalized central series. It is also given a characterization of MC-groups with finite abelian section rank: such class of groups is a suitable enlargement of the class of FC-groups.

Mathematics Subject Classification: 20F24; 20F14

Keywords: PC-hypercentral groups; CC-hypercentral groups; upper central series; chains of normal subgroups

1. Introduction

Let $\mathfrak{X}$ be a class of groups. An element x of a group G is said to be $\mathfrak{X}C$-*central* if $G/C_G(x^G)$ satisfies $\mathfrak{X}$, where the symbol x^G represents the normal closure in G of the subgroup $< x >$. Sometimes the factor $G/C_G(x^G)$ is denoted by $Aut_G(x^G)$ to recall that $G/C_G(x^G)$ is a group of automorphisms of x^G (see [**16**, Chapter 3]).

If $\mathfrak{X}$ has remarkable closure properties, then it is possible to introduce some series which generalize the upper central series. Haimo [**6**] and Nishigôri [**13**] started this kind of study thanks to FC-central series, successively [**2**] and [**11**] have extended such studies to wider classes of groups. [**2**] regards $\mathfrak{X}C$-hypercentral groups, where $\mathfrak{X}$ is the class of polycyclic-by-finite or Chernikov groups, while [**11**] treats $\mathfrak{X}C$-hypercentral groups, where $\mathfrak{X}$ is a suitable Schur class. Here we are interested to improve certain results of [**2**] and [**11**], considering $\mathfrak{X}$ as the class of (soluble minimax)-by-finite groups.

In a more explicit way a group H is said to be $(soluble\ \ minimax) - by - finite$ if it contains a normal subgroup K such that K has a finite characteristic series $1 = K_0 \lhd K_1 \lhd \ldots \lhd K_n = K$ whose factors are abelian minimax and H/K

is finite. Abelian minimax groups are described by [**16**, Lemma 10.31] and consequentely soluble minimax groups are well known (see [**16**, Sections 10.3 and 10.4]).

If G is a group and x is an element of G, then x is called *MC-element* of G if $G/C_G(x^G)$ is (soluble minimax)-by-finite. If x and y are MC-elements of G, then both $G/C_G(x^G)$ and $G/C_G(y^G)$ are (soluble minimax)-by-finite, so $G/(C_G(x^G)\cap C_G(y^G))$ is also (soluble minimax)-by-finite. But the intersection of $C_G(x^G)$ with $C_G(y^G)$ lies in $C_G((xy^{-1})^G)$, so that $G/C_G((xy^{-1})^G)$ is (soluble minimax)-by-finite and xy^{-1} is an MC-element of G. Hence the MC-elements of G form a subgroup $M(G)$ and $M(G)$ is characteristic in G.

This simple remark allows us to define the series

$$1 = M_0 \triangleleft M_1 \triangleleft \ldots \triangleleft M_\alpha \triangleleft M_{\alpha+1} \triangleleft \ldots ,$$

where $M_1 = M(G)$, the factor $M_{\alpha+1}/M_\alpha$ is the subgroup of G/M_α generated by the MC-elements of G/M_α and

$$M_\lambda = \bigcup_{\alpha<\lambda} M_\alpha,$$

with α ordinal and λ limit ordinal. This series is a characteristic ascending series of G and it is called *upper MC-central series* of G. The last term of the upper MC-central series of G is called *MC-hypercenter* of G and it is denoted by $\bar{M}(G)$. If $G = M_\beta$, for some ordinal β, we say that G is an *MC-hypercentral* group of type at most β and this is equivalent to say that $G = \bar{M}(G)$.

The *MC-length* of an MC-hypercentral group is defined to be the least ordinal β such that $G = M_\beta$, in particular when $G = M_c$ for some positive integer c, we say that G is *MC-nilpotent* of length c. A group G is said to be an MC-group if all its elements are MC-elements, that is, if G has MC-length at most 1.

In analogy with FC-groups, the first term $M(G)$ of the upper MC-central series of G is said to be the *MC-center* of G and the α-th term of the upper MC-central series of G is said to be the *MC-center of length α* of G. Roughly speaking, the upper MC-central series of G measures the distance of G to be an MC-group. By definitions, it happens that $Z(G) \le F(G) \le M(G)$, where $F(G)$ is the FC-center of G; i.e.: the subgroup of G generated by the elements which have only a finite number of conjugates.

It is reasonable to expect that MC-hypercentral groups satisfy some classical McLain's Theorems on generalized nilpotent series [**16**,Theorem 4.37, Theorem 4.38], with related finiteness conditions [**2**, Theorem A, Theorem B, Theorem C] or [**16**, Theorem 4.39, Theorem 4.39.1, Theorem 4.39.2]. On the other hand the conditions of hypercentrality, FC-hypercentrality and MC-

hypercentrality can be different in a same group and this fact will be shown by means of examples. In such direction of study we prove:

Theorem 1. *Let G be an MC-hypercentral group.*

(i) *If G satisfies $max-n$, then G is nilpotent-by-(soluble minimax)-by-finite and finitely generated;*

(ii) *If G satisfies $min-n$, then G is nilpotent-by-Chernikov.*

Theorem 2. *Let G be a periodic group with finite abelian section rank. Then the following are equivalent:*

(i) *G is CC-hypercentral;*

(ii) *G is FC-hypercentral;*

(iii) *G is PC-hypercentral;*

(iv) *G is MC-hypercentral.*

Theorem 3. *Let G be an MC-group. G has finite abelian section rank if and only if G is an extension of a soluble minimax group by a CL-group.*

Most of our notation is standard and it is referred to [**16**]. In particular:

- the symbols $F(G)$, $P(G)$, $C(G)$ are used to denote respectively the FC-center, the PC-center, the CC-center of a group G. The definitions related to these subgroups can be found in [**2**,**16**];
- a group G has *finite abelian section rank* if it has no infinite abelian sections of prime exponent;
- a group G satisfies $max-sn$ (respectively $min-sn$) if it satisfies the maximal (respectively the minimal) condition on subnormal subgroups;
- a group G satisfies $max-ab$ (respectively $min-ab$) if it satisfies the maximal (respectively the minimal) condition on abelian subgroups;
- a group G satisfies $max-n$ (respectively $min-n$) if it satisfies the maximal (respectively the minimal) condition on normal subgroups;
- a minimax group is a group which has a series of finite length each whose factors satisfies either max or min. A soluble minimax group is a minimax group which is soluble;

- a group G is said to be (soluble minimax)-by-finite if it contains a normal soluble minimax subgroup N of finite index in G;
- if m is a positive integer or infinity, the *m-th layer* of a group G is the subgroup generated by the elements of G of order m. A group all of whose layers are Chernikov is called CL-group. The structure of such groups is described by [**16**, Theorem 4.43].

2. Proof of Theorem 1

PC-hypercentral and CC-hypercentral groups are MC-hypercentral and it is clear that each periodic MC-hypercentral group is CC-hypercentral, so [**2**] and [**11**] solve partially Theorem 1, Theorem 2 and Theorem 3. For convenience we recall [**2**, Lemma 1] without proof.

Lemma 2.1. *Let G be a group which satisfies $min-n$ and let $1 = G_0 \triangleleft G_1 \triangleleft \ldots \triangleleft G_\alpha \triangleleft G_{\alpha+1} \ldots \triangleleft G_\beta = G$ be an ascending normal series of G. Then the subgroup $I = \bigcap_{\alpha<\beta} C_G(G_{\alpha+1}/G_\alpha)$ is nilpotent.*

Proof of Theorem 1. (i). Since G is MC-hypercentral with $max-n$, its upper MC-central series is $1 = M_0 \triangleleft M_1 \triangleleft \ldots \triangleleft M_c = G$ for some positive integer c. G satisfies $max-n$, hence $M_1 = x_1^G \ldots x_m^G$ for some finite set $\{x_1 \ldots x_m\}$ of elements of M_1. $Aut_G(M_1)$ is (soluble minimax)-by-finite and with a similar argument $Aut_G(M_{i+1}/M_i)$ too, for each $i \in \{1, \ldots c-1\}$. Put $I = \bigcap_{i=1}^{c} C_G(M_{i+1}/M_i)$, I is the stabilizer of the upper MC-central series of G so I is nilpotent, thus G is nilpotent-by-(soluble minimax)-by-finite. Let N be a nilpotent-by-(soluble minimax) normal subgroup of G of finite index. By [**16**, Theorem 5.31], N has $max-n$ and it is finitely generated by a classical Hall's Theorem [**16**, Theorem 5.33], hence G is finitely generated.

(ii). Let G be an MC-hypercentral group with $min-n$ and let x be a nontrivial element of $M(G)$. Then $G/C_G(x^G)$ is (soluble minimax)-by-finite with $min-n$, and so it is Chernikov. This implies that G is a CC-hypercentral group with $min-n$, then, using Lemma 2.1, we may apply the argument in [**2**, Proof of Theorem B] to conclude that G is nilpotent-by-Chernikov. For convenience of the reader we recall the argument of [**2**, Proof of Theorem B].

The upper CC-hypercentral series of G has the form

$$1 = C_0 \triangleleft C_1 \triangleleft \ldots \triangleleft C_\alpha \triangleleft C_{\alpha+1} \ldots \triangleleft C_\beta = G,$$

and $I = \bigcap_{\alpha<\beta} C_G(C_{\alpha+1}/C_\alpha)$ is nilpotent by Lemma 2.1. Since G has $min-n$, there exists a positive integer n such that $I = \bigcap_{j=0}^{n-1} C_G(C_{j+1}/C_j)$. On the other hand, for each $j \in \{0, \dots, n-1\}$, there exists a finite subset X_j of C_{j+1} such that $C_G(C_{j+1}/C_j) = \bigcap_{x \in X_j} C_G(x^G C_{j+1}/C_j)$, so $G/C_G(C_{j+1}/C_j)$ is a Chernikov group. Then G/I is a Chernikov group.□

A natural generalization to MC-hypercentral groups of [**2**, Corollary 1] is the following.

Corollary 2.2. *If G is an MC-hypercentral group with $min - n$, then G is locally finite.*

Proof. Obvious by Theorem 1.□

It is natural to ask what is the structure of an MC-hypercentral group which is neither CC-hypercentral, nor PC-hypercentral. The following example describes an MC-group, which is neither a PC-group, nor a CC-group. This group is neither hypercentral, nor an FC-group so it shows that there are difficulties to extend the classical McLain's Theorems on generalized nilpotent series [**16**,Theorem 4.37,Theorem 4.38] to the $\mathfrak{X}C$-hypercentral series, when $\mathfrak{X}$ is an arbitrary class of groups.

Example 2.3. Let p be a prime, $\mathbb{Q}$ be the additive group of the rational numbers and $U(\mathbb{Q})$ be the group of units of $\mathbb{Q}$. We may consider $G = Q \ltimes A$, where Q is a p-generating subgroup of $U(\mathbb{Q})$ and $A = \mathbb{Q}_p$ is the additive group of rational numbers whose denominators are p-numbers. Fixed a positive integer n, we have $Q = < x_0 > \times < x_1 > \times \dots \times < x_n >$, where $x_0 \in \{-1, 1\}$ and $x_1, \dots, x_n$ generate the free abelian group $< x_1 > \times \dots \times < x_n >$. If $-1 \notin Q$, then $Q = < x_1 > \times \dots \times < x_n >$ and $n = r$ is the Prüfer rank of Q. G is generated by A together with elements $y_1, \dots y_r$, where y_i is a preimage of x_i under the epimorphism $G \to Q$, for all $i \in \{1, \dots, r\}$. Here y_i acts on Q via multiplication by x_i. Also $[y_i, y_j] = c_{ij} \in A$, where c_{ij} satisfy the system of linear equations over $\mathbb{Q}$:

$$\forall i, j, k \in \{1, \dots, r\}, \quad c_{ij} = -c_{ji}, \quad c_{jk}(x_i - 1) + c_{ki}(x_j - 1) + c_{ij}(x_k - 1) = 0.$$

The second equation is the famous Hall-Witt identity. This construction is due to Robinson and Wilson [**15**].

It is easy to verify that each element of G is an MC-element, then $G = M(G)$ and G is an MC-group. On the other hand $Z(G) = C(G) = P(G) = F(G) = 1$, hence G is neither a CC-group nor a PC-group. Moreover G is neither nilpotent nor an FC-group. The triviality of $Z(G)$, $C(G)$, $P(G)$ and $F(G)$ shows also that G is MC-hypercentral, but neither hypercentral, nor CC-hypercentral, nor PC-hypercentral, nor FC-hypercentral.

In order to obtain a group G which has different $Z(G)$, $F(G)$, $P(G)$, $C(G)$, $M(G)$, it is possible to consider just-non-$\mathfrak{X}$ groups, where $\mathfrak{X}$ is a suitable group-theoretical property (see [**10**] for details).

Example 2.4. The group, which is described in [**5**, Proposition 2.2], is nilpotent of class 2 and it has $C(G) = F(G) = Z(G) \neq 1$ but $M(G) = P(G) = G$. This group is CC-nilpotent and FC-nilpotent of class 2 but it is PC-nilpotent and MC-nilpotent of class 1.

[**10**, Corollary 14.19] describes a just-non-Chernikov group G with $F(G) = 1$. Such group has $1 = Z(G) = C(G)$, but the PC-length and the MC-length of G depend by $G/Fit(G)$, where $Fit(G)$ is the Fitting subgroup of G.

3. Some special cases

In this Section we want to analyze some finitary conditions on MC-hypercentral groups, trying to adapt some classical results on FC-hypercentral chains (see [**6**,**13**,**16**]).According to [**1**], next lemma proves that MC-ascendant groups are generalized radical groups, that is, they have an ascending series whose factors are either locally nilpotent or finite.

Lemma 3.1. *If G is a group with nontrivial MC-center, then G contains a nontrivial normal subgroup which is finite or abelian.*

Proof. If $M(G) = P(G)$ or $M(G) = C(G)$, then we finish by [**2**, Lemma 2]. Given $x \in M(G)$, $x^G/Z(x^G)$ is (soluble minimax)-by-finite. If $Z(x^G) \neq 1$, G has a nontrivial normal abelian subgroup; if $Z(x^G) = 1$, then x^G is (soluble minimax)-by-finite and the result is proved.□

Proposition 3.2. *If G is an MC-ascendant group, the following are equivalent:*

(i) *G satisfies $max - ab$ (respectively $min - ab$);*

(ii) *G satisfies $max - sn$ (respectively $min - sn$);*

(iii) *G satisfies max (respectively min);*

(iv) *G is polycyclic-by-finite (respectively Chernikov).*

Proof. Obvious by Lemma 3.1 and [**2**, Proposition 1].□

The examples in [**2**] show that $max - n$ (respectively $min - n$) can not be insert in Proposition 3.2.

Corollary 3.3. *Let G be a group such that one of the conditions of Proposition 3.2 holds. Then the following are equivalent:*

(i) *G is MC-hypercentral;*

(ii) *G is CC-hypercentral;*

(iii) *G is PC-hypercentral;*

(iv) *G is FC-hypercentral.*

Proof. This follows by [**16**, Theorem 4.39.2], [**2**, Proposition 2] and Proposition 3.2.□

4. Proofs of Theorem 2 and Theorem 3

Proof of Theorem 2. (i) implies (ii) by [**11**, Theorem 3.5]. By definitions (ii) implies (iii) and (iii) implies (iv). We must prove that (iv) implies (i). Since a periodic (soluble minimax)-by-finite group is Chernikov, the upper CC-central series and the upper MC-central series coincide and the result is proved.□

Example 2.3 shows that we can not hope in a wider version of Theorem 2 for non-periodic groups. However the torsion-free groups improve Theorem 2 in the following sense.

Corollary 4.1. *Let G be a torsion-free group. Then the following are equivalent:*

(i) *G is hypercentral;*

(ii) *G is FC-hypercentral;*

(iii) *G is CC-hypercentral.*

Proof. By definitions (i) implies (ii) and (ii) implies (iii). We must prove that (iii) implies (i). Let x be a nontrivial element of $C(G)$. By [**16**, Theorem 4.36], x^G is Chernikov-by-cyclic. Since G is torsion-free, x^G is cyclic, then $x^G =< x > [G,x] =< x >$ and $[g,x] = 1$ for each element g of G. Thus x belongs to $Z(G)$ and $C(G) = Z(G)$. If α is a limit ordinal, then it is clear that it must be $G = C_\alpha = Z_\alpha$. Let α be not a limit ordinal, C_α be the general term of the upper CC-central series of G and Z_α be the general term of the upper central series of G. Using the initial argument $C_2/C_1 = C(G/C_1) = Z(G/Z_1) = Z_2/Z_1$, so $C_2 = Z_2$ and by induction we conclude that $C_\alpha = Z_\alpha$ for each α.□

Proof of Theorem 3. Let G be an MC-group with finite abelian section rank, we want to show that there is a soluble minimax normal subgroup H

of G such that G/H is a CL-group. If G is a PC-group or a CC-group, then we finish by [**2**, Proposition 4]. Let G be neither a PC-group, nor a CC-gorup. Put $FratG$ the Frattini's subgroup of G, the section $G/FratG$ has finite rank and [**9**, Corollary 10] implies that there exists a normal soluble minimax subgroup H of G such that G/H is a CC-group. G/H is an extension of an (abelian Chernikov)-by-polycylic group by a CL-group, thanks to [**2**, Proposition 4]. Now G is an extension of a soluble minimax group by an (abelian Chernikov)-by-polycylic group by a CL-group. Since the class of soluble minimax groups is closed with respect to extensions, G is an extension of a soluble minimax group by a CL-group.

Conversely if G is an extension of a soluble minimax group H by a CL-group G/H, both H and G/H have finite abelian section rank (see [**16**, Theorem 4.42] and [**2**, p.3052]). Since the property to have finite abelian section rank is additive on extensions (see [**16**]), we conclude that G has again finite abelian section rank.□

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Received: June 17, 2007