MC-Hypercentral Groups

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Abstract

This paper is devoted to the imposition of some chain conditions on groups having a generalized central series. It is also given a characterization of MC-groups with finite abelian section rank: such class of groups is a suitable enlargement of the class of FC-groups.

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1. Introduction

Let $\mathfrak{X}$ be a class of groups. An element $x$ of a group $G$ is said to be $\mathfrak{X}$C-central if $G/C_G(x^G)$ satisfies $\mathfrak{X}$, where the symbol $x^G$ represents the normal closure in $G$ of the subgroup $\langle x \rangle$. Sometimes the factor $G/C_G(x^G)$ is denoted by $\text{Aut}_G(x^G)$ to recall that $G/C_G(x^G)$ is a group of automorphisms of $x^G$ (see [16, Chapter 3]).

If $\mathfrak{X}$ has remarkable closure properties, then it is possible to introduce some series which generalize the upper central series. Haimo [6] and Nishigôri [13] started this kind of study thanks to FC-central series, successively [2] and [11] have extended such studies to wider classes of groups. [2] regards $\mathfrak{X}$C-hypercentral groups, where $\mathfrak{X}$ is the class of polycyclic-by-finite or Chernikov groups, while [11] treats $\mathfrak{X}$C-hypercentral groups, where $\mathfrak{X}$ is a suitable Schur class. Here we are interested to improve certain results of [2] and [11], considering $\mathfrak{X}$ as the class of (soluble minimax)-by-finite groups.

In a more explicit way a group $H$ is said to be (soluble minimax) – by – finite if it contains a normal subgroup $K$ such that $K$ has a finite characteristic series $1 = K_0 \lhd K_1 \lhd \ldots \lhd K_n = K$ whose factors are abelian minimax and $H/K$
is finite. Abelian minimax groups are described by [16, Lemma 10.31] and consequently soluble minimax groups are well known (see [16, Sections 10.3 and 10.4]).

If $G$ is a group and $x$ is an element of $G$, then $x$ is called $MC$-element of $G$ if $G/C_G(x^G)$ is (soluble minimax)-by-finite. If $x$ and $y$ are $MC$-elements of $G$, then both $G/C_G(x^G)$ and $G/C_G(y^G)$ are (soluble minimax)-by-finite, so $G/(C_G(x^G) \cap C_G(y^G))$ is also (soluble minimax)-by-finite. But the intersection of $C_G(x^G)$ with $C_G(y^G)$ lies in $C_G((xy^{-1})^G)$, so that $G/C_G((xy^{-1})^G)$ is (soluble minimax)-by-finite and $xy^{-1}$ is an $MC$-element of $G$. Hence the $MC$-elements of $G$ form a subgroup $M(G)$ and $M(G)$ is characteristic in $G$.

This simple remark allows us to define the series

$$1 = M_0 \triangleleft M_1 \triangleleft \ldots \triangleleft M_\alpha \triangleleft M_{\alpha+1} \triangleleft \ldots ,$$

where $M_1 = M(G)$, the factor $M_{\alpha+1}/M_\alpha$ is the subgroup of $G/M_\alpha$ generated by the $MC$-elements of $G/M_\alpha$ and

$$M_\lambda = \bigcup_{\alpha < \lambda} M_\alpha ,$$

with $\alpha$ ordinal and $\lambda$ limit ordinal. This series is a characteristic ascending series of $G$ and it is called upper $MC$-central series of $G$. The last term of the upper $MC$-central series of $G$ is called $MC$-hypercenter of $G$ and it is denoted by $\hat{M}(G)$. If $G = M_\beta$, for some ordinal $\beta$, we say that $G$ is an $MC$-hypercentral group of type at most $\beta$ and this is equivalent to say that $G = \hat{M}(G)$.

The $MC$-length of an $MC$-hypercentral group is defined to be the least ordinal $\beta$ such that $G = M_\beta$, in particular when $G = M_c$ for some positive integer $c$, we say that $G$ is $MC$-nilpotent of length $c$. A group $G$ is said to be an $MC$-group if all its elements are $MC$-elements, that is, if $G$ has $MC$-length at most 1.

In analogy with $FC$-groups, the first term $M(G)$ of the upper $MC$-central series of $G$ is said to be the $MC$-center of $G$ and the $\alpha$-th term of the upper $MC$-central series of $G$ is said to be the $MC$-center of length $\alpha$ of $G$. Roughly speaking, the upper $MC$-central series of $G$ measures the distance of $G$ to be an $MC$-group. By definitions, it happens that $Z(G) \leq F(G) \leq M(G)$, where $F(G)$ is the $FC$-center of $G$; i.e.: the subgroup of $G$ generated by the elements which have only a finite number of conjugates.

It is reasonable to expect that $MC$-hypercentral groups satisfy some classical McLain’s Theorems on generalized nilpotent series [16,Theorem 4.37, Theorem 4.38], with related finiteness conditions [2, Theorem A, Theorem B, Theorem C] or [16, Theorem 4.39, Theorem 4.39.1, Theorem 4.39.2]. On the other hand the conditions of hypercentrality, $FC$-hypercentrality and $MC$-
hypercentrality can be different in a same group and this fact will be shown by means of examples. In such direction of study we prove:

**Theorem 1.** Let $G$ be an MC-hypercentral group.

(i) If $G$ satisfies $\text{max} - n$, then $G$ is nilpotent-by-(soluble minimax)-by-finite and finitely generated;

(ii) If $G$ satisfies $\text{min} - n$, then $G$ is nilpotent-by-Chernikov.

**Theorem 2.** Let $G$ be a periodic group with finite abelian section rank. Then the following are equivalent:

(i) $G$ is CC-hypercentral;

(ii) $G$ is FC-hypercentral;

(iii) $G$ is PC-hypercentral;

(iv) $G$ is MC-hypercentral.

**Theorem 3.** Let $G$ be an MC-group. $G$ has finite abelian section rank if and only if $G$ is an extension of a soluble minimax group by a CL-group.

Most of our notation is standard and it is referred to [16]. In particular:

- the symbols $F(G)$, $P(G)$, $C(G)$ are used to denote respectively the FC-center, the PC-center, the CC-center of a group $G$. The definitions related to these subgroups can be found in [2,16];

- a group $G$ has *finite abelian section rank* if it has no infinite abelian sections of prime exponent;

- a group $G$ satisfies $\text{max} - sn$ (respectively $\text{min} - sn$) if it satisfies the maximal (respectively the minimal) condition on subnormal subgroups;

- a group $G$ satisfies $\text{max} - ab$ (respectively $\text{min} - ab$) if it satisfies the maximal (respectively the minimal) condition on abelian subgroups;

- a minimax group is a group which has a series of finite length each whose factors satisfies either $\text{max}$ or $\text{min}$. A soluble minimax group is a minimax group which is soluble;
- a group $G$ is said to be (soluble minimax)-by-finite if it contains a normal soluble minimax subgroup $N$ of finite index in $G$;

- if $m$ is a positive integer or infinity, the $m$-th layer of a group $G$ is the subgroup generated by the elements of $G$ of order $m$. A group all of whose layers are Chernikov is called $CL$-group. The structure of such groups is described by [16, Theorem 4.43].

2. Proof of Theorem 1

$PC$-hypercentral and $CC$-hypercentral groups are $MC$-hypercentral and it is clear that each periodic $MC$-hypercentral group is $CC$-hypercentral, so [2] and [11] solve partially Theorem 1, Theorem 2 and Theorem 3. For convenience we recall [2, Lemma 1] without proof.

**Lemma 2.1.** Let $G$ be a group which satisfies $\text{min} - n$ and let $1 = G_0 \triangleleft G_1 \triangleleft \ldots \triangleleft G_{\alpha} \triangleleft G_{\alpha+1} \ldots \triangleleft G_{\beta} = G$ be an ascending normal series of $G$. Then the subgroup $I = \bigcap_{\alpha<\beta} C_{G}(G_{\alpha+1}/G_{\alpha})$ is nilpotent.

**Proof of Theorem 1.** (i). Since $G$ is $MC$-hypercentral with $\text{max} - n$, its upper $MC$-central series is $1 = M_0 \triangleleft M_1 \triangleleft \ldots \triangleleft M_c = G$ for some positive integer $c$. $G$ satisfies $\text{max} - n$, hence $M_1 = x_1^G \ldots x_m^G$ for some finite set $\{x_1 \ldots x_m\}$ of elements of $M_1$. $\text{Aut}_G(M_1)$ is (soluble minimax)-by-finite and with a similar argument $\text{Aut}_G(M_{i+1}/M_i)$ too, for each $i \in \{1, \ldots c - 1\}$. Put $I = \bigcap_{i=1}^c C_{G}(M_{i+1}/M_i)$, $I$ is the stabilizer of the upper $MC$-central series of $G$ so $I$ is nilpotent, thus $G$ is nilpotent-by-(soluble minimax)-by-finite. Let $N$ be a nilpotent-by-(soluble minimax) normal subgroup of $G$ of finite index. By [16, Theorem 5.31], $N$ has $\text{max} - n$ and it is finitely generated by a classical Hall’s Theorem [16, Theorem 5.33], hence $G$ is finitely generated.

(ii). Let $G$ be an $MC$-hypercentral group with $\text{min} - n$ and let $x$ be a nontrivial element of $M(G)$. Then $G/C_G(x^G)$ is (soluble minimax)-by-finite with $\text{min} - n$, and so it is Chernikov. This implies that $G$ is a $CC$-hypercentral group with $\text{min} - n$, then, using Lemma 2.1, we may apply the argument in [2, Proof of Theorem B] to conclude that $G$ is nilpotent-by-Chernikov. For convenience of the reader we recall the argument of [2, Proof of Theorem B].

The upper $CC$-hypercentral series of $G$ has the form

$$1 = C_0 \triangleleft C_1 \triangleleft \ldots \triangleleft C_{\alpha} \triangleleft C_{\alpha+1} \ldots \triangleleft C_{\beta} = G,$$

and $I = \bigcap_{\alpha<\beta} C_{G}(C_{\alpha+1}/C_{\alpha})$ is nilpotent by Lemma 2.1. Since $G$ has $\text{min} - n$, there exists a positive integer $n$ such that $I = \bigcap_{j=0}^{n-1} C_{G}(C_{j+1}/C_{j})$. On the
other hand, for each $j \in \{0, \ldots, n - 1\}$, there exists a finite subset $X_j$ of $C_{j+1}$ such that $C_G(C_{j+1}/C_j) = \bigcap_{x \in X_j} C_G(x^G C_{j+1}/C_j)$, so $G/C_G(C_{j+1}/C_j)$ is a Chernikov group. Then $G/I$ is a Chernikov group. □

A natural generalization to $MC$-hypercentral groups of [2, Corollary 1] is the following.

**Corollary 2.2.** If $G$ is an $MC$-hypercentral group with $\min - n$, then $G$ is locally finite.

**Proof.** Obvious by Theorem 1. □

It is natural to ask what is the structure of an $MC$-hypercentral group which is neither $CC$-hypercentral, nor $PC$-hypercentral. The following example describes an $MC$-group, which is neither a $PC$-group, nor a $CC$-group. This group is neither hypercentral, nor an $FC$-group so it shows that there are difficulties to extend the classical McLain’s Theorems on generalized nilpotent series [16, Theorem 4.37, Theorem 4.38] to the $XC$-hypercentral series, when $X$ is an arbitrary class of groups.

**Example 2.3.** Let $p$ be a prime, $\mathbb{Q}$ be the additive group of the rational numbers and $U(\mathbb{Q})$ be the group of units of $\mathbb{Q}$. We may consider $G = \mathbb{Q} \times A$, where $A$ is a $p$-generating subgroup of $U(\mathbb{Q})$ and $A = \mathbb{Q}_p$ is the additive group of rational numbers whose denominators are $p$-numbers. Fixed a positive integer $n$, we have $Q = \langle x_0 \rangle \times \langle x_1 \rangle \times \ldots \times \langle x_n \rangle$, where $x_0 \in \{-1, 1\}$ and $x_1, \ldots, x_n$ generate the free abelian group $\langle x_1 \rangle \times \ldots \times \langle x_n \rangle$. If $-1 \notin Q$, then $Q = \langle x_1 \rangle \times \ldots \times \langle x_n \rangle$ and $n = r$ is the Prüfer rank of $Q$. $G$ is generated by $A$ together with elements $y_1, \ldots, y_r$, where $y_i$ is a preimage of $x_i$ under the epimorphism $G \to Q$, for all $i \in \{1, \ldots, r\}$. Here $y_i$ acts on $Q$ via multiplication by $x_i$. Also $[y_i, y_j] = c_{ij} \in A$, where $c_{ij}$ satisfy the system of linear equations over $\mathbb{Q}$:

$$\forall i, j, k \in \{1, \ldots, r\}, \quad c_{ij} = -c_{ji}, \quad c_{jk}(x_i - 1) + c_{ki}(x_j - 1) + c_{ij}(x_k - 1) = 0.$$  

The second equation is the famous Hall-Witt identity. This construction is due to Robinson and Wilson [15].

It is easy to verify that each element of $G$ is an $MC$-element, then $G = M(G)$ and $G$ is an $MC$-group. On the other hand $Z(G) = C(G) = P(G) = F(G) = 1$, hence $G$ is neither a $CC$-group nor a $PC$-group. Moreover $G$ is neither nilpotent nor an $FC$-group. The triviality of $Z(G)$, $C(G)$, $P(G)$ and $F(G)$ shows also that $G$ is $MC$-hypercentral, but neither hypercentral, nor $CC$-hypercentral, nor $PC$-hypercentral.

In order to obtain a group $G$ which has different $Z(G)$, $F(G)$, $P(G)$, $C(G)$, $M(G)$, it is possible to consider just-non-$\mathcal{X}$ groups, where $\mathcal{X}$ is a suitable group-theoretical property (see [10] for details).
Example 2.4. The group, which is described in [5, Proposition 2.2], is nilpotent of class 2 and it has $C(G) = F(G) = Z(G) \neq 1$ but $M(G) = P(G) = G$. This group is $CC$-nilpotent and $FC$-nilpotent of class 2 but it is $PC$-nilpotent and $MC$-nilpotent of class 1.

[10, Corollary 14.19] describes a just-non-Chernikov group $G$ with $F(G) = 1$. Such group has $1 = Z(G) = C(G)$, but the $PC$-length and the $MC$-length of $G$ depend by $G/Fit(G)$, where $Fit(G)$ is the Fitting subgroup of $G$.

3. Some special cases

In this Section we want to analyze some finitary conditions on $MC$-hypercentral groups, trying to adapt some classical results on $FC$-hypercentral chains (see [6,13,16]). According to [1], next lemma proves that $MC$-ascendant groups are generalized radical groups, that is, they have an ascending series whose factors are either locally nilpotent or finite.

**Lemma 3.1.** If $G$ is a group with nontrivial $MC$-center, then $G$ contains a nontrivial normal subgroup which is finite or abelian.

**Proof.** If $M(G) = P(G)$ or $M(G) = C(G)$, then we finish by [2, Lemma 2]. Given $x \in M(G)$, $x^G/Z(x^G)$ is (soluble minimax)-by-finite. If $Z(x^G) \neq 1$, $G$ has a nontrivial normal abelian subgroup; if $Z(x^G) = 1$, then $x^G$ is (soluble minimax)-by-finite and the result is proved. □

**Proposition 3.2.** If $G$ is an $MC$-ascendant group, the following are equivalent:

(i) $G$ satisfies $max - ab$ (respectively $min - ab$);

(ii) $G$ satisfies $max - sn$ (respectively $min - sn$);

(iii) $G$ satisfies $max$ (respectively $min$);

(iv) $G$ is polycyclic-by-finite (respectively Chernikov).

**Proof.** Obvious by Lemma 3.1 and [2, Proposition 1]. □

The examples in [2] show that $max - n$ (respectively $min - n$) can not be insert in Proposition 3.2.

**Corollary 3.3.** Let $G$ be a group such that one of the conditions of Proposition 3.2 holds. Then the following are equivalent:

(i) $G$ is $MC$-hypercentral;
Proof. This follows by [16, Theorem 4.39.2], [2, Proposition 2] and Proposition 3.2. □

4. Proofs of Theorem 2 and Theorem 3

Proof of Theorem 2. (i) implies (ii) by [11, Theorem 3.5]. By definitions (ii) implies (iii) and (iii) implies (iv). We must prove that (iv) implies (i). Since a periodic (soluble minimax)-by-finite group is Chernikov, the upper $CC$-central series and the upper $MC$-central series coincide and the result is proved. □

Example 2.3 shows that we can not hope in a wider version of Theorem 2 for non-periodic groups. However the torsion-free groups improve Theorem 2 in the following sense.

Corollary 4.1. Let $G$ be a torsion-free group. Then the following are equivalent:

(i) $G$ is hypercentral;

(ii) $G$ is $FC$-hypercentral;

(iii) $G$ is $CC$-hypercentral.

Proof. By definitions (i) implies (ii) and (ii) implies (iii). We must prove that (iii) implies (i). Let $x$ be a nontrivial element of $C(G)$. By [16, Theorem 4.36], $x^G$ is Chernikov-by-cyclic. Since $G$ is torsion-free, $x^G$ is cyclic, then $x^G = \langle x \rangle \subseteq [G, x] = \langle x \rangle$ and $[g, x] = 1$ for each element $g$ of $G$. Thus $x$ belongs to $Z(G)$ and $C(G) = Z(G)$. If $\alpha$ is a limit ordinal, then it is clear that it must be $G = C_\alpha = Z_\alpha$. Let $\alpha$ be not a limit ordinal, $C_\alpha$ be the general term of the upper $CC$-central series of $G$ and $Z_\alpha$ be the general term of the upper central series of $G$. Using the initial argument $C_2/C_1 = C(G/C_1) = Z(G/Z_1) = Z_2/Z_1$, so $C_2 = Z_2$ and by induction we conclude that $C_\alpha = Z_\alpha$ for each $\alpha$. □

Proof of Theorem 3. Let $G$ be an $MC$-group with finite abelian section rank, we want to show that there is a soluble minimax normal subgroup $H$
of $G$ such that $G/H$ is a $CL$-group. If $G$ is a $PC$-group or a $CC$-group, then we finish by [2, Proposition 4]. Let $G$ be neither a $PC$-group, nor a $CC$-group. Put Frat$G$ the Frattini’s subgroup of $G$, the section $G/$Frat$G$ has finite rank and [9, Corollary 10] implies that there exists a normal soluble minimax subgroup $H$ of $G$ such that $G/H$ is a $CC$-group. $G/H$ is an extension of an (abelian Chernikov)-by-polycyclic group by a $CL$-group, thanks to [2, Proposition 4]. Now $G$ is an extension of a soluble minimax group by an (abelian Chernikov)-by-polycyclic group by a $CL$-group. Since the class of soluble minimax groups is closed with respect to extensions, $G$ is an extension of a soluble minimax group by a $CL$-group.

Conversely if $G$ is an extension of a soluble minimax group $H$ by a $CL$-group $G/H$, both $H$ and $G/H$ have finite abelian section rank (see [16, Theorem 4.42] and [2, p.3052]). Since the property to have finite abelian section rank is additive on extensions (see [16]), we conclude that $G$ has again finite abelian section rank. □

References


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