On $MC$-Hypercentral Triply Factorized Groups

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Abstract

A group $G$ is called triply factorized in the product of two subgroups $A$, $B$ and a normal subgroup $K$ of $G$, if $G = AB = AK = BK$. This decomposition of $G$ has been studied by several authors, investigating on those properties which can be carried from $A$, $B$ and $K$ to $G$. It is known that if $A$, $B$ and $K$ are $FC$-groups and $K$ has restrictions on the rank, then $G$ is again an $FC$-group. The present paper extends this result to wider classes of $FC$-groups.

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1. Introduction

Let $\mathfrak{X}$ be a class of groups. An element $x$ of a group $G$ is said to be $\mathfrak{X}C$-central, or $\mathfrak{X}C$-element of $G$, if $G/C_G(x^G)$ satisfies $\mathfrak{X}$, where the symbol $x^G$ represents the normal closure in $G$ of the subgroup $<x>$. Sometimes the factor $G/C_G(x^G)$ is denoted by $Aut_G(x^G)$ to recall that $G/C_G(x^G)$ is a group of automorphisms of $x^G$ (see [10, Chapter 3]).

A minimax group is a group $G$ which has a series of finite length each of whose factors satisfies either the maximal condition or the minimal condition on subgroups. The maximal condition on subgroups is often denoted with $max$ and the minimal condition on subgroups is often denoted with $min$. Thus minimax is a finiteness property which generalizes both $max$ and $min$. It is easy to verify that the class of minimax groups is closed with respect to forming, subgroups, images and extensions of its members [10,vol.II, p.166]. A soluble minimax group is a minimax group which is soluble. The structure
of soluble minimax group is described by [10, Lemma 10.31] and [10, Sections 10.3, 10.4]. Finally a group $G$ is said to be (soluble minimax)-by-finite if it contains a normal soluble minimax subgroup $S$ of finite index in $G$.

The definition of $XC$-element of $G$ can be specialized for the class of (soluble minimax)-by-finite groups in the following way. If $G$ is a group and $x$ is an element of $G$, then $x$ is called $MC$-element of $G$ if $G/C_G(x^G)$ is (soluble minimax)-by-finite. If $x$ and $y$ are $MC$-elements of $G$, then both $G/C_G(x^G)$ and $G/C_G(y^G)$ are (soluble minimax)-by-finite, so $G/(C_G(x^G) \cap C_G(y^G))$ is also (soluble minimax)-by-finite. But the intersection of $C_G(x^G)$ with $C_G(y^G)$ lies in $C_G((xy)^{-1}G)$, so that $G/C_G((xy)^{-1}G)$ is (soluble minimax)-by-finite and $xy^{-1}$ is an $MC$-element of $G$. Hence the $MC$-elements of $G$ form a subgroup $M(G)$ and it is easy to check that $M(G)$ is characteristic in $G$.

This remark allows us to define the series

$$1 = M_0 \triangleleft M_1 \triangleleft \ldots \triangleleft M_\alpha \triangleleft M_{\alpha+1} \triangleleft \ldots ,$$

where $M_1 = M(G)$, the factor $M_{\alpha+1}/M_\alpha$ is the subgroup of $G/M_\alpha$ generated by the $MC$-elements of $G/M_\alpha$ and

$$M_\lambda = \bigcup_{\alpha<\lambda} M_\alpha,$$

with $\alpha$ ordinal and $\lambda$ limit ordinal. This series is a characteristic ascending series of $G$ and it is called upper $MC$-central series of $G$. The last term of the upper $MC$-central series of $G$ is called $MC$-hypercenter of $G$ and it is denoted by $\bar{M}(G)$. If $G = M_\beta$, for some ordinal $\beta$, we say that $G$ is an $MC$-hypercentral group of type at most $\beta$ and this is equivalent to say that $G = \bar{M}(G)$.

The $MC$-length of an $MC$-hypercentral group is defined to be the least ordinal $\beta$ such that $G = M_\beta$. When $G = M_c$ for some positive integer $c$, we say that $G$ is $MC$-nilpotent of length $c$. A group $G$ is said to be an $MC$-group if all its elements are $MC$-elements, that is, if $G$ has $MC$-length at most 1.

The method which has been used to introduce the $MC$-upper central series of a group follows the standard method which has been used by Baer and McLain (see [10, Chapter 4]) for $FC$-groups. In analogy with $FC$-groups, the term $M(G)$ of the upper $MC$-central series of $G$ is said to be the $MC$-center of $G$ and the $\alpha$-th term of the upper $MC$-central series of $G$ is said to be the $MC$-center of length $\alpha$ of $G$. Roughly speaking, the upper $MC$-central series of $G$ measures the distance of $G$ to be an $MC$-group. By definitions, it happens that $Z(G) \leq F(G) \leq M(G)$, where $F(G)$ is the $FC$-center of $G$, that is, $F(G)$ is the subgroup of $G$ generated by the elements which have only a finite number of conjugates. Further information on $MC$-hypercentral series of a group can be found also in [11].
The notion of $FC$-hypercentral series has been introduced by McLain (see for instance [10, Theorems 4.37, 4.38]) and restrictions on $FC$-hypercentral groups which satisfy $\max$ or $\min$ are obtained in [5] and [9]. These restrictions are seen to be valid also for weaker finiteness conditions than $\max$ and $\min$ (see for instance [10, Theorems 4.39, 4.39.2]). The results of [5] and [9] are considered classic in Theory of Generalized $FC$-groups and they have been extended by [2] and [8] to classes of groups wider than $FC$-hypercentral groups (see [2, Theorems A,B,C] and [8, Theorems 2.4,3.2.3.5]). [2] regards $XC$-hypercentral groups, where $X$ is either the class of polycyclic-by-finite groups or the class of Chernikov groups. Such groups are called respectively $PC$-hypercentral and $CC$-hypercentral groups. [8] treats $XC$-hypercentral groups, where $X$ is a Schur class. Recall that a class $X$ of groups is said to be a Schur class if for every group $G$ such that $G/Z(G)$ belongs to $X$ also the commutator subgroup $G'$ belongs to $X$.

We recall that a group $G$ is said to be triply factorized in the product of two subgroups $A$, $B$ and a normal subgroup $K$ of $G$, if $G = AB = AK = BK$. As testified in [1, Chapter 6], triply factorized groups are well-known and many properties are often carried from $A$, $B$ and $K$ to $G$ (see for instance [1,Theorems 6.3.4, 6.3.6, 6.3.7, 6.3.8, 6.5.1, 6.5.3, 6.5.4, 6.5.5, 6.5.11, 6.5.12, 6.5.13, 6.6.3, 6.6.6, 6.6.7,6.6.11]). In particular [8,Theorem 4.1] shows that if the group $G = AB = AK = BK$ is triply factorized in the product of two $CC$-hypercentral subgroups $A$ and $B$ and a soluble minimax normal subgroup $K$ of $G$, then $G$ is $CC$-hypercentral. The present paper improves [1, Theorems 6.5.1,6.5.3] and [8,Theorem 4.1], considering the class of $MC$-hypercentral groups. We prove

**Main Theorem.** Let the group $G = AB = AK = BK$ be the product of two subgroups $A$ and $B$ and a soluble minimax normal subgroup $K$ of $G$.

(i) If $A$, $B$ and $K$ are $MC$-nilpotent, then $G$ is $MC$-nilpotent.

(ii) If $A$, $B$ and $K$ are $MC$-hypercentral, then $G$ is $MC$-hypercentral.

Section 2 describes the general properties of $MC$-groups; Section 3 is devoted to the Proof of Main Theorem, ending with some examples to Main Theorem. Most of our notation is standard and can be found in [10]. For the literature on triply factorized groups, we refer to [1]. For the literature on $PC$-groups, $CC$-groups and $MC$-groups we refer to [10] and to [3], [4], [6], [7]. In particular:

- an element $x$ of a group $G$ is said to be a $CC$-element of $G$, if $G/C_G(x^G)$ is a Chernikov group. A group $G$ whose elements are all $CC$-elements is said to be a $CC$-group;
- an element \( x \) of a group \( G \) is said to be a PC-element of \( G \), if \( G/C_G(x^G) \) is a polycyclic-by-finite group. A group \( G \) whose elements are all PC-elements is said to be a PC-group;

- the symbols \( F(G) \), \( P(G) \), \( C(G) \), \( M(G) \) are used to denote respectively the FC-center, the PC-center, the CC-center, the MC-center of \( G \). Clearly the definition of \( P(G) \) and \( C(G) \) can be given, following the definition of \( M(G) \);

- a group \( G \) has finite torsion-free rank if it has a series of finite length whose factors are either periodic or infinite cyclic. The number of infinite cyclic factors in such a series is an invariant of \( G \) called its torsion-free rank;

- a group \( G \) has finite minimax rank if it has a series of finite length whose factors are either finite or infinite cyclic or quasicyclic of type \( p^\infty \), where \( p \) is any prime. The number of infinite factors in such series is an invariant, called minimax rank of \( G \).

- a group \( G \) has finite abelian section rank if it has no infinite abelian sections of prime exponent. It is easy to verify that this condition is equivalent to require that \( G \) has each abelian section of finite rank (see [10,vol.2,p.120]).

2. Preliminaries

For convenience of the reader this Section constitutes a survey on MC-groups.

**Lemma 2.1.** Let \( G \) be an MC-group, \( n \) be a positive integer and \( x_1, \ldots, x_n \) be elements of \( G \). If \( X = \langle x_1, \ldots, x_n \rangle \), then \( X^G \) is (soluble minimax)-by-finite.

**Proof.** This follows by [6, Theorem 2] (or in English [7, Corollary 2.1]). \( \square \)

Lemma 2.1 shows that an MC-group can be covered by normal (soluble minimax)-by-finite subgroups (see [6, p.161-162]). [3, Theorem 2.2] and [10, Theorem 4.36] give the corresponding condition for PC-groups and CC-groups. Another formulation of Lemma 2.1 is shown by the next lemma.

**Lemma 2.2.** If \( G \) is an MC-group, then it is locally-(normal and (soluble minimax)-by-finite). Moreover if \( G \) is an MC-group, then \( G' \) is locally-(normal and (soluble minimax)-by-finite).

**Proof.** This follows by [6, Theorem 2] (or in English [7, Corollary 2.1]). \( \square \)
More interesting is to note what happens for locally nilpotent and locally soluble MC-groups.

**Lemma 2.3.** If $G$ is a locally soluble MC-group, then it is $\omega$-hyperabelian. If $G$ is a locally nilpotent MC-group, then it is $2\omega$-hyperabelian.

**Proof.** This follows by [6, Theorem 3].

A corresponding result for locally nilpotent and locally soluble PC-groups is [3, Theorem 3.2]. A corresponding result for locally nilpotent and locally soluble CC-groups is [4, Theorem 2.2].

The following two lemmas are referred respectively to [8, Theorem 3.2] and [8, Corollary 3.3].

**Lemma 2.4.** If $G$ is an MC-hypercentral group with finite torsion-free rank and has no nontrivial subnormal (soluble minimax)-by-finite subgroups, then $G$ is nilpotent-by-(soluble minimax)-by-finite.

**Proof.** We note that the class of (soluble minimax)-by-finite groups is a Schur class which contains the class of finite groups and it is closed with respect to forming subgroups, images and extensions of its members. Then the result follows by [8, Theorem 3.2].

**Lemma 2.5.** Let $G$ be a CC-hypercentral groups with finite torsion-free rank. If $G$ has no nontrivial periodic normal subgroups, then $G$ is nilpotent-by-finite.

**Proof.** This follows by [8, Corollary 3.3].

We will use the next adaptaments of Lemma 2.5.

**Lemma 2.6.** Let $G$ be an MC-hypercentral group with finite abelian section rank. If $G$ has neither periodic nor (soluble minimax)-by-finite nontrivial normal subgroups, then $G$ is nilpotent-by-finite.

**Proof.** Since $G$ has finite abelian section rank, the subgroup generated by any system of (soluble minimax)-by-finite normal subgroups of $G$ is (soluble minimax)-by-finite, then $G$ has no nontrivial (soluble minimax)-by-finite subnormal subgroups by [10, Lemma 1.31]. This allows us to apply Lemma 2.4, so $G$ is nilpotent-by-(soluble minimax)-by-finite and there exists a normal nilpotent subgroup $N$ of $G$ such that $G/N$ is (soluble minimax)-by-finite. Clearly $N$ must be torsion-free. Let $R/N$ be the finite residual of $G/N$. The group $Z_{i+1}(N)/Z_i(N)$ is torsion-free abelian of finite rank for each non-negative integer $i$. But $G/C_N(Z_{i+1}(N)/Z_i(N))$ is a group of automorphisms of a torsion-free abelian group of finite rank. Since $G$ has finite abelian section rank, $G/C_N(Z_{i+1}(N)/Z_i(N))$ is torsion-free abelian of finite rank or finite. This means that $R$ acts trivially on $Z_{i+1}(N)/Z_i(N)$ for every $i \geq 0$. Put $c$ the nilpotency class of $N$, we have that $N = Z_c(N)$ is contained in $Z_c(R)$, so that
Lemma 2.7. If $G$ is a group with nontrivial $MC$-center, then $G$ contains a nontrivial normal subgroup which is either abelian or finite.

Proof. If $M(G) = P(G)$ or $M(G) = C(G)$, then the result follows by [2, Lemma 2]. Given $x \in M(G)$, $x^G/Z(x^G)$ is (soluble minimax)-by-finite. If $Z(x^G) \neq 1$, $G$ has a nontrivial normal abelian subgroup; if $Z(x^G) = 1$, then $x^G$ is (soluble minimax)-by-finite and the result is proved.

We end with a variation of the famous Hall’s Criterion for the nilpotence of a group (see [10, Theorem 2.27]).

Lemma 2.8. Let $\mathfrak{X}$ be a class of groups which is closed with respect to forming subgroups, images and extensions of its members. If $G$ is a group and $N$ is a nilpotent normal subgroup of $G$ such that $G/N'$ is $\mathfrak{X}C$-nilpotent (respectively $\mathfrak{X}C$-hypercentral), then $G$ is $\mathfrak{X}C$-nilpotent (respectively $\mathfrak{X}C$-hypercentral).

Proof. This follows by [8, Lemma 3.1].

3. Main Theorem

Proof of the Main Theorem. Let $G = AB = AK = BK$ be triply factorized in the product of two $MC$-nilpotent subgroups $A$, $B$ and a normal $MC$-nilpotent minimax soluble subgroup $K$ of $G$. Suppose that the statement (i) is false and $G = AB = AK = BK$ be a counterexample such that the minimax rank of $K$ is minimal.

If $T$ is the largest periodic normal subgroup of $K$, then $T$ is a Chernikov group and so it is contained in the second term of the upper $MC$-central series of $G$. Then we may suppose that $K$ has no nontrivial periodic normal subgroups. If $D$ is the largest (soluble minimax)-by-finite normal subgroup of $K$, again $D$ is contained in the second term of the upper $MC$-central series of $G$. Then we may suppose that $K$ has neither periodic nor (soluble minimax)-by-finite nontrivial normal subgroups. Then $K$ is nilpotent-by-finite by Lemma 2.6. This means that there is a normal nilpotent subgroup $K_0$ of $K$ of finite index $|K : K_0|$. The factorizer $X(K_0) = AK_0 \cap BK_0$ of $K_0$ in $G$ has finite index in $G$ and hence $X$ is not $MC$-nilpotent.

On the other hand $X$ has triple factorization

$$X = A^*B^* = A^*K_0 = B^*K_0,$$

where $A^* = A \cap BK_0$ and $B^* = B \cap AK_0$. From this we may suppose that $K$ is nilpotent and then torsion-free.

Let $K' \neq 1$. The group $K/K'$ has minimax rank less than $K$, so $G/K'$ is an $MC$-nilpotent group and hence, by Lemma 2.8, also $G$ is $MC$-nilpotent. This
contradiction forces $K$ to be abelian. Let $M(A)$ be the $MC$-center of $A$. Then the normal subgroup $M(A) \cap K$ of $G$ lies in $M(G)$. Thus the factor group $G/(M(A) \cap K)$ is not $MC$-nilpotent, and $M(A) \cap K = 1$. Since $A$ is $MC$-nilpotent, we have also $A \cap K = 1$ (see [10, Lemma 2.16]). The subgroup $C_A(K)$ is normal in $G$ and $G/C_A(K)$ is not $MC$-nilpotent, being $C_A(K) \cap K = 1$. But

$$C_{A/C_A(K)}((KC_A(K))/C_A(K)) = 1,$$

then we may suppose that $C_A(K) = 1$, and therefore $A$ is isomorphic with a group of automorphisms of $K$. But $K$ is torsion-free abelian of finite torsion-free rank and $A$ has an ascending normal series whose factors are either abelian or finite by Lemma 2.7. We conclude that $A$ is (soluble minimax)-by-finite. A similar argument can be applied to show that $B$ is a (soluble minimax)-by-finite group. Then $G$ is a (soluble minimax)-by-finite group. This contradiction shows that the statement (i) holds.

The proof of the statement (ii) is similar. □

**Examples and Counterexamples.**

1. The infinite dihedral group. Let $G$ be the group which is presented by

$$< a, x : a^x = a^{-1}, x^2 = 1 >.$$

$G$ has $F(G) = C(G) = Z(G) = 1$ and $M(G) = P(G) = G$. Obviously $G$ is triply factorized by $G = < a, x > < x > = < a, x > < a > = < x > < a >$ and it respects Main Theorem.

2. The group of P.Hall, described in [10, Theorem 5.36], is a 2-generated group such that $Z(G)$ is a quasicyclic $p$-group, where $p$ is any prime, $G''$ is central in $G$ and $G/Z(G)$ is isomorphic to the wreath product of two infinite cyclic groups. Such group is a finitely generated $PC$-nilpotent and $MC$-nilpotent group of length 3. $G$ is triply factorized by $G = AB = AK = BK$, where $K = Z(G)$ is a normal group with $min$ and $A = B = < x > wr < y >$ with $< x >$ and $< y >$ infinite cyclic. Furthermore $A$, $B$ and $K$ are $MC$-nilpotent. This shows the validity of Main Theorem.

3. The group $G = < x > wr < y >$, where $< x >$ and $< y >$ are infinite cyclic, is an example of $PC$-nilpotent and $MC$-nilpotent group of length 2 which is neither an $FC$-hypercentral group, nor a $CC$-hypercentral group, nor a hypercentral group. In fact $F(G) = C(G) = Z(G) = 1$, $P(G) = M(G) = B$ and $P_2(G) = M_2(G) = G$, where $B$ is an abelian free group of countable rank and it denotes the base of $G$. Here we notice that $G$ cannot be written as triply factorized product, since it has no nontrivial soluble minimax normal subgroups. This shows that the converse of Main Theorem cannot hold.
References


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