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On MC-Hypercentral Triply Factorized Groups

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Abstract

A group G is called triply factorized in the product of two subgroups A, B and a normal subgroup K of G, if $G = AB = AK = BK$. This decomposition of G has been studied by several authors, investigating on those properties which can be carried from A, B and K to G. It is known that if A, B and K are FC-groups and K has restrictions on the rank, then G is again an FC-group. The present paper extends this result to wider classes of FC-groups.

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1. Introduction

Let $\mathfrak{X}$ be a class of groups. An element x of a group G is said to be $\mathfrak{X}C$-*central*, or XC-*element of* G, if $G/C_G(x^G)$ satisfies $\mathfrak{X}$, where the symbol x^G represents the normal closure in G of the subgroup $< x >$. Sometimes the factor $G/C_G(x^G)$ is denoted by $Aut_G(x^G)$ to recall that $G/C_G(x^G)$ is a group of automorphisms of x^G (see [10, Chapter 3]).

A *minimax group* is a group G which has a series of finite length each of whose factors satifies either the maximal condition or the minimal condition on subgroups. The maximal condition on subgroups is often denoted with *max* and the minimal condition on subgroups is often denoted with *min*. Thus *minimax* is a finiteness property which generalizes both *max* and *min*. It is easy to verify that the class of minimax groups is closed with respect to forming, subgroups, images and extensions of its members [10,vol.II, p.166]. A *soluble minimax group* is a minimax group which is soluble. The structure of soluble minimax group is described by [10, Lemma 10.31] and [10, Sections 10.3, 10.4]. Finally a group G is said to be (*soluble minimax*)-*by-finite* if it contains a normal soluble minimax subgroup S of finite index in G.

The definition of XC-element of G can be specialized for the class of (soluble minimax)-by-finite groups in the following way. If G is a group and x is an element of G, then x is called *MC-element of G* if $G/C_G(x^G)$ is (soluble minimax)-by-finite. If x and y are MC-elements of G, then both $G/C_G(x^G)$ and $G/C_G(y^G)$ are (soluble minimax)-by-finite, so $G/(C_G(x^G) \cap C_G(y^G))$ is also (soluble minimax)-by-finite. But the intersection of $C_G(x^G)$ with $C_G(y^G)$ lies in $C_G((xy^{-1})^G)$, so that $G/C_G((xy^{-1})^G)$ is (soluble minimax)-by-finite and xy^{-1} is an MC-element of G. Hence the MC-elements of G form a subgroup $M(G)$ and it is easy to check that $M(G)$ is characteristic in G.

This remark allows us to define the series

$$1 = M_0 \lhd M_1 \lhd \ldots \lhd M_\alpha \lhd M_{\alpha+1} \lhd \ldots ,$$

where $M_1 = M(G)$, the factor $M_{\alpha+1}/M_\alpha$ is the subgroup of G/M_α generated by the MC-elements of G/M_α and

$$M_\lambda = \bigcup_{\alpha<\lambda} M_\alpha,$$

with α ordinal and λ limit ordinal. This series is a characteristic ascending series of G and it is called *upper MC-central series* of G. The last term of the upper MC-central series of G is called *MC-hypercenter* of G and it is denoted by $\bar{M}(G)$. If $G = M_\beta$, for some ordinal β, we say that G is an *MC-hypercentral* group of type at most β and this is equivalent to say that $G = \bar{M}(G)$.

The *MC-length* of an MC-hypercentral group is defined to be the least ordinal β such that $G = M_\beta$. When $G = M_c$ for some positive integer c, we say that G is *MC-nilpotent* of length c. A group G is said to be an MC-group if all its elements are MC-elements, that is, if G has MC-length at most 1.

The method which has been used to introduce the MC-upper central series of a group follows the standard method which has been used by Baer and McLain (see [10, Chapter 4]) for FC-groups. In analogy with FC-groups, the term $M(G)$ of the upper MC-central series of G is said to be the *MC-center* of G and the α-th term of the upper MC-central series of G is said to be the *MC-center of length α* of G. Roughly speaking, the upper MC-central series of G measures the distance of G to be an MC-group. By definitions, it happens that $Z(G) \leq F(G) \leq M(G)$, where $F(G)$ is the FC-center of G, that is, $F(G)$ is the subgroup of G generated by the elements which have only a finite number of conjugates. Further information on MC-hypercentral series of a group can be found also in [11].

The notion of FC-hypercentral series has been introduced by McLain (see for instance [10, Theorems 4.37, 4.38]) and restrictions on FC-hypercentral groups which satisfy *max* or *min* are obtained in [5] and [9]. These restrictions are seen to be valid also for weaker finiteness conditions than *max* and *min* (see for instance [10, Theorems 4.39, 4.39.2]). The results of [5] and [9] are considered classic in Theory of Generalized FC-groups and they have been extended by [2] and [8] to classes of groups wider than FC-hypercentral groups (see [2, Theorems A,B,C] and [8, Theorems 2.4,3.2,3.5]). [2] regards $\mathfrak{X}C$-hypercentral groups, where $\mathfrak{X}$ is either the class of polycyclic-by-finite groups or the class of Chernikov groups. Such groups are called respectively *PC-hypercentral* groups and *CC-hypercentral* groups. [8] treats $\mathfrak{X}C$-hypercentral groups, where $\mathfrak{X}$ is a Schur class. Recall that a class $\mathfrak{X}$ of groups is said to be a *Schur class* if for every group G such that $G/Z(G)$ belongs to $\mathfrak{X}$ also the commutator subgroup G' belongs to $\mathfrak{X}$.

We recall that a group G is said to be *triply factorized* in the product of two subgroups A, B and a normal subgroup K of G, if $G = AB = AK = BK$. As testified in [1, Chapter 6], triply factorized groups are well-known and many properties are often carried from A, B and K to G (see for instance [1,Theorems 6.3.4, 6.3.6, 6.3.7, 6.3.8, 6.5.1, 6.5.3, 6.5.4, 6.5.5, 6.5.11, 6.5.12, 6.5.13, 6.6.3, 6.6.6, 6.6.7,6.6.11]). In particular [8,Theorem 4.1] shows that if the group $G = AB = AK = BK$ is triply factorized in the product of two CC-hypercentral subgroups A and B and a soluble minimax normal subgroup K of G, then G is CC-hypercentral. The present paper improves [1, Theorems 6.5.1,6.5.3] and [8,Theorem 4.1], considering the class of MC-hypercentral groups. We prove

Main Theorem. *Let the group $G = AB = AK = BK$ be the product of two subgroups A and B and a soluble minimax normal subgroup K of G.*

(i) *If A, B and K are MC-nilpotent, then G is MC-nilpotent.*

(ii) *If A, B and K are MC-hypercentral, then G is MC-hypercentral.*

Section 2 describes the general properties of MC-groups; Section 3 is devoted to the Proof of Main Theorem, ending with some examples to Main Theorem. Most of our notation is standard and can be found in [10]. For the literature on triply factorized groups, we refer to [1]. For the literature on PC-groups, CC-groups and MC-groups we refer to [10] and to [3], [4], [6], [7].

In particular:

- an element x of a group G is said to be a *CC-element of G*, if $G/C_G(x^G)$ is a Chernikov group. A group G whose elements are all CC-elements is said to be a *CC-group*;

- an element x of a group G is said to be a *PC-element of G*, if $G/C_G(x^G)$ is a polycyclic-by-finite group. A group G whose elements are all PC-elements is said to be a *PC-group*;

- the symbols $F(G)$, $P(G)$, $C(G)$, $M(G)$ are used to denote respectively the *FC-center*, the *PC-center*, the *CC-center*, the *MC-center* of G. Clearly the definition of $P(G)$ and $C(G)$ can be given, following the definition of $M(G)$;

- a group G has *finite torsion-free rank* if it has a series of finite length whose factors are either periodic or infinite cyclic. The number of infinite cyclic factors in such a series is an invariant of G called its *torsion-free rank*;

- a group G has a *finite minimax rank* if it has a series of finite length whose factors are either finite or infinite cyclic or quasicyclic of type p^∞, where p is any prime. The number of infinite factors in such series is an invariant, called *minimax rank* of G.

- a group G has *finite abelian section rank* if it has no infinite abelian sections of prime exponent. It is easy to verify that this condition is equivalent to require that G has each abelian section of finite rank (see [10,vol.2,p.120]).

2. Preliminaries

For convenience of the reader this Section constitutes a survey on MC-groups.

Lemma 2.1. *Let G be an MC-group, n be a positive integer and $x_1, \ldots, x_n$ be elements of G. If $X =< x_1, \ldots, x_n >$, then X^G is (soluble minimax)-by-finite.*

Proof. This follows by [6, Theorem 2] (or in English [7, Corollary 2.1]).□

Lemma 2.1 shows that an MC-group can be covered by normal (soluble minimax)-by-finite subgroups (see [6, p.161-162]). [3, Theorem 2.2] and [10, Theorem 4.36] give the corresponding condition for PC-groups and CC-groups. Another formulation of Lemma 2.1 is shown by the next lemma.

Lemma 2.2. *If G is an MC-group, then it is locally-(normal and (soluble minimax)-by-finite). Moreover if G is an MC-group, then G' is locally-(normal and (soluble minimax)-by-finite).*

Proof. This follows by [6, Theorem 2] (or in English [7, Corollary 2.1]).□

More interesting is to note what happens for locally nilpotent and locally soluble MC-groups.

Lemma 2.3. *If G is a locally soluble MC-group, then it is ω-hyperabelian. If G is a locally nilpotent MC-group, then it is 2ω-hyperabelian.*

Proof. This follows by [6, Theorem 3].□

A corresponding result for locally nilpotent and locally soluble PC-groups is [3, Theorem 3.2]. A corresponding result for locally nilpotent and locally soluble CC-groups is [4, Theorem 2.2].

The following two lemmas are referred respectively to [8,Theorem 3.2] and [8,Corollary 3.3].

Lemma 2.4. *If G is an MC-hypercentral group with finite torsion-free rank and has no nontrivial subnormal (soluble minimax)-by-finite subgroups, then G is nilpotent-by-(soluble minimax)-by-finite .*

Proof. We note that the class of (soluble minimax)-by-finite groups is a Schur class which contains the class of finite groups and it is closed with respect to forming subgroups, images and extensions of its members. Then the result follows by [8, Theorem 3.2].□

Lemma 2.5. *Let G be a CC-hypercentral groups with finite torsion-free rank. If G has no nontrivial periodic normal subgroups, then G is nilpotent-by-finite.*

Proof. This follows by [8, Corollary 3.3].□

We will use the next adaptaments of Lemma 2.5.

Lemma 2.6. *Let G be an MC-hypercentral group with finite abelian section rank. If G has neither periodic nor (soluble minimax)-by-finite nontrivial normal subgroups, then G is nilpotent-by-finite.*

Proof. Since G has finite abelian section rank, the subgroup generated by any system of (soluble minimax)-by-finite normal subgroups of G is (soluble minimax)-by-finite, then G has no nontrivial (soluble minimax)-by-finite subnormal subgroups by [10, Lemma 1.31]. This allows us to apply Lemma 2.4, so G is nilpotent-by-(soluble minimax)-by-finite and there exists a normal nilpotent subgroup N of G such that G/N is (soluble minimax)-by-finite. Clearly N must be torsion-free. Let R/N be the finite residual of G/N. The group $Z_{i+1}(N)/Z_i(N)$ is torsion-free abelian of finite rank for each non negative integer i. But $G/C_G(Z_{i+1}(N)/Z_i(N))$ is a group of automorphisms of a torsion-free abelian group of finite rank. Since G has finite abelian section rank, $G/C_G(Z_{i+1}(N)/Z_i(N))$ is torsion-free abelian of finite rank or finite. This means that R acts trivially on $Z_{i+1}(N)/Z_i(N)$ for every $i \geq 0$. Put c the nilpotency class of N, we have that $N = Z_c(N)$ is contained in $Z_c(R)$, so that

R is nilpotent, and G is nilpotent-by-finite.□

Lemma 2.7. *If G is a group with nontrivial MC-center, then G contains a nontrivial normal subgroup which is either abelian or finite.*

Proof. If $M(G) = P(G)$ or $M(G) = C(G)$, then the result follows by [2, Lemma 2]. Given $x \in M(G)$, $x^G/Z(x^G)$ is (soluble minimax)-by-finite. If $Z(x^G) \neq 1$, G has a nontrivial normal abelian subgroup; if $Z(x^G) = 1$, then x^G is (soluble minimax)-by-finite and the result is proved.□

We end with a variation of the famous Hall's Criterion for the nilpotence of a group (see [10, Theorem 2.27]).

Lemma 2.8. *Let $\mathfrak{X}$ be a class of groups which is closed with respect to forming subgroups, images and extensions of its members. If G is a group and N is a nilpotent normal subgroup of G such that G/N' is $\mathfrak{X}C$-nilpotent (respectively $\mathfrak{X}C$-hypercentral), then G is $\mathfrak{X}C$-nilpotent (respectively $\mathfrak{X}C$-hypercentral).*

Proof. This follows by [8,Lemma 3.1].□

3. Main Theorem

Proof of the Main Theorem. Let $G = AB = AK = BK$ be triply factorized in the product of two MC-nilpotent subgroups A, B and a normal MC-nilpotent minimax soluble subgroup K of G. Suppose that the statement (i) is false and $G = AB = AK = BK$ be a counterexample such that the minimax rank of K is minimal.

If T is the largest periodic normal subgroup of K, then T is a Chernikov group and so it is contained in the second term of the upper MC-central series of G. Then we may suppose that K has no nontrivial periodic normal subgroups. If D is the largest (soluble minimax)-by-finite normal subgroup of K, again D is contained in the second term of the upper MC-central series of G. Then we may suppose that K has neither periodic nor (soluble minimax)-by-finite nontrivial normal subgroups. Then K is nilpotent-by-finite by Lemma 2.6. This means that there is a normal nilpotent subgroup K_0 of K of finite index $|K : K_0|$. The factorizer $X(K_0) = AK_0 \cap BK_0$ of K_0 in G has finite index in G and hence X is not MC-nilpotent.

On the other hand X has triple factorization

$$X = A^*B^* = A^*K_0 = B^*K_0,$$

where $A^* = A \cap BK_0$ and $B^* = B \cap AK_0$. From this we may suppose that K is nilpotent and then torsion-free.

Let $K' \neq 1$. The group K/K' has minimax rank less than K, so G/K' is an MC-nilpotent group and hence, by Lemma 2.8, also G is MC-nilpotent. This contradiction forces K to be abelian. Let $M(A)$ be the MC-center of A. Then the normal subgroup $M(A) \cap K$ of G lies in $M(G)$. Thus the factor group $G/(M(A) \cap K)$ is not MC-nilpotent, and $M(A) \cap K = 1$. Since A is MC-nilpotent, we have also $A \cap K = 1$ (see [10,Lemma 2.16]). The subgroup $C_A(K)$ is normal in G and $G/C_A(K)$ is not MC-nilpotent, being $C_A(K) \cap K = 1$. But

$$C_{A/C_A(K)}((KC_A(K))/C_A(K)) = 1,$$

then we may suppose that $C_A(K) = 1$, and therefore A is isomorphic with a group of automorphisms of K. But K is torsion-free abelian of finite torsion-free rank and A has an ascending normal series whose factors are either abelian or finite by Lemma 2.7. We conclude that A is (soluble minimax)-by-finite. A similar argument can be applied to show that B is a (soluble minimax)-by-finite group. Then G is a (soluble minimax)-by-finite group. This contradiction shows that the statement (i) holds.

The proof of the statement (ii)is similar.□

Examples and Counterexamples.

1. The infinite dihedral group. Let G be the group which is presented by

$$< a, x : a^x = a^{-1}, x^2 = 1 > .$$

G has $F(G) = C(G) = Z(G) = 1$ and $M(G) = P(G) = G$. Obviously G is triply factorized by $G =< a, x >< x >=< a, x >< a >=< x >< a >$ and it respects Main Theorem.

2. The group of P.Hall, described in [10,Theorem 5.36], is a 2-generated group such that $Z(G)$ is a quasicyclic p-group, where p is any prime, G'' is central in G and $G/Z(G)$ is isomorphic to the wreath product of two infinite cyclic groups. Such group is a finitely generated PC-nilpotent and MC-nilpotent group of length 3. G is triply factorized by $G = AB = AK = BK$, where $K = Z(G)$ is a normal group with min and $A = B =< x > wr < y >$ with $< x >$ and $< y >$ infinite cyclic. Furthermore A, B and K are MC-nilpotent. This shows the validity of Main Theorem.

3. The group $G =< x > wr < y >$, where $< x >$ and $< y >$ are infinite cyclic, is an example of PC-nilpotent and MC-nilpotent group of length 2 which is neither an FC-hypercentral group, nor a CC-hypercentral group, nor a hypercentral group. In fact $F(G) = C(G) = Z(G) = 1$, $P(G) = M(G) = B$ and $P_2(G) = M_2(G) = G$, where B is an abelian free group of countable rank and it denotes the base of G. Here we notice that G can not be written as triply factorized product, since it has no nontrivial soluble minimax normal subgroups. This shows that the converse of Main Theorem can not hold.

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