Some Cubic Blaschke Products and Quadratic Rational Functions with Siegel Disks

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Abstract

We show that for any given complex number $\mu$ with modulus at most one and any given real number $\alpha$, there exists a cubic Blaschke product such that the point at infinity is its fixed point with multiplier $\mu$ and its restriction on the unit circle is a critical circle map with rotation number $\alpha$. Moreover if the given real number $\alpha$ is irrational of bounded type, then a modified Blaschke product is quasiconformally conjugate to some quadratic rational function with a Siegel disk whose boundary is a quasicircle containing its critical point and the point at infinity is its fixed point with multiplier $\mu$.

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1 Introduction

Let $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ be a rational function of degree $d \geq 2$ with fixed point of multiplier $e^{2\pi i \alpha}$ at the origin, where $\alpha \in [0, 1]$ is irrational. If $f$ is linearizable at the origin, then there exists a local holomorphic change of coordinate $\Phi : \mathbb{D} \rightarrow \mathbb{C}$ with $0 = \Phi(0)$ such that $\Phi^{-1} \circ f \circ \Phi(z) = e^{2\pi i \alpha}z$, where $\mathbb{D}$ is the unit disk. The Fatou component $\Delta$ of $f$ containing $\Phi(\mathbb{D})$ is called the Siegel disk centered at the origin.

For the irrational number $\alpha$, we consider the continued fraction expansion

$$\alpha = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots}}}$$
of $\alpha$ and then a sequence of rational numbers

$$\frac{p_n}{q_n} = \frac{1}{a_1 + \frac{1}{a_2 + \cdots + \frac{1}{a_n}}}$$

converges to $\alpha$, where $a_n$ is a positive integer uniquely determined by $\alpha$ for all $n \in \mathbb{N}$. The irrational number $\alpha$ is a Diophantine number of order $\kappa \geq 2$ if there exists $\varepsilon > 0$ such that

$$\left| \alpha - \frac{p}{q} \right| > \frac{\varepsilon}{q^\kappa}$$

for all rational numbers $p/q$. The class of Diophantine numbers of order $\kappa$ is denoted by $D_\kappa$. Diophantine numbers of order 2 are said to be of bounded type. The irrational number $\alpha$ is a Bryuno number if the sum

$$\sum_{n=1}^{\infty} \frac{\log q_{n+1}}{q_n}$$

converges. The class of Bryuno numbers is denoted by $B$. Note that for $\kappa > 2$, $D_2 \subset D_\kappa \subset B$ and $D_\kappa$ has full measure on $\mathbb{R}/\mathbb{Z}$ (see [6] or [10]). Bryuno showed that if $\alpha$ is a Bryuno number, then $f$ is linearizable at the origin. Yoccoz showed that for $\lambda = e^{2\pi i \alpha}$ if $\alpha$ is not a Bryuno number, then $P_\lambda(z) = z^2 + \lambda z$ is not linearizable at the origin, that is, $P_\lambda$ is linearizable at the origin if and only if $\alpha$ is a Bryuno number. Moreover the following theorem holds if $\alpha$ is of bounded type. Refer to [9] or [10].

**Theorem 1.1 (Ghys-Douady-Herman-Shishikura-Świątek).** If an irrational number $\alpha \in [0, 1]$ is of bounded type and $\lambda = e^{2\pi i \alpha}$, then the boundary of the Siegel disk $\Delta$ of $P_\lambda$ centered at the origin is a quasicircle containing its critical point $-\lambda/2$.

Moreover if the irrational number $\alpha$ is of bounded type and $\lambda = e^{2\pi i \alpha}$, then the following holds:

(a) (Petersen). The Julia set $J(P_\lambda)$ of $P_\lambda$ is locally connected and has measure zero.

(b) (McMullen). The Hausdorff dimension of $J(P_\lambda)$ is less than 2.

(c) (Graczyk-Jones). The Hausdorff dimension of $\partial \Delta$ is greater than 1.

Conversely, Petersen showed that if $\partial \Delta$ is a quasicircle containing the finite critical point $-\lambda/2$ of $P_\lambda$, then $\alpha \in [0, 1]$ is of bounded type. Zakeri extended Theorem 1.1 to the case of cubic polynomials.
Theorem 1.2 (Zakeri, [11]). Let $P$ be a cubic polynomial with fixed point of multiplier $e^{2\pi i \alpha}$ at the origin. If an irrational number $\alpha \in [0, 1]$ is of bounded type, then the boundary of the Siegel disk $\Delta$ of $P$ centered at the origin is a quasicircle containing one or both critical points.

Geyer showed the following theorem which is extended to some polynomials.

Theorem 1.3 (Geyer, [4]). Let $d \geq 1$ and let $P(z) = e^{2\pi i \alpha} z (1 + z/d)^d$. If an irrational number $\alpha \in [0, 1]$ is of bounded type, then the boundary of the Siegel disk $\Delta$ of $P$ centered at the origin is a quasicircle containing its critical point $-d/(d+1)$.

Let $F_{\lambda, \mu}(z) = z(z + \lambda)/(\mu z + 1)$ with $\lambda \mu \neq 1$. The origin and the point at infinity are fixed points of $F_{\lambda, \mu}$ of multiplier $\lambda$ and $\mu$ respectively. In the case that $\mu = 0$, $F_{\lambda, 0}(z) = P_\lambda(z)$. Therefore the quadratic rational function $F_{\lambda, \mu}$ is considered as a perturbation of the quadratic polynomial $P_\lambda$. In the case that $\lambda = e^{2\pi i \alpha}$ and $\alpha$ is irrational of bounded type, we show the following theorem which is a generalization of Theorem 1.1.

Theorem 1.4. If an irrational number $\alpha \in [0, 1]$ is of bounded type, $\lambda = e^{2\pi i \alpha}$ and $\mu \in \mathbb{D}$ with $\lambda \mu \neq 1$, then the boundary of the Siegel disk $\Delta$ of $F_{\lambda, \mu}$ centered at the origin is a quasicircle containing its critical point.

2 Cubic Blaschke products

2.1 Existence of cubic Blaschke products

We consider a cubic Blaschke product

$$B(z) = e^{2\pi i \theta} z \left( \frac{z - a}{1 - \overline{a} z} \right) \left( \frac{z - b}{1 - \overline{b} z} \right)$$

with $\overline{a} \neq 1$ and $0 < |a| \leq |b| < \infty$. The derivative $B'$ of $B$ is

$$B'(z) = \frac{e^{2\pi i \theta}}{(1 - \overline{a} z)^2 (1 - \overline{b} z)^2} \cdot g(z),$$

where

$$g(z) = \overline{a} \overline{b} z^4 - 2(\overline{a} + \overline{b}) z^3 + \left\{ 3 - |ab|^2 + |a + b|^2 \right\} z^2 - 2(a + b) z + ab.$$ 

So multipliers of fixed points $z = 0$ and $z = \infty$ are $\lambda = ab e^{2\pi i \theta}$ and $\mu = \overline{a} \overline{b} e^{-2\pi i \theta}$ respectively. Let $c_1$, $c_2$, $c_3 = 1/\overline{c}_2$ and $c_4 = 1/\overline{c}_1$ be critical points of $B$. Since
they are solutions of \( g(z) = 0 \), we obtain that

\[
g(z) = \bar{a} b (z - c_1)(z - c_2)(z - c_3)(z - c_4)
= \bar{a} b \left\{ z^4 - C_3 z^3 + C_2 z^2 - C_1 z + C_0 \right\},
\]

where

\[
C_3 = c_1 + \frac{1}{c_1} + c_2 + \frac{1}{c_2},
\]

\[
C_2 = \frac{c_1}{c_1} + \frac{c_2}{c_2} + \left( c_1 + \frac{1}{c_1} \right) \left( c_2 + \frac{1}{c_2} \right),
\]

\[
C_1 = \frac{c_1}{c_1} \left( c_2 + \frac{1}{c_2} \right) + \frac{c_2}{c_2} \left( c_1 + \frac{1}{c_1} \right),
\]

\[
C_0 = \frac{c_1 c_2}{c_1 c_2}.
\]

Comparing coefficients of two representations of \( g(z) \) implies that

\[
c_1 + \frac{1}{c_1} + c_2 + \frac{1}{c_2} = \frac{2(\bar{a} + \bar{b})}{\bar{a} \bar{b}}, \tag{1}
\]

\[
\frac{c_1}{c_1} + \frac{c_2}{c_2} + \left( c_1 + \frac{1}{c_1} \right) \left( c_2 + \frac{1}{c_2} \right) = \frac{3 - |ab|^2 + |a + b|^2}{\bar{a} \bar{b}}, \tag{2}
\]

\[
\frac{c_1}{c_1} \left( c_2 + \frac{1}{c_2} \right) + \frac{c_2}{c_2} \left( c_1 + \frac{1}{c_1} \right) = \frac{2(a + b)}{\bar{a} \bar{b}}, \tag{3}
\]

\[
\frac{c_1 c_2}{c_1 c_2} = \frac{ab}{\bar{a} \bar{b}}. \tag{4}
\]

Eliminating \( c_1 \) and \( \bar{c}_1 \) form equations (1), (2) and (4) gives that

\[
|a + b|^2 - 2 \left( c_2 + \frac{1}{c_2} \right) (\bar{a} + \bar{b})
- \left( \frac{\bar{c}_2}{c_2} \right) ab + \left\{ \left( c_2 + \frac{1}{c_2} \right)^2 - \frac{c_2}{\bar{c}_2} \right\} \bar{a} \bar{b} + 3 - |ab|^2 = 0 \tag{5}
\]
and eliminating \( c_1 \) and \( \bar{c}_1 \) form equations (1), (3) and (4) gives that
\[
\frac{c_2}{c_2} \left( c_2 + \frac{1}{\bar{c}_2} \right) ab + 2 \left( \frac{c_2}{\bar{c}_2} \right) (\bar{a} + \bar{b}) = \frac{c_2}{\bar{c}_2} \left( c_2 + \frac{1}{\bar{c}_2} \right) \bar{a} \bar{b} + 2(a + b). \tag{6}
\]
We obtain that
\[
|a + b|^2 - 4e^{2\pi i \varphi} (\bar{a} + \bar{b}) - e^{2\pi i (-2\varphi)} ab + 3e^{2\pi i 2\varphi} \bar{a} \bar{b} + 3 - |ab|^2 = 0 \tag{7}
\]
and
\[
e^{2\pi i (-2\varphi)} ab + e^{2\pi i \varphi} (\bar{a} + \bar{b}) = e^{2\pi i 2\varphi} \bar{a} \bar{b} + e^{2\pi i (-\varphi)} (a + b) \tag{8}
\]
by substituting \( c_2 = e^{2\pi i \varphi} \) into equations (5) and (6). Eliminating \( ab \) form equations (7) and (8) gives that
\[
|a + b|^2 - 3e^{2\pi i \varphi} (\bar{a} + \bar{b}) - e^{2\pi i (-\varphi)} (a + b) + 2e^{2\pi i 2\varphi} \bar{a} \bar{b} + 3 - |ab|^2 = 0. \tag{9}
\]
Let \( \zeta = a + b \). Then
\[
|\zeta|^2 - 3e^{2\pi i \varphi} \bar{\zeta} - e^{2\pi i (-\varphi)} \zeta + 2e^{2\pi i 2\varphi} \bar{a} \bar{b} + 3 - |ab|^2 = 0. \tag{10}
\]
The real part of the left side of the equation (10) is
\[
x^2 + y^2 - 4x \cos 2\pi \varphi - 4y \sin 2\pi \varphi + 2r \cos 2(\varphi + \theta + \omega) + 3 - r^2 = 0 \tag{11}
\]
and the imaginary part of the left side of the equation (10) is
\[
y \cos 2\pi \varphi - x \sin 2\pi \varphi + r \sin 2(\varphi + \theta + \omega) = 0, \tag{12}
\]
where \( \zeta = x + iy \) and \( \mu = \bar{a} b e^{-2\pi i \theta} = re^{2\pi i \omega} \). One of the solutions of simultaneous equations (11) and (12) are
\[
x = -r \cos 2\pi (3\varphi + \theta + \omega) + 3 \cos 2\pi \varphi
\]
and
\[
y = -r \sin 2\pi (3\varphi + \theta + \omega) + 3 \sin 2\pi \varphi,
\]
and hence
\[
\zeta = re^{2\pi i (3\varphi + \theta + \omega + 1/2)} + 3e^{2\pi i \varphi}
\]
satisfies the equation (10). Conversely, we show the following theorem.

**Theorem 2.1.** Let \( \mu \in re^{2\pi i \omega} \in \mathbb{D} \) and let \( a = a(\theta, \varphi) \) and \( b = b(\theta, \varphi) \) with \( |a| \leq |b| \) be complex numbers satisfying relations \( a + b = re^{2\pi i (3\varphi + \theta + \omega + 1/2)} + 3e^{2\pi i \varphi} \) and \( ab = re^{-2\pi i (\theta + \omega)} \), that is, \( a \) and \( b \) are the solutions of the equation
\[
\xi^2 - \left\{ re^{2\pi i (3\varphi + \theta + \omega + 1/2)} + 3e^{2\pi i \varphi} \right\} \xi + re^{-2\pi i (\theta + \omega)} = 0, \tag{11}
\]
where \((\theta, \varphi) \in [0, 1]^2\). Then the following holds:
(a) In the case that \( r = 0 \), solutions of the equation (†) are \( a = 0 \) and \( b = 3e^{2\pi i\varphi} \).

(b) In the case that \( 0 < r < 1 \), the equation (†) does not have double roots. Moreover \( 0 < |a| < 1 < |b| < \infty \).

(c) In the case that \( r = 1 \) and \( 2\varphi + \theta + \omega \equiv 0 \pmod{1} \), the equation (†) has double roots and \( a = b = e^{2\pi i\varphi} \).

(d) In the case that \( r = 1 \) and \( 2\varphi + \theta + \omega \not\equiv 0 \pmod{1} \), the equation (†) does not have double roots. Moreover \( 0 < |a| < 1 < |b| < \infty \).

(e) In the case (a), (b) or (d),

\[
B(z) = B_{\theta,\varphi}(z) = e^{2\pi i\theta} z \left( \frac{z - a}{1 - \bar{a}z} \right) \left( \frac{z - b}{1 - \bar{b}z} \right)
\]

is a Blaschke product of degree 3 and the point at infinity is a fixed point of \( B \) with multiplier \( \mu \). Moreover \( z = e^{2\pi i\varphi} \) is a critical point of \( B \) and the other two critical points of \( B \) are in \( \hat{\mathbb{C}} \setminus \mathbb{T} \), where \( \mathbb{T} \) is the unit circle. In this case, \( B|_{\mathbb{T}} : T \to \mathbb{T} \) is a homeomorphism.

Proof of (a). It is clear.

Proof of (b). Since \( 0 < r < 1 \), we obtain that \( |a + b| \geq |r - 3| > 2 \) and \( |a||b| < 1 \). In this case, either \( 0 < |a| < 1 \leq |b| < \infty \) or \( 0 < |a| \leq |b| \leq 1 \) hold. If \( 0 < |a| \leq |b| \leq 1 \), then

\[
2 < |a + b| \leq |a| + |b| \leq 2.
\]

This is a contradiction and hence the situation \( 0 < |a| < 1 \leq |b| < \infty \) happens. If \( |b| = 1 \), then

\[
2 < |a + b| \leq |a| + |b| = |a| + 1 \leq 2.
\]

This is a contradiction. Therefore the equation (†) does not have double roots and \( 0 < |a| < 1 < |b| < \infty \).

Proof of (c). If \( r = 1 \) and \( 2\varphi + \theta + \omega \equiv 0 \pmod{1} \), then \( a + b = 2e^{2\pi i\varphi} \) and \( ab = e^{2\pi i:2\varphi} \). Therefore the equation (†) has double roots and \( a = b = e^{2\pi i\varphi} \).

Proof of (d). Since \( |a||b| = r = 1 \), either \( 0 < |a| < 1 < |b| < \infty \) or \( |a| = |b| = 1 \) hold. If \( |a| = |b| = 1 \), then

\[
2 = |r - 3| \leq |a + b| \leq |a| + |b| = 2
\]

and hence \( |a + b| = 2 \). On the other hand,

\[
|a + b| = |r e^{2\pi i(3\varphi+\theta+\omega+1/2)} + 3e^{2\pi i\varphi}| = |e^{2\pi i(2\varphi+\theta+\omega)} - 3|.
\]
So we obtain that \(|e^{2\pi i (2\varphi + \theta + \omega)} - 3| = 2\) and hence \(2\varphi + \theta + \omega \equiv 0 \pmod{1}\). This contradicts that \(2\varphi + \theta + \omega \not\equiv 0 \pmod{1}\). Therefore the equation (1) does not have double roots and \(0 < |a| < 1 < |b| < \infty\).

Proof of (e). In the case that \(r = 0\), critical points of \(B\) are \(z = 0, e^{2\pi i \varphi}\) and \(\infty\). Therefore the assertion holds. We consider the case that \(0 < r \leq 1\) below.

Let
\[
f(z) = f_{\theta, \varphi}(z) = \left(\frac{z - a}{1 - \bar{a}z}\right) \left(\frac{z - b}{1 - \bar{b}z}\right) = \frac{1}{ab} \cdot \frac{z^2 - (a + b)z + ab}{z^2 - \left(\frac{\bar{a} + \bar{b}}{\bar{a}b}\right) z + \frac{1}{\bar{a}b}}.
\]

The necessary and sufficient condition that the degree of the Blaschke product \(B\) be 3 is that the function \(f\) be not constant. So the necessary and sufficient condition that the degree of the Blaschke product \(B\) be 1 is that the function \(f\) be constant, that is,
\[
\frac{z^2 - (a + b)z + ab}{z^2 - \left(\frac{\bar{a} + \bar{b}}{\bar{a}b}\right) z + \frac{1}{\bar{a}b}} = 1 \tag{13}
\]
for all \(z \in \mathbb{C}\). Comparing coefficients of the numerator and the denominator of (13) implies that \(\bar{a}\bar{b}(a + b) = \bar{a} + \bar{b}\) and \(|ab| = 1\). In the case that \(0 < r < 1\), the degree of the Blaschke product \(B\) is 3 since \(|ab| = r < 1\). In the case that \(r = 1\),
\[
\bar{a}\bar{b}(a + b) - (\bar{a} + \bar{b}) = -e^{-2\pi i (3\varphi + \theta + \omega)} \left(e^{2\pi i (2\varphi + \theta + \omega)} - 1\right)^3.
\]
Therefore in the case \(r = 1\) and \(2\varphi + \theta + \omega \not\equiv 0 \pmod{1}\), the degree of the Blaschke product \(B\) is 3. It is clear that the point at infinity is a fixed point of \(B\) with multiplier \(\mu\). Moreover it is clear that \(g(e^{2\pi i \varphi}) = 0\) and hence \(z = e^{2\pi i \varphi}\) is a critical point of \(B\), where
\[
B'(z) = \frac{e^{2\pi i \theta}}{(1 - \bar{a}z)^2(1 - \bar{b}z)^2} \cdot g(z)
\]
and
\[
g(z) = \bar{a}\bar{b}z^4 - 2(\bar{a} + \bar{b})z^3 + \left\{3 - |ab|^2 + |a + b|^2\right\} z^2 - 2(a + b)z + ab.
\]
Finally we show that the other two critical points of \(B\) are in \(\hat{\mathbb{C}} \setminus T\). we factor \(r^{-1}e^{-2\pi i (\theta + \omega)}g(z)\) as
\[
\frac{1}{r} \cdot e^{-2\pi i (\theta + \omega)} \cdot g(z) = (z - e^{2\pi i \varphi})^2 \cdot h(z),
\]
where
\[ h(z) = z^2 + 2e^{2\pi i \varphi} \left\{ e^{-2\pi i (2\varphi + \theta + \omega)} - \frac{3}{r} e^{-2\pi i (2\varphi + \theta + \omega)} + 1 \right\} z + e^{-2\pi i (\varphi + \theta + \omega)}. \]

Let
\[ h_1(z) = 2e^{2\pi i \varphi} \left\{ e^{-2\pi i (2\varphi + \theta + \omega)} - \frac{3}{r} e^{-2\pi i (2\varphi + \theta + \omega)} + 1 \right\} z \]
and
\[ h_2(z) = z^2 + e^{-2\pi i (\varphi + \theta + \omega)}. \]

For \( z \in T \), \( |h_2(z)| \leq 2 \). In the case that \( 0 < r < 1 \), we obtain that
\[ |h_1(z)| \geq 2 \left| \frac{3}{r} - 1 - 1 \right| > 2 \]
on \( T \). In the case that \( r = 1 \), we obtain that
\[ |h_1(z)| = 2 \left| e^{-2\pi i (2\varphi + \theta + \omega)} - 3e^{-2\pi i (2\varphi + \theta + \omega)} + 1 \right| \]
\[ = 2 \left\{ e^{-2\pi i (2\varphi + \theta + \omega)} + 1 \right\}^2 - 5e^{-2\pi i (2\varphi + \theta + \omega)} \]
\[ \geq 2 \left( 5 - \left| e^{-2\pi i (2\varphi + \theta + \omega)} + 1 \right|^2 \right) > 2 \]
on \( T \), since \( 2\varphi + \theta + \omega \neq 0 \) (mod 1). By Rouché’s theorem, the number of roots of \( h(z) = h_1(z) + h_2(z) \) on \( D \) is one since \( |h_1(z)| > 2 \geq |h_2(z)| \) on \( T \) and the number of roots of \( h_1(z) \) on \( D \) is one. So one of critical points of \( B \) other than \( z = e^{2\pi i \varphi} \) is in \( D \). Since critical points of a Blaschke product are symmetric with respect to the unit circle, the other one critical point of \( B \) is in \( \hat{\mathbb{C}} \setminus \mathbb{D} \). In this case, the inverse image \( B^{-1}(T) \) of the unit circle \( T \) is the union of \( T \) and a figure eight which crosses at \( z = e^{2\pi i \varphi} \). Refer to Figure 1. Therefore \( B|_T : T \to T \) is a homeomorphism. \( \square \)

**Remark 2.2.** Two complex numbers \( a = a(\theta, \varphi) \) and \( b = b(\theta, \varphi) \) satisfy that
\[ a(\theta + 1, \varphi) = a(\theta, \varphi) = a(\theta, \varphi + 1) \]
and
\[ b(\theta + 1, \varphi) = b(\theta, \varphi) = b(\theta, \varphi + 1). \]
2.2 Rotation numbers of cubic Blaschke products

Let \( f : \mathbb{T} \to \mathbb{T} \) be an orientation preserving homeomorphism and let \( \tilde{f} : \mathbb{R} \to \mathbb{R} \) be a lift of \( f \) via \( x \mapsto e^{2\pi i x} \) which satisfies \( \tilde{f}(x + 1) = \tilde{f}(x) + 1 \) for all \( x \in \mathbb{R} \). A lift \( \tilde{f} \) of \( f \) is unique up to addition of an integer constant. The rotation number \( \rho(\tilde{f}) \) of \( \tilde{f} \) is defined as

\[
\rho(\tilde{f}) = \lim_{n \to \infty} \frac{\tilde{f}^n(x)}{n},
\]

which is independent of \( x \in \mathbb{R} \). The rotation number \( \rho(f) \) is defined as the residue class of \( \rho(\tilde{f}) \) modulo \( \mathbb{Z} \). Poincaré showed that the rotation number is rational with denominator \( q \) if and only if \( f \) has a periodic point with period \( q \). The following theorem is important (see [5]).

**Theorem 2.3.** Let \( \mathcal{F} \) be the set of all orientation preserving homeomorphisms form the unit circle onto itself with the topology of uniform conver-
gence. Then the rotation number function \( \rho : \mathcal{F} \to \mathbb{R}/\mathbb{Z} \) defined as \( f \mapsto \rho(f) \) is continuous.

If the cubic Blaschke product \( B_{\theta, \varphi} \) as in Theorem 2.1 is an orientation preserving homeomorphism on \( \mathbb{T} \), the rotation number function \( (\theta, \varphi) \mapsto \rho(B_{\theta, \varphi}|_{\mathbb{T}}) \) is continuous. In order to show that \( B_{\theta, \varphi}|_{\mathbb{T}} : \mathbb{T} \to \mathbb{T} \) is an orientation preserving homeomorphism, we show the following lemma.

**Lemma 2.4.** Let \( a(\theta, \varphi) \) and \( b(\theta, \varphi) \) be as in Theorem 2.1. Then for any \( (\theta, \varphi) \in [0, 1]^2 \), a loop \( \Gamma_1[\theta, \varphi] : [0, 1] \to \mathbb{T} \) defined as

\[
\Gamma_1[\theta, \varphi](x) = \left( \frac{e^{2\pi i x} - a(\theta, \varphi)}{1 - a(\theta, \varphi)e^{2\pi i x}} \right) \left( \frac{e^{2\pi i x} - b(\theta, \varphi)}{1 - b(\theta, \varphi)e^{2\pi i x}} \right)
\]

is homotopic to a constant loop \( x \mapsto e^{2\pi i \cdot 2\varphi} \).

*Proof.* Note that \( \Gamma_1[\theta, \varphi](x) = e^{2\pi i \cdot 2\varphi} \) for all \( x \in \mathbb{R} \) if \( r = 1 \) and \( 2\varphi + \theta + \omega \equiv 0 \) (mod 1). Let

\[
H_1[\theta, \varphi](x, t) = \left( \frac{e^{2\pi i (1-t)x + t\varphi} - a(\theta, \varphi)}{1 - a(\theta, \varphi)e^{2\pi i (1-t)x + t\varphi}} \right) \left( \frac{e^{2\pi i (1-t)x + t\varphi} - b(\theta, \varphi)}{1 - b(\theta, \varphi)e^{2\pi i (1-t)x + t\varphi}} \right).
\]

Then \( H_1[\theta, \varphi](x, 0) = \Gamma_1[\theta, \varphi](x) \) and \( H_1[\theta, \varphi](x, 1) = e^{2\pi i \cdot 2\varphi} \). Therefore \( H_1[\theta, \varphi] \) is a homotopy between the loop \( \Gamma_1[\theta, \varphi] \) and the constant loop \( \theta \mapsto e^{2\pi i \cdot 2\varphi} \). \( \square \)

The following two lemmas play important roles in the proof of Theorem 2.7.

**Lemma 2.5.** Let \( a(\theta, \varphi) \) and \( b(\theta, \varphi) \) be as in Theorem 2.1. Then for any \( z \in \mathbb{T} \) and \( \varphi \in [0, 1] \), a loop \( \Gamma_2[z, \varphi] : [0, 1] \to \mathbb{T} \) defined as

\[
\Gamma_2[z, \varphi](\theta) = \left( \frac{z - a(\theta, \varphi)}{1 - a(\theta, \varphi)z} \right) \left( \frac{z - b(\theta, \varphi)}{1 - b(\theta, \varphi)z} \right)
\]

is homotopic to a constant loop \( \theta \mapsto e^{2\pi i \cdot 2\varphi} \).

*Proof.* Note that \( \Gamma_2[e^{2\pi i \varphi}, \varphi](\theta) = e^{2\pi i \cdot 2\varphi} \) for all \( \theta \in [0, 1] \) and hence \( \Gamma_2[e^{2\pi i \varphi}, \varphi] \) is a constant loop \( e^{2\pi i \cdot 2\varphi} \). Let

\[
H_2[z, \varphi](\theta, t) = \left( \frac{z - a(\theta, \varphi, t)}{1 - a(\theta, \varphi, t)z} \right) \left( \frac{z - b(\theta, \varphi, t)}{1 - b(\theta, \varphi, t)z} \right),
\]

where

\[
a(\theta, \varphi, t) = (1-t)a(\theta, \varphi) + te^{2\pi i \varphi}
\]

and

\[
b(\theta, \varphi, t) = (1-t)b(\theta, \varphi) + te^{2\pi i \varphi}.
\]

Then \( H_2[z, \varphi](\theta, 0) = \Gamma_2[z, \varphi](\theta) \) and \( H_2[z, \varphi](\theta, 1) = e^{2\pi i \cdot 2\varphi} \). Therefore \( H_2[z, \varphi] \) is a homotopy between the loop \( \Gamma_2[z, \varphi] \) and the constant loop \( \theta \mapsto e^{2\pi i \cdot 2\varphi} \). \( \square \)
**Lemma 2.6.** Let $a(\theta, \varphi)$ and $b(\theta, \varphi)$ be as in Theorem 2.1. Then for any $z \in \mathbb{T}$ and $\theta \in [0, 1]$, a loop $\Gamma_3[z, \theta] : [0, 1] \to \mathbb{T}$ defined as

$$\Gamma_3[z, \theta](\varphi) = \left( \frac{z - a(\theta, \varphi)}{1 - a(\theta, \varphi)z} \right) \left( \frac{z - b(\theta, \varphi)}{1 - b(\theta, \varphi)z} \right)$$

is homotopic to a loop $\varphi \mapsto e^{2\pi i \cdot 2\varphi}$.

**Proof.** Note that $\Gamma_3[e^{2\pi i \varphi}, \theta](\varphi) = e^{2\pi i \cdot 2\varphi}$. Let $H_3[z, \theta](\varphi, t) = \left( \frac{z - a(\theta, \varphi, t)}{1 - a(\theta, \varphi, t)z} \right) \left( \frac{z - b(\theta, \varphi, t)}{1 - b(\theta, \varphi, t)z} \right)$, where

$$a(\theta, \varphi, t) = (1 - t)a(\theta, \varphi) + te^{2\pi i \varphi}$$

and

$$b(\theta, \varphi, t) = (1 - t)b(\theta, \varphi) + te^{2\pi i \varphi}.$$ 

Then $H_3[z, \theta](\varphi, 0) = \Gamma_3[z, \theta](\varphi)$ and $H_3[z, \theta](\varphi, 1) = e^{2\pi i \cdot 2\varphi}$. Therefore $H_3[z, \theta]$ is a homotopy between the loop $\Gamma_3[z, \theta]$ and the the loop $\varphi \mapsto e^{2\pi i \cdot 2\varphi}$. □

Let

$$\Gamma(x, \theta, \varphi) = \left( \frac{e^{2\pi i x} - a(\theta, \varphi)}{1 - a(\theta, \varphi)e^{2\pi i x}} \right) \left( \frac{e^{2\pi i x} - b(\theta, \varphi)}{1 - b(\theta, \varphi)e^{2\pi i x}} \right).$$

Then $\Gamma(x, \theta, \varphi) = \Gamma_1[\theta, \varphi](x) = \Gamma_2[e^{2\pi i x}, \varphi](\theta) = \Gamma_3[e^{2\pi i x}, \theta](\varphi)$. Lemma 2.4 and Lemma 2.5 imply that

$$\arg(\Gamma(x + 1, \theta, \varphi)) = \arg(\Gamma(x, \theta, \varphi)) = \arg(\Gamma(x, \theta + 1, \varphi))$$

and Lemma 2.6 implies that

$$\frac{1}{2\pi} \arg(\Gamma(x, \theta, \varphi + 1)) = \frac{1}{2\pi} \arg(\Gamma(x, \theta, \varphi)) + 2.$$

**Theorem 2.7.** Let $\alpha \in [0, 1]$ and let $\mu = re^{2\pi i \omega} \in \mathbb{D}$, $a = a(\theta, \varphi)$ and $b = b(\theta, \varphi)$ be as in Theorem 2.1. Then for the Blaschke product

$$B_{\theta, \varphi}(z) = e^{2\pi i \theta} z \left( \frac{z - a}{1 - az} \right) \left( \frac{z - b}{1 - bz} \right),$$

$B_{\theta, \varphi}|_T : \mathbb{T} \to \mathbb{T}$ is an orientation preserving homeomorphism. Moreover

(a) In the case that $0 \leq r < 1$, there exists $(\theta_0, \varphi_0) \in [0, 1]^2$ such that $\rho(B_{\theta_0, \varphi_0}|_T) = \alpha$. 


(b) In the case that $r = 1$, if $\alpha + \omega \not\equiv 0 \pmod{1}$, then there exists $(\theta_0, \varphi_0) \in [0, 1]^2$ such that $\rho(B_{\theta_0, \varphi_0}|_T) = \alpha$ and $2\varphi_0 + \theta_0 + \omega \not\equiv 0 \pmod{1}$.

Proof. In the case that $r = 1$ and $2\varphi + \theta + \omega \equiv 0 \pmod{1}$, $B_{\theta, \varphi}|_T : T \to T$ is an orientation preserving homeomorphism and its rotation number satisfies that $\rho(B_{\theta, \varphi}|_T) \equiv -\omega \pmod{1}$. In the other cases, we consider a lift $\tilde{B}_{\theta, \varphi}(x) = \theta + x + \frac{1}{2\pi} \arg(\Gamma(x, \theta, \varphi))$ of $B_{\theta, \varphi}|_T : T \to T$ via $x \mapsto e^{2\pi ix}$. By Lemma 2.4,

$$\tilde{B}_{\theta, \varphi}(x + 1) = \theta + x + 1 + \frac{1}{2\pi} \arg(\Gamma(x + 1, \theta, \varphi)) = \tilde{B}_{\theta, \varphi}(x) + 1$$

for all $x \in \mathbb{R}$. This implies that $B_{\theta, \varphi}|_T : T \to T$ is an orientation preserving homeomorphism. Consequently the rotation number of $\rho(B_{\theta, \varphi})$ is well defined. By Lemma 2.5, we obtain that $\tilde{B}_{1, \varphi}^n(x) = \tilde{B}_{0, \varphi}^n(x) + n$ and hence

$$\rho(\tilde{B}_{1, \varphi}) = \rho(\tilde{B}_{0, \varphi}) + 1. \quad (14)$$

Moreover by Lemma 2.6, we obtain that $\tilde{B}_{\theta, 1}^n(x) = \tilde{B}_{\theta, 0}^n(x) + 2n$ and hence

$$\rho(\tilde{B}_{\theta, 1}) = \rho(\tilde{B}_{\theta, 0}) + 2. \quad (15)$$

These two equation (14) and (15) imply that

$$\rho(\tilde{B}_{1, 1}) = \rho(\tilde{B}_{0, 0}) + 3.$$ 

Therefore in the case that $0 \leq r < 1$, there exists $(\theta_0, \varphi_0) \in [0, 1]^2$ such that

$$\alpha = \rho(B_{\theta_0, \varphi_0}|_T) \equiv \rho(\tilde{B}_{\theta_0, \varphi_0}) \pmod{1}$$

since the rotation number function function $(\theta, \varphi) \mapsto \rho(B_{\theta, \varphi}|_T)$ is continuous. In the case that $r = 1$, if $2\varphi + \theta + \omega \equiv 0 \pmod{1}$, then $\rho(B_{\theta, \varphi}|_T) \equiv -\omega \pmod{1}$. Hence if $\alpha + \omega \not\equiv 0 \pmod{1}$, then there exists $(\theta_0, \varphi_0) \in [0, 1]^2$ such that

$$\alpha = \rho(B_{\theta_0, \varphi_0}|_T) \equiv \rho(\tilde{B}_{\theta_0, \varphi_0}) \pmod{1}$$

and $2\varphi_0 + \theta_0 + \omega \not\equiv 0 \pmod{1}$.

Remark 2.8. By theorem 2.1, the degree of $B_{\theta_0, \varphi_0}$ is 3.
Some Blaschke products and quadratic rational functions

Figure 3: The Julia set of some cubic Blaschke product $B_{\theta,\varphi}$ (left) and “the filled-in Julia set” of some modified Blaschke product $B_{\theta,\varphi}$ (right).

3 Quadratic rational functions with Siegel disks

In this section, we show Theorem 1.4. Let $f : \mathbb{R} \to \mathbb{R}$ be a homeomorphism. The map $f$ is $k$-quasisymmetric if there exists $k \geq 1$ such that

$$\frac{1}{k} \leq \left| \frac{f(x+t) - f(x)}{f(x) - f(x-t)} \right| \leq k$$

for all $x \in \mathbb{R}$ and all $t \geq 0$. A homeomorphism $h : T \to T$ is $k$-quasisymmetric if its lift $\tilde{h} : \mathbb{R} \to \mathbb{R}$ is $k$-quasisymmetric. By the theorem of Beurling and Ahlfors, any $k$-quasisymmetric homeomorphism $f : \mathbb{R} \to \mathbb{R}$ is extended to a $K$-quasiconformal map $F : \mathbb{H} \to \mathbb{H}$, where $\mathbb{H}$ is the upper half plain (More precisely $F : \mathbb{C} \to \mathbb{C}$). The dilatation $K$ of $F$ depends only on $k$. Therefore if a homeomorphism $h : T \to T$ is $k$-quasisymmetric, then we can extend $h$ to a $K$-quasiconformal map $H : D \to D$ whose dilatation depends only on $k$.

**Theorem 3.1 (Herman-Świątek).** The rotation number $\rho(f)$ of a real analytic orientation preserving homeomorphism $f : T \to T$ is of constant type if and only if $f$ is quasisymmetrically linearizable, that is, there exists a quasisymmetric homeomorphism $h : T \to T$ such that $h \circ f \circ h^{-1}(z) = e^{2\pi i \rho(f)}z$.

Let $F_{\lambda,\mu}(z) = z(z + \lambda)/(\mu z + 1)$ with $\lambda \mu \neq 1$. Any quadratic rational function with fixed points of multipliers $\lambda$ and $\mu$ with $\lambda \mu \neq 1$ is conjugate to $F_{\lambda,\mu}$ (see [7]).

**Proof of Theorem 1.4.** By Theorem 2.7, there exist $(\theta, \varphi) \in [0, 1]^2$ such that the degree of $B_{\theta,\varphi}$ is 3 and $\rho(B_{\theta,\varphi}|_T) = \alpha$. Since $\alpha$ is of bounded type, there exists a quasisymmetric homeomorphism $h : T \to T$ such that $h \circ B_{\theta,\varphi}|_T \circ h^{-1}(z) = R_{\alpha}(z) = e^{2\pi i \alpha}z$. By the theorem of Beurling and Ahlfors, $h$ has a
Figure 4: Golden Siegel disks of $F_{\lambda, \mu}$ centered at the origin, where $\lambda = e^{2\pi i (\sqrt{5} - 1)/2}$ and $\mu = re^{2\pi i (\sqrt{5} - 1)/2}$. In the case $r = 1$, the point at infinity is the center of another golden Siegel disk.
quasiconformal extension $H : \overline{D} \to \overline{D}$ with $H(0) = 0$. We define a new map $\mathfrak{B}_{\theta, \varphi}$ as

\[
\mathfrak{B}_{\theta, \varphi} = \begin{cases} 
B_{\theta, \varphi} & \text{on } \hat{\mathbb{C}} \setminus \mathbb{D}, \\
H^{-1} \circ R_{\alpha} \circ H & \text{on } \mathbb{D}.
\end{cases}
\]

The map $\mathfrak{B}_{\theta, \varphi}$ is quasiregular on $\hat{\mathbb{C}}$ since $T$ is an analytic curve. Moreover $\mathfrak{B}_{\theta, \varphi}$ is a degree 2 branched covering of $\hat{\mathbb{C}}$. We define a conformal structure $\sigma_{\theta, \varphi}$ as

\[
\sigma_{\theta, \varphi} = \begin{cases} 
H^* \sigma_0 & \text{on } \mathbb{D}, \\
\left( \mathfrak{B}_{\theta, \varphi}^{-n} \right) \sigma_0 & \text{on } \mathfrak{B}_{\theta, \varphi}^{-n} (\mathbb{D}) \setminus \mathbb{D} \text{ for all } n \in \mathbb{N}, \\
\sigma_0 & \text{on } \hat{\mathbb{C}} \setminus \bigcup_{n=1}^{\infty} \mathfrak{B}_{\theta, \varphi}^{-n} (\mathbb{D}),
\end{cases}
\]

where $\sigma_0$ is the standard conformal structure on $\hat{\mathbb{C}}$. The conformal structure $\sigma_{\theta, \varphi}$ is invariant under $\mathfrak{B}_{\theta, \varphi}$ and its maximal dilatation is the dilatation of $H$ since $H$ is quasiconformal and $B_{\theta, \varphi}$ is holomorphic. By the measurable Riemann mapping theorem, there exists a quasiconformal homeomorphism $\Psi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $\Psi^* \sigma_0 = \sigma_{\theta, \varphi}$. Therefore $\Psi \circ \mathfrak{B}_{\theta, \varphi} \circ \Psi^{-1}$ is a quadratic rational map. We normalize $\Psi$ by $\Psi(0) = 0$, $\Psi(\frac{1}{\bar{b}}) = -1/\mu$ and $\Psi(\infty) = \infty$. Therefore we obtain that $F_{\lambda, \mu} = \Psi \circ \mathfrak{B}_{\theta, \varphi} \circ \Psi^{-1}$ since multipliers of fixed points are invariant under conjugation. The quadratic rational map $F_{\lambda, \mu}$ has a Siegel disk $\Delta = \Psi(\mathbb{D})$ with a critical point $\Psi(e^{2\pi i \varphi}) \in \partial \Delta$. Moreover $\partial \Delta = \Psi(T)$ is a quasicircle since $\Psi$ is quasiconformal. \hfill \Box

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