Subschemes of Multi-Projective Spaces and the Generators of Their Multi-Homogeneous Ideal

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Abstract. Let $Y \subset \Pi := \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_k}$ be a closed subscheme and $F$ a vector bundle on $\Pi$. Here we give Castelnuovo-Mumford’s style results on the multi-degrees of generators of the multi-graded module $\bigoplus(a_1,\ldots,a_k)H^0(\Pi, \mathcal{I}_Y \otimes F(a_1,\ldots,a_k))$.

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1. Introduction

Fix integers $k \geq 2$, $n_i > 0$, $1 \leq i \leq k$, and an infinite field $K$. Let $Z \subset \Pi := \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_k}$ be a closed subscheme. Choose homogeneous coordinates $x_{i,j}$, $1 \leq i \leq k$, $0 \leq j \leq n_i$ of each factor of $\Pi$. Let $e_i$, $1 \leq i \leq k$ denote the basic vector $(0,\ldots,1,\ldots,0)$ of $\mathbb{N}^k$. Set $\deg(x_{i,j}) = e_i$ for all $i,j$. With these degrees the polynomial ring $R := K[x_{1,0},\ldots,x_{k,n_k}]$ is an $\mathbb{N}^k$-graded ring which will be called the multi-graded ring of $\Pi$ ([2]). For any $a = (a_1,\ldots,a_k) \in \mathbb{N}^k$ the $K$-vector space $H^0(\mathcal{O}_\Pi(a))$ is the set of all $f \in R$ with multi-degree $a$. For any $a = (a_1,\ldots,a_k) \in \mathbb{Z}^k$ and $b = (b_1,\ldots,b_k) \in \mathbb{Z}^k$ we will write $a \leq b$ (resp. $a \geq b$) if and only if $a_i \leq b_i$ (resp. $a_i \geq b_i$) for all $i$.

To extend some of the results of [2] we will first prove some general results (see Theorem 1 and Proposition 1).

**Theorem 1.** Let $Y$ be an integral projective variety, $Z \subset Y$ a zero-dimensional subscheme, $M$, $R$ spanned line bundles on $Y$ and $F$ a coherent sheaf on $Y$ which is locally free outside $Z_{\text{red}}$. Let $V \subseteq H^0(Y,M)$ (resp. $W \subseteq H^0(Y,R)$) be a linear subspace spanning $M$ (resp. $R$). Let $\mu_W : W \otimes H^0(Y,F \otimes M) \to H^0(Y,F \otimes M \otimes R)$ and $\mu : V \otimes H^0(Y,\mathcal{I}_Z \otimes F \otimes R) \oplus W \otimes H^0(Y,\mathcal{I}_Z \otimes F \otimes M) \to H^0(Y,\mathcal{I}_Z \otimes A \otimes M \otimes R)$ denote the multiplication maps. Assume

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\[ h^1(Y, \mathcal{I}_Z \otimes F \otimes R \otimes M) = 0, \quad h^1(Y, \mathcal{I}_Z \otimes F \otimes M \otimes R^*) = 0 \] and that \( \mu_W \) is surjective. Then \( \mu \) is surjective.

**Remark 1.** Notice that in the statement of Theorem 1 we allow the case \( M = R \). Hence Theorem 1 and Proposition 1 below cover both [2], Theorem 3, and [2], Theorem 7, and extend them to the case in which \( Z \) is not reduced.

Then we will prove the following result on higher dimensional subvarieties of \( \Pi \).

**Theorem 2.** Let \( T \subset \Pi \) be a pure \( c \)-dimensional subscheme, \( 0 < c < \sum_{i=1}^{k} n_k \).
Let \( F \) be a locally free sheaf on \( \Pi \) and \( d \) such that \( h^i(\Pi, F(u)) = h^i(T, F(u)|T) = 0 \) for all \( i > 0 \) and all \( u \geq d \). Let \( \Pi' \subset \Pi \) be the multi-projective space spanned by \( T \). Assume that for every projection \( \alpha \) of \( \Pi' \) onto one of its factors, say \( \mathbf{P}^s \), \( \dim(\alpha(E)) = \min\{s, c\} \) for all irreducible components \( E \) of \( T_{\text{red}} \).
Fix any \( a = (a_1, \ldots, a_k) \geq d \) such that \( h^i(\Pi, T \otimes F(m)) = 0 \) for all pairs \( (t, m = (m_1, \ldots, m_k)) \) such that \( 1 \leq t \leq c + 1 \) and \( a_i - t + 1 \leq m_i \leq a_i \) for all \( 1 \leq i \leq k \). Then the \( \mathbb{R}\)-module \( \bigoplus_{x \in \mathbf{Z}} H^0(\Pi, T \otimes F(h)) \) is generated by its components with multi-degree \( b = (b_1, \ldots, b_k) \) such that \( b_i \leq a_i + c + 1 \) for all \( 1 \leq i \leq c + 1 \).

**2. Proofs and Proposition 1.**

**Proposition 1.** Fix an integer \( t \geq 0 \). Let \( Y \) be an integral projective variety, \( Z \subset Y \) a zero-dimensional subscheme, a spanned line bundle \( M \) on \( Y \) and \( F \) a coherent sheaf on \( Y \) which is locally free at each point of \( Z_{\text{red}} \) and \( h^1(Y, F \otimes M^\otimes z) = 0 \) for all \( z \geq t + 1 \). Set \( V := H^0(Y, M) \). Fix a general \( D \in |M| \) and let \( W \) be the image of \( V \) in \( H^0(D, M|D) \).
For all integers \( x \) let \( \mu_x : V \otimes H^0(Y, \mathcal{I}_Z \otimes A \otimes M^\otimes x) \rightarrow H^0(Y, \mathcal{I}_Z \otimes A \otimes M^\otimes(x+1)) \) denote the multiplication map. Assume \( h^1(Y, \mathcal{I}_Z \otimes A \otimes M^\otimes x) = 0 \) for \( x = t, t+1 \) and that for all integers \( x \geq t \) the multiplication maps \( \alpha_x : V \otimes H^0(Y, A \otimes M^\otimes x) \rightarrow H^0(Y, A \otimes M^\otimes(x+1)) \) and \( \beta_x : W \otimes H^0(D, A \otimes M^\otimes x|D) \rightarrow H^0(D, A \otimes M^\otimes(x+1)|D) \) is surjective. Then \( h^2(Y, \mathcal{I}_Z \otimes A \otimes M^\otimes x) = 0 \) for all \( x \geq t + 2 \) and \( \mu_x \) is surjective for all \( x \geq t + 1 \).

**Proof.** By induction on \( t \) it is sufficient to do the case \( x = t+2 \) of the vanishing statement and the surjectivity of \( \mu_{t+1} \). Since \( M \) is spanned, \( D \) is general and \( Z \) is zero-dimensional, \( D \cap Z_{\text{red}} = \emptyset \). Hence for all integers \( y \) there is an exact sequence

\[
0 \rightarrow \mathcal{I}_Z \otimes F \otimes M^\otimes(y-1) \rightarrow \mathcal{I}_Z \otimes F \otimes M^\otimes y \rightarrow (F \otimes M^\otimes y)|D \rightarrow 0
\]

From (1) and induction on \( x \) we get the vanishing statement. Now we will prove that \( \mu_{t+1} \) is surjective. Let \( f \in H^0(Y, M) \) be an equation of \( D \). Fix \( u \in H^0(Y, \mathcal{I}_Z \otimes F \otimes M^\otimes t+2) \) and consider \( u|D \).
Since \( \beta_{t+1} \) is surjective, there are \( w_i \in W \) and \( u_i \in H^0(D, A \otimes M^\otimes(t+1)|D) \) such that \( u|D = \sum w_i u_i \). Since \( W \) is the image of \( V \) and \( h^1(Y, \mathcal{I}_Z \otimes A \otimes M^\otimes(t+1)) = 0 \), there are \( v_i \in V \) and \( a_i \in H^0(Y, \mathcal{I}_Z \otimes A \otimes M^\otimes t+2) \) such that \( v_i|D = w_i \) and \( a_i|D = u_i \). Hence
Since $c(f, h_b) = 1$ and each $E_i$ of $E$ of $F \subset W$, the surjectivity of $\mu_W$ gives the surjectivity of the multiplication map $\mu_{W,D} : W_D \otimes H^0(D, (F|D) \otimes (M|D)) \rightarrow H^0(D, (F|D) \otimes (M|D) \otimes (R|D))$. Thus there are finitely many $A_i \in W_D$ and $B_i \in H^0(D, (F|D) \otimes (M|D) \otimes (R|D))$ such that $u|D = \sum_i A_i B_i$. Take $A'_i \in W$ such that $A'_i|D = A_i$. Since $h^1(Y, \mathcal{I}_Z \otimes F \otimes M \otimes R^*) = 0$, there is $B'_i \in H^0(Y, \mathcal{I}_Z \otimes F \otimes M)$ such that $B'_i|D = B_i$. Hence $(u - \sum_i A'_i B'_i)|D = 0$. Since $h^1(Y, \mathcal{I}_Z \otimes F \otimes R \otimes M^*) = 0$, there is $G \in H^0(Y, \mathcal{I}_Z \otimes F \otimes R)$ such that $u - \sum_i A'_i B'_i = fG$ (use (2) with $\alpha = 0$ and $\beta = 1$). Hence $u \in \text{Im}(\mu)$. 

Proof of Theorem 2. Without losing generality we may assume $\Pi' = \Pi$. We will use induction on $c$, the case $c = 0$ being true by Theorem 1. First assume $c = 1$ and take $i \in \{1, \ldots, k\}$ such that $\dim(\pi_i(E)) > 0$ for all irreducible components $E$ of $T_{\text{red}}$. Fix a general hyperplane $H \subset \mathbb{P}^n$ and set $T_1 := T \cap \pi_i^{-1}(H)$. Bertini’s theorem ([1], Th. 6.3) gives that $T_1$ is zero-dimensional. Since $\dim(T) = 1$ and each $\mathcal{O}_Y(e_i)$ is spanned, if $h^1(T, (F|T)(b)) = 0$, then $h^1(T, (F|T)(c)) = 0$ for all $c \geq b$. If $b \in \mathbb{N}^k$, then $h^0(T, \mathcal{O}_Y(b)) = 0$ if and only if $h^2(\Pi, T_1(b)) = 0$. Since $\dim(T_1) = 0$, if $h^1(\Pi, T_1(b)) = 0$, then $b \in \mathbb{N}^k$ and $h^1(\Pi, T_1(c)) = 0$ for all $c \geq b$. Assume $h^1(\Pi, F(u)) = 0$ for all $u \geq \underline{0} := (0, \ldots, 0)$. Since $\dim(T_1) = 0$, if $h^1(\Pi, T_1(b)) = 0$ and $b \in \mathbb{N}^k$, then $h^1(\Pi, T_1(c)) = 0$ for all $c \geq b$. We have an exact sequence

\begin{equation}
0 \rightarrow \mathcal{I}_T \otimes F(a - e_i) \rightarrow \mathcal{I}_T \otimes F(a) \rightarrow \mathcal{I}_{T_1, \Pi} \otimes (F|\Pi_1)(a) \rightarrow 0
\end{equation}

From (3) we get that the following conditions are equivalent:

(a) $h^1(\Pi, \mathcal{I}_T \otimes F(a)) = 0$ and the map $i_{2,a} : h^2(\Pi, \mathcal{I}_T \otimes (a - e_i)) \rightarrow h^2(\Pi, \mathcal{I}_T \otimes F(a))$ is injective;

(b) $h^1(\Pi, \mathcal{I}_T \otimes F(a)) = 0$ and $h^1(\Pi, \mathcal{I}_{T_1} \otimes F(a)) = 0$.

From now on we assume $h^1(\Pi, F(u)) = 0$ for all $u \geq \underline{0} := (0, \ldots, 0)$. If $a \geq 0$ and $h^1(\Pi, \mathcal{I}_T \otimes F(a)) = 0$, then $h^1(\Pi, \mathcal{I}_{T_1} \otimes F(b)) = 0$ for all $b \geq a$. Hence if $a \geq 0$ and (b) is satisfied then $i_{2,a}$ is injective for all $b \geq a$. Since $(1, \ldots, 1)$ is ample, we get that if (a) holds and $a \geq 0$, then $h^2(\Pi, \mathcal{I}_T \otimes (b - e_i)) = 0$ for all $b \geq a$. Now assume $a \geq 0$ and that (a) is satisfied. Theorem 1 and Proposition 1 give that the $R$-module $T_1(F, a) := \oplus_{b \geq a} H^0(\Pi_1, \mathcal{I}_{T_1} \otimes F(b))$ is generated by
its components in degree $a$ and $a + e_j$, $1 \leq j \leq k$. Theorem 1 and Proposition 1 give that the $R$-module $T(F, a) := \oplus_{b \geq a} H^0(I, I_T \otimes F(b))$ is generated by its components in degree $a, a + e_j$, $1 \leq j \leq k$, and $a + e_j + e_m$, $1 \leq j \leq m \leq k$, proving the case $\dim(T) = 1$. The inductive proof of the general case require only notational modifications and hence it is omitted.

REFERENCES


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