A Note on Browkin-Brzezinski’s Conjecture

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Abstract
In this paper we prove Browkin-Brzezinski’s conjecture for a class of polynomials.

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1 Introduction

Let $F$ be a fixed algebraically closed field of characteristic 0. Let $f(z)$ be a polynomial non-constants with coefficients in $F$ and let $r(f)$ be the number of distinct zeros of $f$. Then we have the following.

Mason’s theorem ([3]). Let $a(z), b(z), c(z)$ be relatively prime polynomials in $F$ and not all constants such that $a + b = c$. Then

$$\max\{\deg(a), \deg(b), \deg(c)\} \leq r(abc) - 1.$$  

For different purposes some generalizations of the theorem are given (see [2, 4]). In [1], Browkin and Brzezinski suggested the following conjecture.

Conjecture 1.1 Let $f_0, ..., f_n+1$ be $n + 2$ polynomials not all constants in $F[x]$ and have no common zeros such that

$$f_0 + \cdots + f_n + f_{n+1} = 0.$$  

Then

$$\max_{0 \leq j \leq n+1} \deg(f_j) \leq (2n - 1) (r(f_0 \cdots f_{n+1}) - 1).$$

In this paper we prove the following theorem.

Theorem 1.1 Browkin-Brzezinski’s conjecture is true for polynomials $f_0, ..., f_{n+1}$ such that $\gcd(f_i, f_j, f_k) = 1$ for all distinct $i, j, k \in \{0, ..., n + 1\}$. 
2 Proof of the main theorem

Let $f$ be a rational function, we write $f$ in the form:

$$f = \frac{f_1}{f_2},$$

Where $f_1, f_2$ are polynomial functions are non-zero and relatively prime on $F[x]$. The degree of $f$, denoted by $\deg f$, is defined to be $\deg f_1 - \deg f_2$.

Let $a \in F$, we write $f$ in the form:

$$f = (x - a)^l \frac{g_1}{g_2},$$

and $g_1(a)g_2(a) \neq 0$, then $l$ is called the order of $f$ at $a$ and is denoted by $\mu^a_f$.

We have the following easily proved properties of $\mu^a_f$.

**Lemma 2.1** Let $f, g$ be two polynomials and $a \in F$, we have

a) $\mu^a_{f+g} \geq \min(\mu^a_f, \mu^a_g)$,

b) $\mu^a_{fg} = \mu^a_f + \mu^a_g$,

c) $\mu^a_{\frac{f}{g}} = \mu^a_f - \mu^a_g$.

**Lemma 2.2** Let $\varphi$ be a rational function on $F$ and let the derivatives order $k$ of $\varphi$ satisfy the following $\varphi^{(k)} \neq 0$. Then

$$\mu^a_{\varphi^{(k)}} \geq -k + \mu^a_{\varphi}.$$  

**Proof.** Let $\varphi(x) = (x - \alpha)^m \frac{f(x)}{g(x)}$, where $f(x), g(x)$ are relatively prime and $f(\alpha)g(\alpha) \neq 0$. Then, we have

$$\varphi'(x) = (x - \alpha)^{m-1} \frac{mf(x)g(x) + (x - \alpha)(f'(x)g(x) - f(x)g'(x))}{g^2(x)}.$$  

By $\mu^a_g = 0$, we have

$$\mu^a_{\varphi'} \geq m - 1.$$  

Therefore

$$\mu^a_{\varphi^{(k)}} \geq -1 + \mu^a_{\varphi}.$$  

From this we obtain

$$\mu^a_{\varphi^{(k)}} \geq -k + \mu^a_{\varphi}.$$  

**Lemma 2.3** Browkin-Brzczinski’s conjecture is true for polynomials $f_0, \ldots, f_{n+1}$ such that $\gcd(f_i, f_j, f_k) = 1$ for all distinct $i, j, k \in \{0, \ldots, n + 1\}$ and $f_0, \ldots, f_n$ are linearly independent.
Proof. By the hypothesis \( f_0, \ldots, f_n \) are linearly independent, we have the Wronskian \( W \) of \( f_0, \ldots, f_n \) does not vanish. We set

\[
P = \frac{W(f_0, \ldots, f_n)}{f_0 \cdots f_n},
\]

\[
Q = \frac{f_0 \cdots f_{n+1}}{W(f_0, \ldots, f_n)}.
\]

Hence we have

\[
f_{n+1} = PQ.
\] (2)

We first prove that

\[
\deg Q \leq (2n - 1)r(f_0 \cdots f_{n+1}).
\]

Suppose that \( \alpha \) is a zero of \( f_0 f_1 \cdots f_{n+1} \), by the hypothesis there exists \( \nu, 0 \leq \nu \leq n + 1 \) such that \( f_{\nu} \neq 0 \). By the hypothesis \( f_0 + \cdots + f_n = f_{n+1} \) we have

\[
\mu_{f_0 \cdots f_{n+1}}^{(n)} = \mu_{f_0 \cdots f_{n+1}}^{(n)} - \sum_{j=0}^{n+1} \mu_{f_j}^{(n)} - \mu_{W(f_0, \ldots, f_{n+1})}^{(n)}
\]

\[
W(f_0, \ldots, f_{\nu-1}, f_{\nu+1}, \ldots, f_{n+1}) \text{ is the sum of follow terms}
\]

\[
\delta f_{\alpha_0}(f_{\alpha_1})' \cdots (f_{\alpha_n})^{(n)},
\]

Where \( \alpha_i \in \{0, \ldots, n+1\} \setminus \{\nu\} \), \( \delta = \pm 1 \). We suppose there exists \( k \) functions \( f_j \) such that \( f_j(\alpha) = 0 \). By the hypothesis \( gdc(f_i, f_j, f_k) = 1 \) for all distinct \( i, j, k \in \{0, \ldots, n+1\} \) we have \( k \leq 2 \). From this and Lemma 2.1 and we have

\[
\mu_{f_0 \cdots f_{n+1}}^{(n)} \geq \sum_{f_j(\alpha)=0}^{n+1} \mu_{f_{\alpha_j}}^{(n)} - (n + (n - 1))
\]

\[
= \mu_{f_0 \cdots f_{n+1}}^{(n)} - (2n - 1)
\]

By Lemma 2.2 we have

\[
\mu_{W(f_0, \ldots, f_{n+1})}^{(n)} \geq \mu_{f_0 \cdots f_{n+1}}^{(n)} - (2n - 1).
\]

Hence

\[
\mu_{f_0 \cdots f_{n+1}}^{(n)} \leq 2n - 1,
\]
By the definition of degree of a rational function, we have:

$$\deg Q \leq (2n - 1)r(f_0 \cdots f_{n+1}).$$  \hspace{1cm} (3)

Next, we will prove that

$$\deg P \leq -\frac{n(n+1)}{2}. \hspace{1cm} (4)$$

At here, we have $P$ as the logarithmic Wronskian corresponding to $I = \{0, 1, \ldots, n\}$ which is

$$\begin{vmatrix}
1 & 1 & \cdots & 1 \\
\frac{f_0'}{f_0} & \frac{f_1'}{f_1} & \cdots & \frac{f_n'}{f_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{f_0^{(n)}}{f_0} & \frac{f_1^{(n)}}{f_1} & \cdots & \frac{f_n^{(n)}}{f_n}
\end{vmatrix}$$

The determinant $P$ is a summa of following terms

$$\delta \frac{f_{\beta_1}' f_{\beta_2}'' \cdots f_{\beta_n}^{(n)}}{f_{\beta_1} f_{\beta_2} \cdots f_{\beta_n}}.$$ 

For every term, we have

$$\deg \left( \frac{f_{\beta_1}' f_{\beta_2}'' \cdots f_{\beta_n}^{(n)}}{f_{\beta_1} f_{\beta_2} \cdots f_{\beta_n}} \right) = \deg \left( \frac{f_{\beta_1}'}{f_{\beta_1}} \right) + \deg \left( \frac{f_{\beta_2}'}{f_{\beta_2}} \right) + \cdots + \deg \left( \frac{f_{\beta_n}^{(n)}}{f_{\beta_n}} \right)$$

$$= -(1 + 2 + \ldots + n) = -\frac{n(n+1)}{2}.$$ 

Therefore

$$\deg P \leq -\frac{n(n+1)}{2}.$$  \hspace{1cm} (4)

From (2), (3), (4) we have

$$\deg f_{n+1} = \deg P + \deg Q \leq (2n - 1)r(f_0 \cdots f_{n+1}) - \frac{n(n+1)}{2}$$

$$\leq (2n - 1) \left( r(f_0 \cdots f_{n+1}) - 1 \right).$$

Similar arguments apply to the polynomial $f_0, f_1, \ldots, f_n$, we have

$$\max_{0 \leq i \leq n+1} (\deg f_i) \leq (2n - 1) \left( r(f_0 \cdots f_{n+1}) - 1 \right).$$

**Proof of theorem 1.1.** The proof proceed by induction on $n$. For $n = 1$, it is true by Mason’s theorem. Suppose that the theorem is true for all case $m, 1 \leq m \leq n - 1$. If $f_0, \ldots, f_n$ are linearly independent, then this is Lemma 2.3. If $f_0, \ldots, f_n$ are linearly dependent, rewriting (1) as

$$-f_{n+1} = f_0 + \cdots + f_n,$$  \hspace{1cm} (5)
Let $f_{i_1}, ..., f_{i_q}, q < n+1$ be a maximal linearly independent subset of the $f_j, j = 0, ..., n$, since $n \geq 1$ and $gcd(f_i, f_j, f_k) = 1$ for all distinct $i, j, k \in \{0, ..., n+1\}$ it follows that $q \geq 2$. Then each $f_j, 0 \leq j \leq n, j$ not one of $i_k$, is a linear combination of the $f_{i_k}$, of the form

$$f_j = \lambda_1 f_{j_1} + \cdots + \lambda_q f_{i_q} \quad (6)$$

where the $\lambda_k \in F$, and at least two of these $\lambda_k$ are not zero. Using our inductive hypothesis we apply the theorem to (6). This yields that if $\lambda_k \neq 0$ then

$$\text{deg}(f_{i_k}) \leq (2q - 1)(r(f_j \prod_{k=1}^{q} f_{i_k}) - 1). \quad (7)$$

So that

$$\text{deg}(f_{i_k}) \leq (2n - 1)(r(\prod_{k=0}^{n+1} f_k) - 1). \quad (8)$$

From (6) the same estimate as in (8) follows for $\text{deg} f_j$. Thus the theorem is proved for such $f_j$ and $f_{i_k}$. Inserting all the relations of the form (6) into the right side of (5) yields an equation of the form

$$f_{n+1} = \kappa_1 f_{j_1} + \cdots + \kappa_q f_{i_q}, \quad (9)$$

where the $\kappa_j \in F$. Moreover, if one of these $\kappa_v = 0$ then the corresponding $f_{i_v}$ must have appeared in one of the equations (6) with a non-zero $\lambda_v$. Hence (7) has been established for this $f_{i_v}$. Finally, for those $\kappa_v \neq 0$, we treat (9) exactly as we did (6), (note that $q + 1 < n + 1$), and obtain the estimate (8) for $\text{deg} f_{i_v}$, and for $\text{deg} f_{n+1}$. This completes the induction.

References


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