A Note on Browkin-Brzezinski's Conjecture

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Abstract

In this paper we prove Browkin-Brzezinski's conjecture for a class of polynomials.

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1 Introduction

Let F be a fixed algebraically closed field of characteristic 0. Let $f(z)$ be a polynomial non - constants with coefficients in F and let $r(f)$ be the number of distinct zeros of f . Then we have the following.

Mason's theorem ([3]). Let $a(z), b(z), c(z)$ be relatively prime polynomials in F and not all constants such that $a + b = c$. Then

$$\max \{ \deg(a), \deg(b), \deg(c) \} \leq r(abc) - 1.$$

For different purposes some generalizations of the theorem are given (see [2, 4]). In [1], Browkin and Brzezinski suggested the following conjecture.

Conjecture 1.1 *Let $f_0, \dots, f_{n+1}$ be $n + 2$ polynomials not all constants in $F[x]$ and have no common zeros such that*

$$f_0 + \dots + f_n + f_{n+1} = 0. \tag{1}$$

Then

$$\max_{0 \leq j \leq n+1} \deg(f_j) \leq (2n - 1) (r(f_0 \cdots f_{n+1}) - 1).$$

In this paper we prove the following theorem.

Theorem 1.1 *Browkin-Brzezinski's conjecture is true for polynomials $f_0, \dots, f_{n+1}$ such that $\gcd(f_i, f_j, f_k) = 1$ for all distinct $i, j, k \in \{0, \dots, n + 1\}$.*

2 Proof of the main theorem

Let f is a rational function, we write f in the form:

$$f = \frac{f_1}{f_2},$$

Where f_1, f_2 are polynomial functions are non-zero and relatively prime on $F[x]$. The degree of f , denoted by $\deg f$, is define to be $\deg f_1 - \deg f_2$.

Let $a \in F$, we write f in the form:

$$f = (x - a)^l \frac{g_1}{g_2},$$

and $g_1(a)g_2(a) \neq 0$, then l is called the *order of f at a* and is denoted by μ_f^a . We have the following easily proved properties of μ_f^a .

Lemma 2.1 *Let f, g be two polynomials and $a \in F$, we have*

- a) $\mu_{f+g}^a \geq \min(\mu_f^a, \mu_g^a)$,
- b) $\mu_{fg}^a = \mu_f^a + \mu_g^a$,
- c) $\mu_{\frac{f}{g}}^a = \mu_f^a - \mu_g^a$,

Lemma 2.2 *Let φ be a the rational function on F and let the derivatives order k of φ satisfy the following $\varphi^{(k)} \not\equiv 0$. Then*

$$\mu_{\varphi^{(k)}}^\alpha \geq -k + \mu_\varphi^\alpha.$$

Proof. Let $\varphi(x) = (x - \alpha)^m \frac{f(x)}{g(x)}$, where $f(x), g(x)$ are relatively prime and $f(\alpha)g(\alpha) \neq 0$. Then, we have

$$\varphi'(x) = (x - \alpha)^{m-1} \frac{mf(x)g(x) + (x - \alpha)(f'(x)g(x) - f(x)g'(x))}{g^2(x)}.$$

By $\mu_g^\alpha = 0$, we have

$$\mu_{\varphi'}^\alpha \geq m - 1.$$

Therefore

$$\mu_{\varphi'}^\alpha \geq -1 + \mu_\varphi^\alpha.$$

From this we obtain

$$\mu_{\varphi^{(k)}}^\alpha \geq -k + \mu_\varphi^\alpha.$$

Lemma 2.3 *Browkin-Brzezinski's conjecture is true for polynomials $f_0, \dots, f_{n+1}$ such that $\gcd(f_i, f_j, f_k) = 1$ for all distinct $i, j, k \in \{0, \dots, n+1\}$ and $f_0, \dots, f_n$ are linearly independent.*

Proof. By the hypothesis $f_0, \dots, f_n$ are linearly independent, we have the Wronskian W of $f_0, \dots, f_n$ does not vanish. We set

$$P = \frac{W(f_0, \dots, f_n)}{f_0 \cdots f_n},$$

$$Q = \frac{f_0 \cdots f_{n+1}}{W(f_0, \dots, f_n)}.$$

Hence we have

$$f_{n+1} = PQ. \quad (2)$$

We first prove that

$$\deg Q \leq (2n - 1)r(f_0 \cdots f_{n+1}).$$

Suppose that α is a zero of $f_0 f_1 \cdots f_{n+1}$, by the hypothesis there exists $\nu, 0 \leq \nu \leq n + 1$ such that $f_\nu \neq 0$. By the hypothesis $f_0 + \cdots + f_n = f_{n+1}$ we have

$$\begin{aligned} \mu_{\frac{f_0 \cdots f_{n+1}}{W(f_0, \dots, f_n)}}^\alpha &= \mu_{\frac{f_0 \cdots f_{\nu-1} f_{\nu+1} \cdots f_{n+1}}{W(f_0, \dots, f_{\nu-1}, f_{\nu+1}, \dots, f_{n+1})}}^\alpha \\ &= \sum_{j=0}^{n+1} \mu_{f_j}^\alpha - \mu_{W(f_0, \dots, f_{\nu-1}, f_{\nu+1}, \dots, f_{n+1})}^\alpha \end{aligned}$$

$W(f_0, \dots, f_{\nu-1}, f_{\nu+1}, \dots, f_{n+1})$ is the sum of follow terms

$$\delta f_{\alpha_0} (f_{\alpha_1})' \cdots (f_{\alpha_n})^{(n)},$$

Where $\alpha_i \in \{0, \dots, n + 1\} \setminus \{\nu\}, \delta = \pm 1$. We suppose there exists k functions f_j such that $f_j(\alpha) = 0$. By the hypothesis $\gcd(f_i, f_j, f_k) = 1$ for all distinct $i, j, k \in \{0, \dots, n + 1\}$ we have $k \leq 2$. From this and Lemma 2.1 and we have

$$\begin{aligned} &\mu_{f_{\alpha_0} (f_{\alpha_1})' \cdots (f_{\alpha_n})^{(n)}}^\alpha \\ &\geq \sum_{f_j(\alpha)=0} \mu_{f_{\alpha_j}}^\alpha - (n + (n - 1)) \\ &= \mu_{\prod_{j=0}^{n+1} f_{\alpha_j}}^\alpha - (2n - 1) \end{aligned}$$

By Lemma 2.2 we have

$$\mu_{W(f_0, \dots, f_{\nu-1}, f_{\nu+1}, \dots, f_{n+1})}^\alpha \geq \mu_{\prod_{j=0}^{n+1} f_{\alpha_j}}^\alpha - (2n - 1).$$

Hence

$$\mu_{\frac{f_0 \cdots f_{n+1}}{W(f_0, \dots, f_n)}}^\alpha \leq 2n - 1,$$

By the definition of degree of a rational function, we have:

$$\deg Q \leq (2n-1)r(f_0 \cdots f_{n+1}). \quad (3)$$

Next, we will prove that

$$\deg P \leq -\frac{n(n+1)}{2}.$$

At here, we have P as the logarithmic Wronskian corresponding to $I = \{0, 1, \dots, n\}$ which is

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \frac{f'_0}{f_0} & \frac{f'_1}{f_1} & \cdots & \frac{f'_n}{f_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{f_0^{(n)}}{f_0} & \frac{f_1^{(n)}}{f_1} & \cdots & \frac{f_n^{(n)}}{f_n} \end{vmatrix}$$

The determinant P is a summa of following terms

$$\delta \frac{f'_{\beta_1} f''_{\beta_2} \cdots f_{\beta_n}^{(n)}}{f_{\beta_1} f_{\beta_2} \cdots f_{\beta_n}}.$$

For every term, we have

$$\begin{aligned} \deg \left(\frac{f'_{\beta_1} f''_{\beta_2} \cdots f_{\beta_n}^{(n)}}{f_{\beta_1} f_{\beta_2} \cdots f_{\beta_n}} \right) &= \deg \left(\frac{f'_{\beta_1}}{f_{\beta_1}} \right) + \deg \left(\frac{f''_{\beta_2}}{f_{\beta_2}} \right) + \cdots + \deg \left(\frac{f_{\beta_n}^{(n)}}{f_{\beta_n}} \right) \\ &= -(1 + 2 + \cdots + n) = -\frac{n(n+1)}{2}. \end{aligned}$$

Therefore

$$\deg P \leq -\frac{n(n+1)}{2}. \quad (4)$$

From (2), (3), (4) we have

$$\begin{aligned} \deg f_{n+1} &= \deg P + \deg Q \leq (2n-1)r(f_0 \cdots f_{n+1}) - \frac{n(n+1)}{2} \\ &\leq (2n-1)(r(f_0 \cdots f_{n+1}) - 1). \end{aligned}$$

Similar arguments apply to the polynomial $f_0, f_1, \dots, f_n$, we have

$$\max_{0 \leq i \leq n+1} (\deg f_i) \leq (2n-1)(r(f_0 \cdots f_{n+1}) - 1).$$

Proof of theorem 1.1. The proof proceed by induction on n . For $n = 1$, it is true by Mason's theorem. Suppose that the theorem is true for all case $m, 1 \leq m \leq n-1$. If $f_0, \dots, f_n$ are linearly independent, then this is Lemma 2.3. If $f_0, \dots, f_n$ are linearly dependent, rewriting (1) as

$$-f_{n+1} = f_0 + \cdots + f_n, \quad (5)$$

Let $f_{i_1}, \dots, f_{i_q}, q < n+1$ be a maximal linearly independent subset of the $f_j, j = 0, \dots, n$, since $n \geq 1$ and $\gcd(f_i, f_j, f_k) = 1$ for all distinct $i, j, k \in \{0, \dots, n+1\}$ it follows that $q \geq 2$. Then each $f_j, 0 \leq j \leq n, j$ not one of i_k , is a linear combination of the f_{i_k} , of the form

$$f_j = \lambda_1 f_{i_1} + \dots + \lambda_q f_{i_q} \quad (6)$$

where the $\lambda_k \in F$, and at least two of these λ_k are not zero. Using our inductive hypothesis we apply the theorem to (6). This yields that if $\lambda_k \neq 0$ then

$$\deg(f_{i_k}) \leq (2q-1)(r(f_j \prod_{k=1}^q f_{i_k}) - 1). \quad (7)$$

So that

$$\deg(f_{i_k}) \leq (2n-1)(r(\prod_{k=0}^{n+1} f_k) - 1). \quad (8)$$

From (6) the same estimate as in (8) follows for $\deg f_j$. Thus the theorem is proved for such f_j and f_{i_k} . Inserting all the relations of the form (6) into the right side of (5) yields an equation of the form

$$f_{n+1} = \kappa_1 f_{i_1} + \dots + \kappa_q f_{i_q}, \quad (9)$$

where the $\kappa_j \in F$. Moreover, if one of these $\kappa_v = 0$ then the corresponding f_{i_v} must have appeared in one of the equations (6) with a non-zero λ_v . Hence (7) has been established for this f_{i_v} . Finally, for those $\kappa_v \neq 0$, we treat (9) exactly as we did (6), (note that $q+1 < n+1$), and obtain the estimate (8) for $\deg f_{i_v}$, and for $\deg f_{n+1}$. This completes the induction.

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