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On λ-Nuclear Maps

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Abstract

In this paper, we prove that if the map T is a λ-nuclear map from a Banach space E into a Banach space F and if F contains no (isomorphic) copy of c_0, then T is a compact map. Also, we give an example to show that the assumption that F does not contain a copy of c_0 is essential.

Keywords: sequence spaces, λ-nuclear map, convergent and unconditionally convergent, compact map

1 Basic Concepts.

A sequence space is a vector space of sequences of scalars (real or complex) which includes every finitely non-zero sequence. If λ is a sequence space then its **Köthe dual** is the sequence space

$$\lambda^{\times} = \{\zeta : \textstyle\sum_n |\eta_n \zeta_n| < +\infty \quad \forall \eta \in \lambda\}.$$

A linear map T from a normed space E into a normed space F is called a **λ-nuclear map** if there exist a sequence (α_n) in λ and sequences (a_n) and (y_n) in E' and F respectively such that (a_n) is bounded and (y_n) has the property that for each $b \in F'$, $(\langle y_n, b\rangle) \in \lambda^{\times}$ and such that

$$Tx = \textstyle\sum_n \alpha_n \langle x, a_n\rangle y_n,$$

for all $x \in E$ [2,3].

We say that a Banach space E has the **Schur property** if every weakly convergent sequence in E is norm convergent.

A series $\sum_n x_n$ in a Banach space E is called **weakly unconditionally Cauchy** if

$$\textstyle\sum_n |\langle x_n, a\rangle| < +\infty,$$

for all $a \in E'$.

The following well-known results in Banach space theory are crucial for our subsequent arguments.

Theorem 1.1 [1] *Every weakly unconditionally Cauchy series* $\sum_n x_n$ *in a Banach space* E *is unconditionally convergent if and only if* E *contains no (isomorphic) copy of* c_0.

Theorem 1.2 [1] *(Schur's Theorem) If a sequence* (x_n) *in* ℓ_1 *is weakly Cauchy (which means that* $\lim_n \langle x_n, y\rangle$ *exists for each* $y \in \ell_\infty$*), then* (x_n) *is norm convergent.*

Remark. According to Schur's Theorem, the space ℓ_1 has Schur's property.

2 Main results.

We start our work by proving the following technical Lemma.

Lemma 2.1 *Let* $\sum_n x_n$ *be an unconditionally convergent series in a Banach space* E*. Define a map* $S : \ell_\infty \to E$ *by putting* $S\eta = \sum_n \eta_n x_n$*. Then* S *is a compact map. In fact, we have* $\lim_n ||S - S_n|| = 0$*, where* S_n *is the finite rank linear map from* ℓ_∞ *to* E *defined by* $S_n\eta = \sum_{k=1}^n \eta_k x_k$.

Proof. To prove this, let $R : c_0 \to E$ be the restriction of S to the subspace c_0 of ℓ_∞. Then its adjoint $R' : E' \to \ell_1$ is given by $R'a = (\langle x_n, a\rangle)$. Indeed, writing $\eta_n = \langle x_n, a\rangle$, we have, for all $\zeta \in c_0$,

$$\begin{aligned} \langle \zeta, R'a\rangle &= \langle R\zeta, a\rangle \\ &= \sum\nolimits_n \zeta_n \langle x_n, a\rangle \\ &= \sum\nolimits_n \zeta_n \eta_n = \langle \zeta, \eta\rangle, \end{aligned}$$

and hence $R'a = \eta$. Clearly R' is continuous when both E' and ℓ_1 are equipped with weak*-topology. We claim that R' is also continuous when E' is equipped with weak*-topology while ℓ_1 is equipped with the weak topology. To establish this claim, it suffices to check that, for each $\zeta \in \ell_\infty$. The linear functional $a \mapsto \langle \zeta, R'a\rangle$ defined on E' is continuous with respect to the weak*-topology. This follows immediately from the identity $\langle \zeta, R'a\rangle = \langle S\zeta, a\rangle$ which is easily checked as follows:

$$\begin{aligned} \langle \zeta, R'a\rangle &= \sum\nolimits_n \zeta_n \langle x_n, a\rangle \\ &= \left\langle \sum\nolimits_n \zeta_n x_n, a \right\rangle = \langle S\zeta, a\rangle \end{aligned}$$

Now, the closed unit ball E'_1 is compact with respect to the weak*-topology. So, by the continuity of R' we have proved, its image $R'E'_1$ in ℓ_1 is weakly compact. By a version of Schur's Theorem, we see that $R'E'_1$ is compact in the norm topology. This shows that R' is compact. Hence its adjoint $R'' : \ell_\infty \to E''$ is also compact. Clearly $S : \ell_\infty \to E$ and $R'' : \ell_\infty \to E''$ coincide on ℓ_∞. So S is also compact. Let $R_n : \ell_1 \to \ell_1$ be the projection defined by

$$R_n\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n, 0, 0, \ldots).$$

Using the compactness of $R'E_1'$ in the norm topology we can show that $\sup_{\zeta\in R'E_1'} ||\zeta - R_n\zeta|| \to 0$ as $n \to \infty$. Thus $\lim_n ||R' - R_nR'|| = 0$, which gives $\lim_n ||R'' - R''R_n'|| = 0$. It is easy to check that $R''R_n'$ coincides with S_n on ℓ_∞. Therefore $\lim_n ||S - S_n|| = 0$. ■

The following result is essential in proving our main result.

Lemma 2.2 [2] *Let F be a Banach space, λ sequence space and $(\alpha_n) \in \lambda$. Let (y_n) be a sequence in F such that $(\langle y_n, b\rangle) \in \lambda^\times$ for each $b \in F'$. Then*

$$\sup_{b\in F',||b||\le 1} \sum\nolimits_n |\alpha_n \langle y_n, b\rangle| < +\infty.$$

Remark. Let (α_n) be an element of a sequence space λ, (y_n) be a sequence in a Banach space F, and $(\langle y_n, b\rangle) \in \lambda^\times$ for all $b \in F$. Then by using Lemma 2.1, and Theorem 1.1, we know that series $\sum_n \alpha_n y_n$ is unconditionally convergent if F has no copy of c_0.

To this end, we have furnished the necessary back ground to give our main result.

Theorem 2.1 *If T is a λ-nuclear map from a Banach space E into a Banach space F and if F contains no copy of c_0, then T is a compact map.*

Proof. Since T is a λ-nuclear map, there exists a sequence (α_n) in λ and sequences (a_n) and (y_n) in E' and F respectively such that (a_n) is bounded in E' and $(\langle y_n, b\rangle) \in \lambda^\times$ for all $b \in F'$ and such that

$$Tx = \sum\nolimits_n \alpha_n \langle x, a_n\rangle y_n.$$

Define linear maps $R : \lambda \to F$, $L : E \to \ell_\infty$ and $D\alpha : \ell_\infty \to \lambda$ by putting $R\zeta = \sum_n \zeta_n y_n$, $Lx = (\langle x, a_n\rangle)$, and $D\alpha\eta = (\alpha_n\eta_n)$. Then clearly $T = RD\alpha L$. Since $\sum_n \alpha_n y_n$ is unconditionally convergent series in F, by Lemma 2.1, the linear map from ℓ_∞ to F sending η to $\sum_n \eta_n\alpha_n y_n$ is compact. It is easy to check that $RD\alpha\eta = \sum_n \alpha_n\eta_n y_n$ and hence $RD\alpha$ is compact. Therefore T is compact as well. ■

Now we give an example to show that the assumption that F does not contain a copy of c_0 in Theorem 2.1 is essential.

Example 2.1 *Let $E = F = c_0$ and $\lambda = \ell_\infty$. Let T be the identity map on c_0. Then T is λ-nuclear map which is noncompact.*

Proof. Let (e_n) denoted to the standard base of c_0 and (e'_n) denoted to the standard base of ℓ_1. Then $Tx = \sum_n \alpha_n \langle x, a_n\rangle y_n$, where $\alpha_n = 1$, $a_n = e'_n$, and $y_n = e_n$. Notice that $\alpha \in \ell_\infty = \lambda$, (a_n) is bounded sequence in $c'_0 = \ell_1$ and $(\langle y_n, b\rangle) \in \lambda^\times = \ell_1$ for all $b \in c'_0 = \ell_1$. Therefore T is λ-nuclear. Clearly T is noncompact. ■

Our main result is a generalization to the following result.

Corollary 2.1 [1] *Every λ-nuclear map from a Banach space E into a reflexive Banach space F is compact.*

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