On $\lambda$-Nuclear Maps

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Abstract

In this paper, we prove that if the map $T$ is a $\lambda$-nuclear map from a Banach space $E$ into a Banach space $F$ and if $F$ contains no (isomorphic) copy of $c_0$, then $T$ is a compact map. Also, we give an example to show that the assumption that $F$ does not contain a copy of $c_0$ is essential.

Keywords: sequence spaces, $\lambda$-nuclear map, convergent and unconditionally convergent, compact map

1 Basic Concepts.

A sequence space is a vector space of sequences of scalars (real or complex) which includes every finitely non-zero sequence. If $\lambda$ is a sequence space then its Köthe dual is the sequence space

$$\lambda^\times = \{ \zeta : \sum_n |\eta_n \zeta_n| < +\infty \ \forall \eta \in \lambda \}.$$ 

A linear map $T$ from a normed space $E$ into a normed space $F$ is called a $\lambda$-nuclear map if there exist a sequence $(\alpha_n)$ in $\lambda$ and sequences $(a_n)$ and $(y_n)$ in $E'$ and $F$ respectively such that $(a_n)$ is bounded and $(y_n)$ has the property that for each $b \in F'$, $(\langle y_n, b \rangle) \in \lambda^\times$ and such that

$$Tx = \sum_n \alpha_n \langle x, a_n \rangle y_n,$$

for all $x \in E$ [2,3].

We say that a Banach space $E$ has the Schur property if every weakly convergent sequence in $E$ is norm convergent.

A series $\sum_n x_n$ in a Banach space $E$ is called weakly unconditionally Cauchy if

$$\sum_n |\langle x_n, a \rangle| < +\infty,$$

for all $a \in E'$.

The following well-known results in Banach space theory are crucial for our subsequent arguments.
Theorem 1.1 [1] Every weakly unconditionally Cauchy series $\sum_n x_n$ in a Banach space $E$ is unconditionally convergent if and only if $E$ contains no (isomorphic) copy of $c_0$.

Theorem 1.2 [1] (Schur’s Theorem) If a sequence $(x_n)$ in $\ell_1$ is weakly Cauchy (which means that $\lim_n \langle x_n, y \rangle$ exists for each $y \in \ell_\infty$), then $(x_n)$ is norm convergent.

Remark. According to Schur’s Theorem, the space $\ell_1$ has Schur’s property.

2 Main results.

We start our work by proving the following technical Lemma.

Lemma 2.1 Let $\sum_n x_n$ be an unconditionally convergent series in a Banach space $E$. Define a map $S : \ell_\infty \to E$ by putting $S\eta = \sum_n \eta_n x_n$. Then $S$ is a compact map. In fact, we have $\lim_n ||S - S_n|| = 0$, where $S_n$ is the finite rank linear map from $\ell_\infty$ to $E$ defined by $S_n\eta = \sum_{k=1}^n \eta_k x_k$.

Proof. To prove this, let $R : c_0 \to E$ be the restriction of $S$ to the subspace $c_0$ of $\ell_\infty$. Then its adjoint $R' : E' \to \ell_1$ is given by $R'a = (\langle x_n, a \rangle)$. Indeed, writing $\eta_n = \langle x_n, a \rangle$, we have, for all $\zeta \in c_0$,

$$\langle \zeta, R'a \rangle = \langle R\zeta, a \rangle = \sum_n \zeta_n \langle x_n, a \rangle = \sum_n \zeta_n \eta_n = \langle \zeta, \eta \rangle,$$

and hence $R'a = \eta$. Clearly $R'$ is continuous when both $E'$ and $\ell_1$ are equipped with weak*-topology. We claim that $R'$ is also continuous when $E'$ is equipped with weak*-topology while $\ell_1$ is equipped with the weak topology. To establish this claim, it suffices to check that, for each $\zeta \in \ell_\infty$. The linear functional $a \mapsto \langle \zeta, R'a \rangle$ defined on $E'$ is continuous with respect to the weak*-topology. This follows immediately from the identity $\langle \zeta, R'a \rangle = \langle S\zeta, a \rangle$ which is easily checked as follows:

$$\langle \zeta, R'a \rangle = \sum_n \zeta_n \langle x_n, a \rangle = \langle \sum_n \zeta_n x_n, a \rangle = \langle S\zeta, a \rangle.$$

Now, the closed unit ball $E'_1$ is compact with respect to the weak*-topology. So, by the continuity of $R'$ we have proved, its image $R'E'_1$ in $\ell_1$ is weakly compact. By a version of Schur’s Theorem, we see that $R'E'_1$ is compact in the norm topology. This shows that $R'$ is compact. Hence its adjoint $R'' : \ell_\infty \to E''$ is
also compact. Clearly \( S : \ell_\infty \rightarrow E \) and \( R'' : \ell_\infty \rightarrow E'' \) coincide on \( \ell_\infty \). So \( S \) is also compact. Let \( R_n : \ell_1 \rightarrow \ell_1 \) be the projection defined by

\[
R_n \zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n, 0, 0, \ldots).
\]

Using the compactness of \( R'E'_1 \) in the norm topology we can show that \( \sup_{\zeta \in R'E'_1} ||\zeta - R_n \zeta|| \rightarrow 0 \) as \( n \rightarrow \infty \). Thus \( \lim_n ||R' - R'_n|| = 0 \), which gives \( \lim_n ||R'' - R''R'_n|| = 0 \). It is easy to check that \( R''R'_n \) coincides with \( S_n \) on \( \ell_\infty \). Therefore \( \lim_n ||S - S_n|| = 0 \).

The following result is essential in proving our main result.

**Lemma 2.2** [2] Let \( F \) be a Banach space, \( \lambda \) sequence space and \((\alpha_n) \in \lambda\). Let \((y_n)\) be a sequence in \( F \) such that \( (\langle y_n, b \rangle) \in \lambda^\times \) for each \( b \in F' \). Then

\[
\sup_{b \in F', ||b|| \leq 1} \sum_n |\alpha_n(y_n, b)| < +\infty.
\]

**Remark.** Let \((\alpha_n)\) be an element of a sequence space \( \lambda \), \((y_n)\) be a sequence in a Banach space \( F \), and \( (\langle y_n, b \rangle) \in \lambda^\times \) for all \( b \in F \). Then by using Lemma 2.1, and Theorem 1.1, we know that series \( \sum_n \alpha_n y_n \) is unconditionally convergent if \( F \) has no copy of \( c_0 \).

To this end, we have furnished the necessary back ground to give our main result.

**Theorem 2.1** If \( T \) is a \( \lambda \)-nuclear map from a Banach space \( E \) into a Banach space \( F \) and if \( F \) contains no copy of \( c_0 \), then \( T \) is a compact map.

**Proof.** Since \( T \) is a \( \lambda \)-nuclear map, there exists a sequence \((\alpha_n)\) in \( \lambda \) and sequences \((a_n)\) and \((y_n)\) in \( E' \) and \( F \) respectively such that \((\alpha_n)\) is bounded in \( E' \) and \( (\langle y_n, b \rangle) \in \lambda^\times \) for all \( b \in F' \) and such that

\[
Tx = \sum_n \alpha_n \langle x, a_n \rangle y_n.
\]

Define linear maps \( R : \lambda \rightarrow F \), \( L : E \rightarrow \ell_\infty \) and \( D\alpha : \ell_\infty \rightarrow \lambda \) by putting \( R\zeta = \sum_n \zeta_n y_n \), \( Lx = (\langle x, a_n \rangle) \), and \( D\alpha \eta = (\alpha_n \eta_n) \). Then clearly \( T = RD\alpha L \).

Since \( \sum_n \alpha_n y_n \) is unconditionally convergent series in \( F \), by Lemma 2.1, the linear map from \( \ell_\infty \) to \( F \) sending \( \eta \) to \( \sum_n \eta_n \alpha_n y_n \) is compact. It is easy to check that \( RD\alpha \eta = \sum_n \alpha_n \eta_n y_n \) and hence \( RD\alpha \) is compact. Therefore \( T \) is compact as well.

Now we give an example to show that the assumption that \( F \) does not contain a copy of \( c_0 \) in Theorem 2.1 is essential.
Example 2.1 Let $E = F = c_0$ and $\lambda = \ell_\infty$. Let $T$ be the identity map on $c_0$. Then $T$ is $\lambda$-nuclear map which is noncompact.

Proof. Let $(e_n)$ denoted to the standard base of $c_0$ and $(e'_n)$ denoted to the standard base of $\ell_1$. Then $Tx = \sum_n \alpha_n \langle x, a_n \rangle y_n$, where $\alpha_n = 1$, $a_n = e'_n$, and $y_n = e_n$. Notice that $\alpha \in \ell_\infty = \lambda$, $(a_n)$ is bounded sequence in $c'_0 = \ell_1$ and $(\langle y_n, b \rangle) \in \lambda^* = \ell_1$ for all $b \in c'_0 = \ell_1$. Therefore $T$ is $\lambda$-nuclear. Clearly $T$ is noncompact.

Our main result is a generalization to the following result.

Corollary 2.1 [1] Every $\lambda$-nuclear map from a Banach space $E$ into a reflexive Banach space $F$ is compact.

References


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