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A New Curvature of Surface

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Abstract

The purpose of this paper is to show that a classical approach of the definition of curvature associated to a regular point of a surface, leads to give in a natural way a new fundamental form, the so called the third form.

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1 Introduction

Inspired by the definition of a curvature of a curve at point M_0, given by $K = \dfrac{\theta}{\|\overrightarrow{M_0M}\|}$ where θ is the angle between the two tangent lines at M_0 and M (M is infinitely close to M_0), we give the same approach for a regular surface (S), by considering the ratio $\dfrac{\theta}{\|\overrightarrow{M_0M}\|}$ where, in this case, θ represents the angle between the two tangent plans at M_0 and M.

2 Parametrical surfaces

We consider a surface S embeded in the euclidean space$\mathbb{R}^3$. we suppose that there exists a regular card f of an open set $\hat{S}$ of $\mathbb{R}^2$ provided with the orthonormal system (o, u, v) with values in $\mathbb{R}^3$, where $\mathbb{R}^3$ is provided by the orthonormal system (o, x, y, z) such that $S = f(\hat{S}\ )$.

We consider a curvilinear coordinates system on S, $m \in \hat{S}$ and $M \in S$ with

$$M = f(m) = f(u,v) = \begin{cases} x(u,v) \\ y(u,v) \\ z(u,v) \end{cases}$$

For each point $M \in S$, we $a_1(m) = \dfrac{\partial f}{\partial u}(u,v)$ and $a_2(m) = \dfrac{\partial f}{\partial v}(u,v)$

The vectors $a_1(m)$, $a_2(m)$ are supposed to be linearly independent for any point $M \in S$. They generate the tangent plan at M, denoted $T_M(S)$. For each point $M \in S$, we define the normal given by

$$\overrightarrow{\mathcal{N}} = a_1(m) \wedge a_2(m)$$

2.1 Theorem

Let $M_0 = f(m_0)$ be a point and let $M = f(m)$ be any point of the surface, infinitely close to M_0 and θ the angle between the two tangent plans at M_0 and M.

The ratio $\dfrac{\theta}{\|\overrightarrow{M_0M}\|}$ have a limit when M tends to M_0.

Proof

We denote by $a_1 = a_1(m_0), a_2 = a_2(m_0)$ and $\overrightarrow{\mathcal{N}}_0 = a_1(m_0) \wedge a_2(m_0)$.

We have :

$$\theta \simeq tg\theta = \frac{\|\overrightarrow{\mathcal{N}_0} \wedge \overrightarrow{\mathcal{N}}\|}{\|\overrightarrow{\mathcal{N}_0}\| \, \|\overrightarrow{\mathcal{N}}\|} \qquad \text{and} \qquad \|\overrightarrow{M_0M}\| = \| f(m) - f(m_0)\|$$

Now we develop $a_1(m)$, $a_2(m)$ and $\overrightarrow{\mathcal{N}}$:

$$a_1(m) = a_1 + \frac{\partial^2 f}{\partial u^2} u + \frac{\partial^2 f}{\partial u \partial v} v + \cdots$$

$$a_2(M) = a_2 + \frac{\partial^2 f}{\partial u \partial v} u + \frac{\partial^2 f}{\partial v^2} v + \cdots$$

$$\overrightarrow{\mathcal{N}} = a_1 \wedge a_2 + (a_1 \wedge \frac{\partial^2 f}{\partial u \partial v} - a_2 \wedge \frac{\partial^2 f}{\partial u^2})u + (a_1 \wedge \frac{\partial^2 f}{\partial v^2} - a_2 \wedge \frac{\partial^2 f}{\partial u \partial v})v + \cdots$$

Then we form the vector product;

$$\overrightarrow{\mathcal{N}_0} \wedge \overrightarrow{\mathcal{N}} = \left[(a_1, a_2, \frac{\partial^2 f}{\partial u \partial v})a_1 - (a_1, a_2, \frac{\partial^2 f}{\partial u^2})a_2\right] u + \left[(a_1, a_2, \frac{\partial^2 f}{\partial v^2})a_1 - (a_1, a_2, \frac{\partial^2 f}{\partial u \partial v})a_2\right] v + \cdots$$

By using the classical notations

$E = \|a_1\|^2$, $F = a_1 \cdot a_2$, $G = \|a_2\|^2$

$L = \frac{1}{\|\mathcal{N}_0\|}(a_1, a_2, \frac{\partial^2 f}{\partial u^2})$, $M = \frac{1}{\|\mathcal{N}_0\|}(a_1, a_2, \frac{\partial^2 f}{\partial u \partial v})$, $N = \frac{1}{\|\mathcal{N}_0\|}(a_1, a_2, \frac{\partial^2 f}{\partial v^2})$

We obtain :

$$\frac{\left\|\overrightarrow{\mathcal{N}_0} \wedge \overrightarrow{\mathcal{N}}\right\|^2}{\|\mathcal{N}_0\|^2} = G(Lu - Mv)^2 + E(Mu - Nv)^2 - 2F(MLu^2 + (M^2 + LN)uv + MNv^2) + ...$$

and $\|\overrightarrow{M_0 M}\|^2 = Eu^2 + Fuv + Gv^2 + \cdots$

We have the fisrt fundamental form

$$\Pi_1 = Eu^2 + Fuv + Gv^2$$

Which gives the equivalence

$$\left(\frac{\theta}{\|\overrightarrow{M_0 M}\|}\right)^2$$

$$\approx \frac{G(Lu - Mv)^2 + E(Mu - Nv)^2 - 2F(MLu^2 + (M^2 + LN)uv + MNv^2)}{(EG - F^2)(Eu^2 + Fuv + Gv^2)}$$

2.2 Definition 1

We say a new fundamental form, the expression given by

$$\Pi_3 = \frac{G(Lu - Mv)^2 + E(Mu - Nv)^2 - 2F(MLu^2 + (M^2 + LN)uv + MNv^2)}{EG - F^2}$$

2.3 Definition 2

We say the curvature in a regular point of a surface the expression $K = \lim_{M\longrightarrow M_0}\left(\frac{\theta}{\|\overrightarrow{M_0M}\|}\right)^2 = \frac{\Pi_3}{\Pi_1}$

$$\frac{G(Lu-Mv)^2+E(Mu-Nv)^2-2F(MLu^2+(M^2+LN)uv+MNv^2)}{EG-F^2}$$

3 Cartesian surfaces

When surface is defined by its cartesian equation $z = g(x, y)$, the Monge notations are the following

$$p = \frac{\partial g}{\partial x} \quad , \quad q = \frac{\partial g}{\partial y}\,, \quad r = \frac{\partial^2 g}{\partial x^2}\,, \quad s = \frac{\partial^2 g}{\partial x \partial y} \quad , \quad t = \frac{\partial^2 g}{\partial y^2}$$

$$a_1 = (1, 0,\ p) \qquad a_2 = (0, 1,\ q) \qquad \mathcal{N} = (-p, -q, 1)$$

$$\frac{\partial^2 f}{\partial u^2} = (0, 0, r) \qquad \frac{\partial^2 f}{\partial u \partial v} = (0, 0, s) \qquad \frac{\partial^2 f}{\partial v^2} = (0, 0, t)$$

$$L = \frac{r}{\sqrt{1+p^2+q^2}} \qquad M = \frac{s}{\sqrt{1+p^2+q^2}} \qquad N = \frac{t}{\sqrt{1+p^2+q^2}}$$

Thus, we deduct

$$K = \frac{(1+q^2)(rx-sy)^2+(1+p^2)(sx-ty)^2-2pq(rsx^2+(s^2+rs)xy+y^2)}{(1+p^2+q^2)^4((1+p^2+q^2)x^2+pqxy+(1+q^2)y^2)}$$

3.1 Particular Case:

When surface S is defined by its canonical expression

$$z = \frac{1}{2}(rx^2 + 2sxy + ty^2) + \cdots$$

The first quadratic form is

$$\Pi_1 = x^2 + y^2$$

The new quadratic form is

$$\Pi_3 = (r^2 + s^2)\, x^2 + 2s(r+t)\, xy + (s^2 + t^2) y^2$$

Then, the curvature is the ratio: $K = \dfrac{\Pi_3}{\Pi_1}$

Then, we have the expression of a new curvature

$$K = (r^2 + s^2)\, a^2 + 2s(r+s)\, ab + (s^2 + t^2) b^2$$

With $a^2 + b^2 = 1$, K is a quadratic form, the associated matrix is the as follows

$$M = \begin{pmatrix} r^2 + s^2 & s(r+t) \\ s(r+t) & s^2 + t^2 \end{pmatrix}$$

The equation of the extrema is given by

$$\rho^2 - \left(r^2 + 2s^2 + t^2\right)\ \rho + \left(rt - s^2\right)^2 = 0$$

The new Gauss curvature is the product of both extrema

$$K_G = \left(rt - s^2\right)^2$$

The average curvature (arithmetic mean)

$$K_M = \frac{r^2 + 2s^2 + t^2}{2}$$

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