A New Curvature of Surface

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Abstract
The purpose of this paper is to show that a classical approach of the
definition of curvature associated to a regular point of a surface, leads
to give in a natural way a new fundamental form, the so called the third
form.

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1 Introduction
Inspired by the definition of a curvature of a curve at point $M_0$, given by
$K = \frac{\theta}{\|\overrightarrow{M_0M}\|}$ where $\theta$ is the angle between the two tangent lines at $M_0$ and
$M$ ( $M$ is infinitely close to $M_0$ ), we give the same approach for a regular
surface $(S)$, by considering the ratio $\frac{\theta}{\|\overrightarrow{M_0M}\|}$ where, in this case, $\theta$ represents
the angle between the two tangent plans at $M_0$ and $M$.

2 Parametrical surfaces
We consider a surface $S$ embeded in the euclidean space $\mathbb{R}^3$. we suppose that
there exists a regular card $f$ of an open set $\tilde{S}$ of $\mathbb{R}^2$ provided with the or-
thonormal system $(o, u, v)$ with values in $\mathbb{R}^3$, where $\mathbb{R}^3$ is provided by the
orthonormal system $(o, x, y, z)$ such that $S = f(\tilde{S})$. 
We consider a curvilinear coordinates system on $S$, $m \in \hat{S}$ and $M \in S$ with

$$M = f(m) = f(u, v) = \begin{cases} x(u, v) \\ y(u, v) \\ z(u, v) \end{cases}$$

For each point $M \in S$, we $a_1(m) = \frac{\partial f}{\partial u}(u, v)$ and $a_2(m) = \frac{\partial f}{\partial v}(u, v)$.

The vectors $a_1(m)$, $a_2(m)$ are supposed to be linearly independent for any point $M \in S$. They generate the tangent plane at $M$, denoted $T_M(S)$. For each point $M \in S$, we define the normal given by

$$\vec{N} = a_1(m) \wedge a_2(m)$$

### 2.1 Theorem

Let $M_0 = f(m_0)$ be a point and let $M = f(m)$ be any point of the surface, infinitely close to $M_0$ and $\theta$ the angle between the two tangent planes at $M_0$ and $M$.

The ratio $\frac{\theta}{\|M_0M\|}$ have a limit when $M$ tends to $M_0$.

#### Proof

We denote by $a_1 = a_1(m_0), a_2 = a_2(m_0)$ and $\vec{N}_0 = a_1(m_0) \wedge a_2(m_0)$. We have:

$$\theta \simeq \tan \theta = \frac{\|\vec{N}_0 \wedge \vec{N}\|}{\|\vec{N}_0\| \|\vec{N}\|} \quad \text{and} \quad \|M_0M\| = \|f(m) - f(m_0)\|$$

Now we develop $a_1(m)$, $a_2(m)$ and $\vec{N}$:

$$a_1(m) = a_1 + \frac{\partial^2 f}{\partial u^2} u + \frac{\partial^2 f}{\partial u \partial v} v + \cdots$$

$$a_2(M) = a_2 + \frac{\partial^2 f}{\partial u \partial v} u + \frac{\partial^2 f}{\partial v^2} v + \cdots$$

$$\vec{N} = a_1 \wedge a_2 + (a_1 \wedge \frac{\partial^2 f}{\partial u \partial v} - a_2 \wedge \frac{\partial^2 f}{\partial u^2}) u + (a_1 \wedge \frac{\partial^2 f}{\partial v^2} - a_2 \wedge \frac{\partial^2 f}{\partial u \partial v}) v + \cdots$$
Then we form the vector product;

\[ \vec{N}_0 \wedge \vec{N} = \left[ \left( a_1, a_2, \frac{\partial^2 f}{\partial u \partial v} \right) a_1 - \left( a_1, a_2, \frac{\partial^2 f}{\partial u^2} \right) a_2 \right] u + \left[ \left( a_1, a_2, \frac{\partial^2 f}{\partial v^2} \right) a_1 - \left( a_1, a_2, \frac{\partial^2 f}{\partial u \partial v} \right) a_2 \right] v + \cdots \]

By using the classical notations

\[ E = \| a_1 \|^2, \quad F = a_1 \cdot a_2, \quad G = \| a_2 \|^2 \]

\[ L = \frac{1}{\| \vec{N}_0 \|} (a_1, a_2, \frac{\partial^2 f}{\partial u^2}), \quad M = \frac{1}{\| \vec{N}_0 \|} (a_1, a_2, \frac{\partial^2 f}{\partial u \partial v}), \quad N = \frac{1}{\| \vec{N}_0 \|} (a_1, a_2, \frac{\partial^2 f}{\partial v^2}) \]

We obtain:

\[ \left( \frac{\vec{N}_0 \wedge \vec{N}}{\| \vec{N}_0 \|^2} \right)^2 = G(Lu - Mv)^2 + E(Mu - Nv)^2 - 2F(MLu^2 + (M^2 + LN)uv + Mnv^2) + \cdots \]

and \( \| \vec{M}_0 \vec{M} \|^2 = Eu^2 + Fuv + Gv^2 + \cdots \)

We have the first fundamental form

\[ \Pi_1 = Eu^2 + Fuv + Gv^2 \]

Which gives the equivalence

\[ \left( \frac{\theta}{\| \vec{M}_0 \vec{M} \|} \right)^2 \approx \frac{G(Lu - Mv)^2 + E(Mu - Nv)^2 - 2F(MLu^2 + (M^2 + LN)uv + Mnv^2)}{(EG - F^2)(Eu^2 + Fuv + Gv^2)} \]

### 2.2 Definition 1

We say a new fundamental form, the expression given by

\[ \Pi_3 = \frac{G(Lu - Mv)^2 + E(Mu - Nv)^2 - 2F(MLu^2 + (M^2 + LN)uv + Mnv^2)}{EG - F^2} \]
2.3 Definition 2

We say the curvature in a regular point of a surface the expression

\[ K = \frac{\theta}{\frac{\Phi}{\Phi_0}} \left( \frac{\theta}{\Phi_0} \right)^2 = \Pi_3 \left( \frac{\Phi}{\Phi_0} \right) \frac{G(Lu - Mv)^2 + E(Mu - Nv)^2 - 2F(MLu^2 + (M^2 + LN)uv + MNv^2)}{EG - F^2} \]

3 Cartesian surfaces

When surface is defined by its cartesian equation \( z = g(x, y) \), the Monge notations are the following

\[ p = \frac{\partial g}{\partial x}, \quad q = \frac{\partial g}{\partial y}, \quad r = \frac{\partial^2 g}{\partial x^2}, \quad s = \frac{\partial^2 g}{\partial x \partial y}, \quad t = \frac{\partial^2 g}{\partial y^2} \]

\[ a_1 = (1, 0, p) \quad a_2 = (0, 1, q) \quad N = (-p, -q, 1) \]

\[ \frac{\partial^2 f}{\partial u^2} = (0, 0, r) \quad \frac{\partial^2 f}{\partial u \partial v} = (0, 0, s) \quad \frac{\partial^2 f}{\partial v^2} = (0, 0, t) \]

\[ L = \frac{r}{\sqrt{1 + p^2 + q^2}} \quad M = \frac{s}{\sqrt{1 + p^2 + q^2}} \quad N = \frac{t}{\sqrt{1 + p^2 + q^2}} \]

Thus, we deduct

\[ K = \frac{(1 + q^2)(rx - sy)^2 + (1 + p^2)(sx - ty)^2 - 2pq(rsx^2 + (s^2 + rs)xy + y^2)}{(1 + p^2 + q^2)^4((1 + p^2 + q^2)x^2 + pqxy + (1 + q^2)y^2)} \]

3.1 Particular Case:

When surface \( S \) is defined by its canonical expression

\[ z = \frac{1}{2}(rx^2 + 2sxy + ty^2) + \cdots \]

The first quadratic form is
A new curvature of surface

\[ \Pi_1 = x^2 + y^2 \]

The new quadratic form is

\[ \Pi_3 = (r^2 + s^2) x^2 + 2s(r + t) xy + (s^2 + t^2)y^2 \]

Then, the curvature is the ratio:  \( K = \frac{\Pi_3}{\Pi_1} \)
Then, we have the expression of a new curvature

\[ K = (r^2 + s^2) a^2 + 2s(r + s) ab + (s^2 + t^2)b^2 \]

With \( a^2 + b^2 = 1 \), \( K \) is a quadratic form, the associated matrix is the as follows

\[
M = \begin{pmatrix}
  r^2 + s^2 & s(r + t) \\
  s(r + t) & s^2 + t^2
\end{pmatrix}
\]

The equation of the extrema is given by

\[
\rho^2 - (r^2 + 2s^2 + t^2) \rho + (rt - s^2)^2 = 0
\]

The new Gauss curvature is the product of both extrema

\[ K_G = (rt - s^2)^2 \]

The average curvature (arithmetic mean)

\[ K_M = \frac{r^2 + 2s^2 + t^2}{2} \]

References


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