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An Approximation for the Mumford-Shah Functional

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Abstract

We approximate, in the sense of Γ-convergence, the Mumford-Shah functional by means of a sequence of non-local integral functionals depending on the average of the absolute value of the gradient.

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1 Introduction

In the variational approach to many problems in computer vision (image segmentation, signal processing and so on) an important rôle has been played by the Mumford-Shah functional, which is the most famous example of a free discontinuity functional (terminology introduced by DeGiorgi in [11]). The Mumford-Shah functional is given by

$$MS(u) = \int_\Omega |\nabla u|^2 \, dx + c\mathcal{H}^{n-1}(S_u)$$

where $u \in SBV(\Omega)$, the space of special functions of bounded variation; S_u is the approximate discontinuity set of u and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. Several approximation methods are known for the Mumford-Shah functional and, more in general, for free discontinuity functionals: the Ambrosio & Tortorelli approximation (see [1] and [3]) via elliptic functionals, the Gobbino's approximation by finite difference methods (see [12]) and many others (see [6], [7], [9], [10]).

In [5] Braides & Dal Maso approximate the Mumford-Shah functional by means of a sequence of non-local integral functionals given by

$$F_\varepsilon(u) = \frac{1}{\varepsilon}\int_\Omega f\left(\varepsilon \fint_{B_\varepsilon(x)\cap\Omega} |\nabla u|^2\,dy\right)\,dx \tag{1}$$

with $u \in H^1(\Omega)$ and, for instance, $f(t) = t \wedge 1/2$. A variant of this method is investigated in [14], [15] and [13] where the problem of the convergence of

$$F_\varepsilon(u) = \frac{1}{\varepsilon}\int_\Omega f_\varepsilon\left(\varepsilon \fint_{B_\varepsilon(x)\cap\Omega} |\nabla u|\,dy\right)\,dx \tag{2}$$

is considered; here f_ε is a convex-concave function with $f_\varepsilon(\varepsilon t)/\varepsilon \to \phi(t)$, as $(\varepsilon, t) \to (0, 0)$, where ϕ has linear growth at infinity and plays the rôle of the bulk energy density in the limit of F_ε. Then, under the assumption on f_ε in [13], the Mumford-Shah functional cannot be recovered by the Γ-convergence of F_ε, since the bulk term in MS is given by

$$\int_\Omega |\nabla u|^2\,dx$$

and it has superlinear growth at infinity. A question arise: is it possibile to recover the Mumford-Shah functional from (2) instead of (1)? The aim of this paper is to prove an approximation results for the Mumford-Shah functional, obtained adapting the results contained in [14], [15] and [13], by means of a sequence of functionals of type (2). The core of the proof is Theorem 4.1 where the lower bound for the Γ-limit is optimized by a sup of measures argument, while the upper bound descends from standard density results and general properties of Minkowsky content.

2 Preliminary Notes

Functions of bounded variation. For a thorough treatment of BV functions we refer to [2]. Let Ω be an open subset of $\mathbb{R}^n$; the space $BV(\Omega)$ of real *functions of bounded variation* is the space of the functions $u \in L^1(\Omega)$ whose distributional derivative is representable by a measure $\mathbb{R}^n$-valued measure Du on Ω. We denote by S_u the *approximate discontinuity set* of u and by J_u the set of approximate jump points of u.

For a function $u \in BV(\Omega)$ let $Du = D^a u + D^s u$ be the (Lebesgue) decomposition of Du into absolutely continuous and singular part. We denote by ∇u the density of $D^a u$; the measures $D^j u := D^s u \llcorner J_u$, $D^c u := D^s u \llcorner (\Omega \setminus S_u)$ are called the *jump part* and the *Cantor part* of the derivative, respectively.

We say that a function $u \in BV(\Omega)$ is a *special function of bounded variation* ($u \in SBV(\Omega)$) if $|D^c u|(\Omega) = 0$; moreover we say that a function $u \in L^1(\Omega)$ is a *generalized special function of bounded variation* ($u \in GSBV(\Omega)$) if $u^T := (-T) \vee u \wedge T$ belongs to $SBV(\Omega)$ for every $T \geq 0$. If $u \in GSBV(\Omega)$, the function ∇u given by $\nabla u = \nabla u^T$ for $\mathcal{L}^n$-a.e. on $\{|u| \leq T\}$ turns out to be well-defined. Moreover, the set function $T \mapsto S_{u^T}$ is monotone increasing; therefore, we set $S_u = \bigcup_{T>0} S_{u^T}$.

Supremum of measures. We recall the following useful result from measure theory, which can be found in [4].

Lemma 2.1 (supremum of measures) *Let Ω be an open subset of $\mathbb{R}^n$ and denote by $\mathcal{A}(\Omega)$ the family of its open subsets. Let λ be a positive Borel measure on Ω, and $\mu\colon \mathcal{A}(\Omega) \to [0,+\infty)$ a set function which is superadditive on open sets with disjoint compact closures (i.e. if $A, B \subset\subset \Omega$ and $\overline{A} \cap \overline{B} = \emptyset$, then $\mu(A \cup B) \geq \mu(A) + \mu(B)$). Let $(\psi_i)_{i\in I}$ be a family of positive Borel functions. Suppose that*

$$\mu(A) \geq \int_A \psi_i \, d\lambda \quad \textit{for every } A \in \mathcal{A}(\Omega) \textit{ and } i \in I;$$

then

$$\mu(A) \geq \int_A \sup_i \psi_i \, d\lambda \quad \textit{for every } A \in \mathcal{A}(\Omega).$$

3 Main Results

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary, and consider the family $(F_\varepsilon)_{\varepsilon>0}$ of non-local functionals $L^1(\Omega) \to [0, +\infty]$ given by

$$F_\varepsilon(u) = \begin{cases} \dfrac{1}{\varepsilon} \displaystyle\int_\Omega f_\varepsilon \left(\varepsilon \fint_{B_\varepsilon(x)\cap\Omega} |\nabla u| \, dy \right) dx & \text{if } u \in W^{1,1}(\Omega) \\ +\infty & \text{otherwise,} \end{cases} \tag{3}$$

where $f_\varepsilon\colon [0, +\infty) \to [0, +\infty)$ is requested to satisfy the following conditions:

(A1) for every $\varepsilon > 0$, f_ε is a non-decreasing continuous function with $f_\varepsilon(0) = 0$; moreover, there exists $a_\varepsilon > 0$ such that $a_\varepsilon \to 0$ as $\varepsilon \to 0$ and f_ε is concave in $(a_\varepsilon, +\infty)$.

(A2) $$\lim_{(\varepsilon,t)\to(0,0)} \frac{\varepsilon f_\varepsilon(t)}{t^2} = 1\,.$$

(A3) $f_\varepsilon \nearrow f$ uniformly on the compact subsets of $(0, +\infty)$, where $f(t) = f_\infty > 0$ is a constant function.

A possibile choice for f_ε is $f_\varepsilon(t) = (t^2/\varepsilon) \wedge f_\infty$.

Remark 3.1 *Let $\delta \in (0,1)$; by* **(A2)** *there exists $t_\delta > 0$ and $\varepsilon_\delta > 0$ such that $f_\varepsilon(t) \le (1+\delta)t^2/\varepsilon$ for any $0 \le t \le t_\delta$ and $0 \le \varepsilon \le \varepsilon_\delta$. Since $\phi(t) = t^2$ is convex and f_ε is concave in $(a_\varepsilon, +\infty)$, with $a_\varepsilon \to 0$, we get $f_\varepsilon(t) \le (1+\delta)t^2/\varepsilon$ for any $t \ge 0$ and ε sufficiently small. Then $f_\varepsilon(\varepsilon t)/\varepsilon \le (1+\delta)t^2$ for any $t \ge 0$.*

The main result is the following convergence result.

Theorem 3.2 *Let $(F_\varepsilon)_{\varepsilon>0}$ be as in (3), with f_ε satisfying conditions* **(A1)-(A2)-(A3)**. *Then (F_ε) Γ-converges, w.r.t. the strong L^1-topology, as $\varepsilon \to 0$, to $\mathcal{F}\colon L^1(\Omega) \to [0,+\infty]$ given by*

$$\mathcal{F}(u) = \begin{cases} \displaystyle\int_\Omega |\nabla u|^2\,dx + 2f_\infty \mathcal{H}^{n-1}(S_u) & \text{if } u \in GSBV(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

Moreover we have a compactness property:

Theorem 3.3 (compactness) *Let (ε_j) be a positive infinitesimal sequence and let (u_j) be a sequence in $L^1(\Omega)$ such that $||u_j||_\infty \le M$, and $F_{\varepsilon_j}(u_j) \le M$ for a suitable constant M independent of j; then there exists a subsequence (u_{j_k}) converging in $L^1(\Omega)$ to a function $u \in SBV(\Omega)$.*

For the sequel we will need a "localization" of F_ε: for every open subset A of Ω, we set

$$F_\varepsilon(u, A) = \begin{cases} \displaystyle\frac{1}{\varepsilon}\int_A f_\varepsilon\left(\varepsilon \fint_{B_\varepsilon(x)\cap\Omega} |\nabla u|\,dy\right)dx & \text{if } u \in W^{1,1}(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

Clearly, $F_\varepsilon\big(\cdot,\Omega\big)$ coincides with the functional F_ε defined in (3). The lower and upper Γ-limits of $\big(F_\varepsilon(\cdot,A)\big)$ will be denoted by $F'(\cdot,A)$ and $F''(\cdot,A)$, respectively.

4 Lower bound and compactness

Theorem 4.1 *For any $u \in GSBV(\Omega)$ and for any open subset A of Ω*

$$F'(u,A) \ge \int_A |\nabla u|^2\,dx + 2f_\infty \mathcal{H}^{n-1}(S_u \cap A).$$

Proof. *Step 1.* First we show that

$$F'(u,A) \ge \int_\Omega |\nabla u|^2\,dx + 2f_\infty \mathcal{H}^{n-1}(S_u \cap A), \quad \forall u \in SBV(\Omega).$$

Fix $\delta \in (0,1)$, $T > 0$ and $\eta > 0$ small; consider the family $(g_\varepsilon)_{\varepsilon>0}$ given by

$$g_\varepsilon(t) = (1-\delta)\varepsilon\phi^T\left(\frac{t}{\varepsilon}\right)$$

if $0 \le t < \sqrt{\varepsilon}$ and

$$g_\varepsilon(t) = \left\{(1-\delta)\left[\varepsilon\phi^T\left(\frac{\sqrt{\varepsilon}}{\varepsilon}\right) + (\phi^T)'\left(\frac{\sqrt{\varepsilon}}{\varepsilon}\right)(t-\sqrt{\varepsilon})\right]\right\} \wedge (f_\infty - \eta)$$

if $t \ge \sqrt{\varepsilon}$, with $\phi^T(t) = t^2$ if $0 \le t < T$ and $\phi^T(t) = 2Tt - T^2$ if $t \ge T$. The function g_ε depends on ε, δ, T and η, but, for simplicity, we drop the dependence by δ, T and η. By **(A2)** there exists $t_\delta > 0$ such that, for ε sufficiently small, $f_\varepsilon(t) \ge (1-\delta)\varepsilon\phi^T(t/\varepsilon)$ whenever $0 \le t \le t_\delta$; from convexity of ϕ^T and from uniform convergence of f_ε on compact subsets of $(0,+\infty)$ we get $f_\varepsilon \ge g_\varepsilon$, for ε sufficiently small. Thus:

(1) for every $\varepsilon > 0$, g_ε is a non-decreasing continuous function with $g_\varepsilon(0) = 0$; moreover, there exists $a_\varepsilon > 0$ ($a_\varepsilon = \sqrt{\varepsilon}$) such that $a_\varepsilon \to 0$ as $\varepsilon \to 0$ and g_ε is concave in $(a_\varepsilon, +\infty)$.

(2) $$\lim_{(\varepsilon,t)\to(0,0)} \frac{g_\varepsilon(t)}{(1-\delta)\varepsilon\phi^T(t/\varepsilon)} = 1\,.$$

Moreover it turns out that, denoting by $g(t) = 2(1-\delta)Tt \wedge (f_\infty - \eta)$,

(3) $g_\varepsilon \to g$ uniformly on the compact subsets of $[0,+\infty)$.

(4) There exists $L > 0$ such that

$$|g_\varepsilon(s) - g_\varepsilon(t)| \le L|s-t|, \quad \forall s,t > 0.$$

Then, since

$$F_\varepsilon(u,A) \ge \frac{1}{\varepsilon}\int_A g_\varepsilon\left(\varepsilon ⨍_{B_\varepsilon(x)\cap\Omega} |\nabla u|\,dy\right)dx, \qquad u \in W^{1,1}(\Omega), \tag{4}$$

we get, by Theorem 3.1 in [13],

$$F'(u,A) \ge (1-\delta)\int_A \phi^T(|\nabla u|)\,dx + 2\int_{S_u\cap A}\int_0^1 \vartheta(x,t)\,dt\,d\mathcal{H}^{n-1}(x) + 2(1-\delta)T|D^c u|(\Omega)$$

for all $u \in BV(\Omega)$, where

$$\vartheta(x,t) = g\left(\frac{\omega_{n-1}}{\omega_n}|u^+(x) - u^-(x)|(\sqrt{1-t^2})^{n-1}\right).$$

By arbitrariness of $\delta \in (0,1)$ we have

$$F'(u,A) \geq \int_A \phi^T(|\nabla u|)\,dx + 2\int_{(S_u\cap A)\times[0,1]} \vartheta(x,t)\,dt\,d\mathcal{H}^{n-1}(x) + 2T|D^c u|(\Omega). \tag{5}$$

As $\sup_T \phi^T(t) = t^2$ and $\sup_{T,\eta}[Tt(f_\infty - \eta)] = f_\infty$, for $t > 0$, by Lemma 2.1 we obtain

$$\begin{aligned} F'(u,A) &\geq \int_A |\nabla u|^2\,dx + 2\int_{(S_u\cap A)\times[0,1]} f_\infty\,dt\,d\mathcal{H}^{n-1}(x) \\ &= \int_A |\nabla u|^2\,dx + 2f_\infty \mathcal{H}^{n-1}(S_u \cap A). \end{aligned}$$

Step 2. Let $u \in GSBV(\Omega)$, and $T > 0$. By definition, $u^T \in SBV(\Omega)$ and $|\nabla u| \geq |\nabla u^T|$. Thus for every sequence $u_j \to u$ in $L^1(\Omega)$ we get $F'(u,A) \geq \liminf_{j\to+\infty} F_{\varepsilon_j}(u_j^T)$. By *Step 1*, as $u_j^T \to u^T$ in $L^1(\Omega)$, we obtain

$$F'(u,A) \geq \int_A |\nabla u^T|^2\,dx + 2f_\infty \mathcal{H}^{n-1}(S_{u^T} \cap A).$$

By taking the limit as $T \to +\infty$ and recalling the definition of ∇u and S_u we conclude. ■

Proof of Theorem 3.3. Let (ε_j) be a positive infinitesimal sequence and let (u_j) be a sequence in $L^1(\Omega)$ such that $||u_j||_\infty \leq M$, and $F_{\varepsilon_j}(u_j) \leq M$ for a suitable constant M independent of j. Then by (4) and by compactness Theorem 3.2 in [13], there exists a subsequence (u_{j_k}) converging to $u \in BV(\Omega)$. Suppose $|D^c u|(\Omega) \neq 0$; then, by taking the limit as $T \to +\infty$ in (5), $F'(u)$ would be $+\infty$, which contradicts $F_{\varepsilon_j}(u_j) \leq M$. Thus $|D^c u|(\Omega) = 0$ and then $u \in SBV(\Omega)$. ■

5 Upper bound

In this last section we conclude the proof of Theorem 3.2. As usual, first we will take into account a suitable dense subset of $SBV(\Omega)$: let $\mathcal{W}(\Omega)$ be the space of all functions $w \in SBV(\Omega)$ satisfying the following properties:

i) $\mathcal{H}^{n-1}(\overline{S}_w \setminus S_w) = 0$;

ii) $\overline{S}_w$ is the intersection of Ω with the union of a finite member of $(n-1)$-dimensional simplexes;

iii) $w \in W^{k,\infty}(\Omega \setminus \overline{S}_w)$ for every $k \in \mathbb{N}$

where $SBV^2(\Omega) = \{u \in SBV(\Omega) : \ |\nabla u| \in L^2(\Omega),\ \mathcal{H}^{n-1}(S_u) < +\infty\}$. In [8] the density property of $\mathcal{W}(\Omega)$ in $SBV(\Omega)$ is proved. More precisely:

Theorem 5.1 *Assume that $\partial\Omega$ is Lipschitz. Let $u \in SBV^2(\Omega) \cap L^\infty(\Omega)$. Then there exists a sequence (w_j) in $\mathcal{W}(\Omega)$ such that $w_j \to u$ strongly in $L^1(\Omega)$, $\nabla w_j \to \nabla u$ strongly in $L^2(\Omega, \mathbb{R}^n)$, $\limsup_h \|w_j\|_\infty \le \|u\|_\infty$ and*

$$\limsup_{j\to+\infty} \int_{S_{w_j}} \phi(w_j^+, w_j^-, \nu_{w_j})\, d\mathcal{H}^{n-1} \le \int_{S_u} \phi(u^+, u^-, \nu_u)\, d\mathcal{H}^{n-1}$$

for every upper semicontinuous function ϕ such that $\phi(a,b,\nu) = \phi(b,a,-\nu)$ whenever $a, b \in \mathbb{R}$ and $\nu \in S^{n-1}$.

Theorem 5.2 *Let $u \in GSBV(\Omega)$; then*

$$F''(u) \le \int_\Omega |\nabla u|^2\, dx + 2 f_\infty \mathcal{H}^{n-1}(S_u).$$

Proof. Since the upper Γ-limit of F_ε coincides with the upper Γ-limit of the relaxed functional $\overline{F}_\varepsilon$, we get $F''(u) \le \limsup_{\varepsilon\to 0} \overline{F}_\varepsilon(u)$. It can be easily seen (see [15], Proposition 3.6) that, for $\varepsilon > 0$ fixed, we have

$$\overline{F}_\varepsilon(u) = \frac{1}{\varepsilon}\int_\Omega f_\varepsilon\left(\frac{\varepsilon}{|B_\varepsilon(x)\cap\Omega|}|Du|(B_\varepsilon(x)\cap\Omega)\right) dx. \tag{6}$$

Step 1. First we consider the case $u \in \mathcal{W}(\Omega)$. Let $S_\varepsilon = \{x \in \Omega : d(x, S_u) < \varepsilon\}$; then we can split $\overline{F}_\varepsilon$ as follows:

$$\begin{aligned}\overline{F}_\varepsilon(u) = &\frac{1}{\varepsilon}\int_{\Omega\setminus S_\varepsilon} f_\varepsilon\left(\frac{\varepsilon}{|B_\varepsilon(x)\cap\Omega|}|Du|(B_\varepsilon(x)\cap\Omega)\right) dx\\ &+ \frac{1}{\varepsilon}\int_{S_\varepsilon} f_\varepsilon\left(\frac{\varepsilon}{|B_\varepsilon(x)\cap\Omega|}|Du|(B_\varepsilon(x)\cap\Omega)\right) dx.\end{aligned}$$

Since $u \in W^{1,1}(\Omega\setminus S_\varepsilon)$, the first integral becomes

$$\frac{1}{\varepsilon}\int_{\Omega\setminus S_\varepsilon} f_\varepsilon\left(\varepsilon \fint_{B_\varepsilon(x)\cap\Omega}|\nabla u| dy\right) dx \le \frac{1}{\varepsilon}\int_\Omega f_\varepsilon\left(\varepsilon \fint_{B_\varepsilon(x)\cap\Omega}|\nabla u| dy\right) dx.$$

Moreover since

$$\fint_{B_\varepsilon(x)\cap\Omega}|\nabla u| dy \to |\nabla u(x)|$$

a.e. $x \in \Omega$, by **(A2)** and from the dominated convergence Theorem (see Remark 3.1) we get

$$\frac{1}{\varepsilon}\int_\Omega f_\varepsilon\left(\varepsilon \fint_{B_\varepsilon(x)\cap\Omega}|\nabla u|\, dy\right) dx \to \int_\Omega |\nabla u|^2\, dx.$$

We estimate now the second integral

$$\frac{1}{\varepsilon}\int_{S_\varepsilon} f_\varepsilon\left(\frac{\varepsilon}{|B_\varepsilon(x)\cap\Omega|}|Du|(B_\varepsilon(x)\cap\Omega)\right)\,dx.$$

Let $\eta > 0$ small; by uniform convergence of f_ε on compact subsets of $(0, +\infty)$ and by monotonicity property of f_ε, for ε sufficiently small ti holds $f_\varepsilon(t) \leq f_\infty + \eta$, for any $t \geq 0$. Thus

$$\frac{1}{\varepsilon}\int_{S_\varepsilon} f_\varepsilon\left(\frac{\varepsilon}{|B_\varepsilon(x)\cap\Omega|}|Du|(B_\varepsilon(x)\cap\Omega)\right)\,dx \leq \frac{|S_\varepsilon|}{\varepsilon}(f_\infty+\eta).$$

Since S_u is the union of $(n-1)$-dimensional simplexes, by standard results on Minkowsky content we have $|S_\varepsilon|/2\varepsilon \to \mathcal{H}^{n-1}(S_u)$, and then

$$\frac{|S_\varepsilon|}{\varepsilon}(f_\infty+\eta) \to 2(f_\infty+\eta)\mathcal{H}^{n-1}(S_u).$$

We conclude by arbitrariness of η.

Step 2. In the case $u \in SBV^2(\Omega) \cap L^\infty(\Omega)$ the thesis descends from Theorem 5.1 and from lower semicontinuity of F''. Finally it is easy to conclude by truncation arguments and again by lower semicontinuity of F''. ■

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