Behavior of the Trinomial Arcs

\( I(p, k, r, n) \) when \( 0 < \alpha < 1 \)

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Abstract

In this paper, we deal with the family \( I(p, k, r, n) \) of trinomial arcs defined as the set of roots of the trinomial equation \( z^n = \alpha z^k + (1 - \alpha) \), with \( z = \rho e^{i\theta} \) is a complex number, \( \alpha \) is a real number between 0 and 1 and \( k \) is an integer such that \( k = (2p + 1)n/(2r + 1) \), where \( n, p \) and \( r \) are three integers satisfying some conditions. These arcs \( I(p, k, r, n) \) are continuous arcs inside the unit disk, expressed in polar coordinates \((\rho, \theta)\). The question is to prove that \( \rho \) changes monotonically with respect to \( \theta \) and that \( \rho(\theta) \) is a decreasing function, for each trinomial arc \( I(p, k, r, n) \).

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1 Introduction

Consider the trinomial equation

\[ z^n = \alpha z^k + (1 - \alpha) \]  \hspace{1cm} (1)

where \( z \) is a complex number, \( n \) and \( k \) are two integers such that \( k = 1, 2, ..., n-1 \) and \( \alpha \) is a real number. Noting that the first discussion of the behavior of the roots of trinomial equation was fulfilled by Fell [4]. She has established
a large description of the trajectories of these roots, called **trinomial arcs**. These arcs can be expressed in polar coordinates \((\rho, \theta)\) by a function \(\rho(\theta)\) and are continuous arcs corresponding to a number \(\alpha\) which is whether between 0 and 1, or between 1 and \(+\infty\), or also between \(-\infty\) and 0. In [4], Fell has studied equally the monotonicity of the function \(\alpha(\theta)\) and gave one bound for the modulus of roots. However, she could not establish the monotonicity of \(\rho\) as a function of \(\theta\). In fact, the descriptive results of Fell [4] gave us the information about the form and the localization of the trinomial arcs. However, these types of arcs are not well-defined, in order to be studied. In this paper, we will restrict our attention to a family of trinomial arcs, solutions of equation (1) with \(0 < \alpha < 1\), inside the unit disk \(D_u = \{ z ; |z| \leq 1 \}\), denoted by \(I(p, k, r, n)\), where \(p, k, r\) and \(n\) satisfy some conditions. By first, we formulate and define this family of trinomial curves. Notice that Dubuc and Zaoui were interested in [3] in some particular trinomial arcs denoted by \(B_m\) and which are part of this family of arcs \(I(p, k, r, n)\). Next, we prove in this work that \(\rho(\theta)\) is a derivable function for these arcs. With a view to solving the problem of monotonicity of \(\rho(\theta)\) for the trinomial arcs \(I(p, k, r, n)\), two important intermediate results are showed. At last, this study allows us to prove that \(\rho(\theta)\) is a decreasing function.

## 2 Study of the trinomial equation

In the equation (1), fix \(n\) and \(k\). For \(z = \rho e^{i\theta}\) in (1), one has \(\rho^n e^{in\theta} = \alpha \rho^k e^{ik\theta} + (1 - \alpha)\). Separating real and imaginary parts, one gets \(\rho^n \sin n\theta = \alpha \rho^k \sin k\theta\) and \(\rho^n \cos n\theta = \alpha \rho^k \cos k\theta + (1 - \alpha)\). So, when \(\theta \neq l\pi/n\) where \(l\) is an integer, we get

\[
\rho^{n-k} = \alpha \frac{\sin k\theta}{\sin n\theta}
\]  

(2)

On the other side, divide (1) by \(z^n\) and consider the imaginary part. When \(\alpha \neq 0\) and \(\theta \neq l\pi/(n - k)\) where \(l\) is an integer, we obtain that

\[
\rho^k = (1 - 1/\alpha) \frac{\sin n\theta}{\sin(n - k)\theta}
\]  

(3)

Therefore, we have the next equation of the trajectories of roots of (1):

\[
\rho^{n-k} \sin n\theta - \rho^n \sin(n - k)\theta = \sin k\theta
\]  

(4)

In fact, Fell has studied in [4] the trinomial equation

\[
\lambda z^n + (1 - \lambda) z^k - 1 = 0,
\]  

(5)

where \(z\) is a complex number, \(n\) and \(k\) are two integers such that \(k = 1, 2, \ldots, n - 1\) and \(\lambda\) is a real number. Substituting into equation (5) the ex-
pression given for \( z^n \) by equation (1), we get \((z^k - 1) [1 - \lambda (1 - \alpha)] = 0\). So, \( z^k = 1 \) or \( \lambda (1 - \alpha) = 1 \). As \( z \) is a complex number, it follows that \( \alpha = 1 - 1/\lambda \). Hence, in order to pass from (1) to (5), we can set \( \alpha = 1 - 1/\lambda \). From this equality stems easily that the case \( 0 \leq \alpha \leq 1 \) of (1) corresponds to the case \( 1 \leq \lambda < +\infty \) of (5).

In this work, we are interested in the case \( 0 \leq \alpha \leq 1 \), we have so

\[
\text{sign} \left( \sin n\theta \right) = \text{sign} \left( \sin k\theta \right) = -\text{sign} \left( \sin (n - k)\theta \right) \quad (6)
\]

**Definition 2.1** An angle \( \theta \) which fulfills (6) will be called a \((n, k)\)-feasible angle for the trinomial equation (1) with \( 0 \leq \alpha \leq 1 \).

Moreover, in view of the next lemma of [2], the trajectories of roots of (1) with \( 0 \leq \alpha \leq 1 \) are inside the unit disk.

**Lemma 2.2** For any \((n, k)\)-feasible angle \( \theta \) for the equation (1) with \( 0 \leq \alpha \leq 1 \), the function of \( \rho, -\rho^n \{\sin (n - k)\theta / \sin k\theta\} + \rho^{n-k} \{\sin n\theta / \sin k\theta\} - 1 \), is increasing and vanishes for one and only one positive value of \( \rho \), which is not larger than 1.

**Remark 2.3** The upper and lower half-planes are symmetrical. Then, we will restrict our study of trinomial arcs to the upper half-plane.

3 Description and definition of trinomial arcs

\( I(p, k, r, n) \)

Notice that for \( \alpha = 0 \), the equation (1) has \( n \) roots; the \( n^{th} \) roots of unity. In [4], Fell tells us that the trajectories of the \( n \) roots can be described as trajectories of particles starting at these \( n \) roots. As \( \alpha \) changes from 0 to 1, they move continuously until \( \alpha = 1 \), \((n - k)\) of them have moved into \((n - k)^{th}\) roots of unity and \( k \) of them have collapsed to 0. There are \( k \) trajectories going to 0, the \( k \) tangents being lines going through 0 and one \( k^{th}\) root of \(-1\). Consider \( C = \{n^{th} \text{ roots of unity}\}, D = \{(n - k)^{th} \text{ roots of unity}\} \) and \( E = \{k^{th} \text{ roots of } -1\} \). Let \( \gamma \) be in \( C \) and \( \delta \) be the unique nearest neighbor of \( \gamma \) in \( D \cap E \). Fell ([4]) asserts that, in the case \( \delta \in D \cap E \) with \( 0 \leq \alpha \leq 1 \), there exists \( \gamma' \) in \( C \) such that \( \delta \) is equidistant from \( \gamma \) and from \( \gamma' \). There exists also \( \alpha_0 \) in \([0, 1]\) such that the trajectories of two particles starting at \( \gamma \) and \( \gamma' \) when \( \alpha = 0 \) are continuous arcs until the point of their meeting on the line segment \( \theta = \arg (\delta) \) when \( \alpha = \alpha_0 \). When \( \alpha \) moves from \( \alpha_0 \) to 1, the two roots remain on the segment \( \theta = \arg (\delta) \), one of them goes to 0 and the other tends to \( \delta \). Fell shows in [4] that all the trinomial arcs solutions of (1) in the case \( 0 \leq \alpha \leq 1 \) with \( \delta \in D \cap E \) are such that the feasible angles \( \theta \) belong to intervals of length
less than or equal to $\pi/n$ and bounded on the one side by $\arg(\delta)$ with $\delta$ is both an $k^{th}$ root of $-1$ and an $(n-k)^{th}$ root of unity and on the other side by $\arg(\gamma)$ with $\gamma$ is an $n^{th}$ root of unity. There are so two types of arcs in this case; the first type is such that $\theta$ belongs to $[\arg(\gamma), \arg(\delta)]$ where $\gamma \in C$ and the second type is such that $\theta$ belongs to $[\arg(\delta), \arg(\gamma')]$ where $\gamma' \in C$, such that $\delta$ is equidistant from $\gamma$ and from $\gamma'$. Then, we can set $\arg(\gamma) = 2\pi r/n$ where $r$ is a nonzero integer, it follows that $\arg(\gamma') = 2(r+1)\pi/n$. Moreover, we can put $\arg(\delta) = (2p+1)\pi/k = 2\pi q/(n-k)$ where $p$ is an integer and $q$ is a nonzero integer.

**Lemma 3.1** For any trinomial arc solutions of equation (1) with $0 \leq \alpha \leq 1$ and the feasible angles are bounded by $\arg(\gamma)$ and $\arg(\delta)$, where $\gamma$ is an $n^{th}$ root of unity and $\delta$ is both an $k^{th}$ root of $-1$ and an $(n-k)^{th}$ root of unity, the integer $k$ satisfy that $k = (2p+1)n/(2r+1) = (2p+1)n/(2[p + q] + 1)$, where $p$ is an integer and $q$ and $r$ are nonzero integers such that $\arg(\delta) = (2p+1)\pi/k = 2\pi q/(n-k)$ and $\arg(\gamma) = 2\pi r/n$.

**Proof.** We assume that $\arg(\delta) = (2p+1)\pi/k = 2\pi q/(n-k)$ and $\arg(\gamma) = 2\pi r/n$, where $p$ is an integer and $q$ and $r$ are nonzero integers. By first, from the equality $(2p+1)\pi/k = 2\pi q/(n-k)$ stems immediately that the integer $k$ verify $k = (2p+1)n/(2[p + q] + 1)$. In addition, according to Fell [4], there exists an $n^{th}$ root of unity $\gamma'$ such that $\delta$ is equidistant from $\gamma$ and from $\gamma'$ We have so $\arg(\gamma') = 2(r+1)\pi/n$ such that $(2p+1)\pi/k = 2\pi r/n = 2(r+1)\pi/n - (2p+1)\pi/k$. Then, we deduce that the integer $k$ satisfy $k = (2p+1)n/(2r+1)$.

**Remark 3.2** By Lemma 3.1, the integer $k$ verify that $k = (2p+1)n/(2r+1) = (2p+1)n/(2[p + q] + 1)$. Therefore, $q = r - p$. Because $q$ is a nonzero integer, we deduce that the integers $p$ and $r$ satisfy the condition $r \geq p + 1$.

In [3], Dubuc and Zaoui were interested in some particular trinomial arcs denoted by $B_m$ and defined as the set of roots of (1) with $0 \leq \alpha \leq 1$, $n = m$, $k = m - 2$, where $m$ is an odd integer larger than 2 and the feasible angles belong to the interval $[\pi - \pi/m, \pi]$. They have showed in [3] that $\rho(\theta)$ is a decreasing function on $[\pi - \pi/m, \pi]$ for the arcs $B_m$. Because $m$ is an odd integer, we can say that $\gamma$ such that $\arg(\gamma) = \pi - \pi/m$ is an $n^{th}$ root of unity and $\delta$ such that $\arg(\delta) = \pi$ is both an $k^{th}$ root of $-1$ and an $(n-k)^{th}$ root of unity. Dubuc and Zaoui have so solved the problem of monotonicity of $\rho(\theta)$, pointed out in [4], for some particular trinomial arcs, namely $B_m$, solutions of (1) in the case $0 \leq \alpha \leq 1$ with $\delta \in D \cap E$ and $\theta \in [\arg(\gamma), \arg(\delta)]$. In this paper, our objective is to study the monotonicity of $\rho(\theta)$ for all trinomial arcs corresponding to this case. So, these arcs, which will be denoted by $I(p, k, r, n)$, will be defined on the intervals of the form $[2\pi r/n, (2p+1)\pi/k]$ where $p$ is an integer and $r$ is a nonzero integer.
Remark 3.3 The cases $\alpha = 0$ and $\alpha = 1$ are two particular cases for the trinomial equation (1). When $\alpha = 0$, equation (1) becomes $z^n = 1$. So, its solutions are the $n^{th}$ roots of unity. In the case $\alpha = 1$, (1) becomes $z^k [z^{n-k} - 1] = 0$. Then, the $n$ roots of (1) are the $(n - k)^{th}$ roots of unity, which are simple roots and 0; a root of multiplicity $k$.

When $n = 2$, the trajectories of roots of equation (1) with $0 < \alpha < 1$ are linear, then we define the family of trinomial arcs $I(p, k, r, n)$ as follows:

Definition 3.4 If $n$ is an integer greater than or equal to 3, so $I(p, k, r, n)$ is the set of roots of equation (1) with $0 < \alpha < 1$ and the feasible angles belong to the interval $[2\pi r/n, (2p + 1)\pi/k]$, where $p$ is an integer, $r$ is a nonzero integer verifying $r \geq p + 1$ and $k$ is an integer such that $k = (2p + 1)n/(2r + 1)$.

Remark 3.5 In the definition of $I(p, k, r, n)$, we use that $\arg(\delta) = (2p + 1)\pi/k$. Notice that all the next results for the arcs $I(p, k, r, n)$ which will be showed in this paper can be proved by using $\arg(\delta) = 2\pi q/(n - k)$ where $q$ is a nonzero integer.

This family of arcs $I(p, k, r, n)$ (see the picture below) exists in view of the following lemma.

Trinomial arcs $I(p, k, r, n)$ inside the upper half unit disk

Lemma 3.6 If $n$ is an integer greater than or equal to 3 and $0 < \alpha < 1$, then in the trinomial equation (1) with the integer $k$ verify $k = (2p + 1)n/(2r + 1)$, where $p$ is an integer, $r$ is a nonzero integer such that $r \geq p + 1$, any angle of the interval $[2\pi r/n, (2p + 1)\pi/k]$ is feasible.

Proof. Let $k$ be an integer satisfying $k = (2p + 1)n/(2r + 1)$. Let be $2\pi r/n < \theta < (2p + 1)\pi/k$. It follows that $2\pi r < n\theta < (2r + 1)\pi$ and that
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\[ \sin n\theta > 0. \] On the other side, we have \( 2\pi rk/n < k\theta < (2p + 1)\pi \). Because \( r \geq p + 1 \), we get \( 2\pi p < 2r(2p + 1)\pi/(2r + 1) = 2\pi rk/n \), so \( k\theta > 0 \). Finally, we have \( 2\pi r(1 - k/n) < (n - k)\theta < (2p + 1)\pi(n/k - 1) \). As \( k = (2p + 1)n/(2r + 1) \), one has \( 4r(r - p)\pi/(2r + 1) < (n - k)\theta < 2(r - p)\pi \). Since \( [2(r - p) - 1]\pi < 4r(r - p)\pi/(2r + 1) \), then \( \sin(n - k)\theta < 0 \). The conditions (6) are so fulfilled.

**Remark 3.7** From the proof of Lemma 3.6, for each trinomial arc \( I(p, k, r, n) \), we have \( \sin n\theta > 0, \sin k\theta > 0 \) and \( \sin(n - k)\theta < 0 \) for any \( \theta \) in the interval \( ]2\pi r/n, (2p + 1)\pi/k[ \).

### 4 Derivability of the function \( \rho(\theta) \) for the arcs \( I(p, k, r, n) \)

Now, we will prove that the derivative \( d\rho/d\theta \) exists and it is well-defined for the trinomial arcs \( I(p, k, r, n) \).

**Proposition 4.1** For each trinomial arc \( I(p, k, r, n) \), the function \( \rho(\theta) \) is derivable for any feasible angle in the interval \( ]2\pi r/n, (2p + 1)\pi/k[ \).

**Proof.** Let \( I(p, k, r, n) \) be a trinomial arc. By equation (3), we have \( \rho^k(\theta) = (1 - 1/\alpha) \sin n\theta/\sin(n - k)\theta \). According to Remark 3.7, the feasible angles \( \theta \) are such that \( \sin n\theta > 0 \) and \( \sin(n - k)\theta < 0 \). If we put \( f(\theta) = (1 - 1/\alpha) \sin n\theta/\sin(n - k)\theta \) and as \( 0 < \alpha < 1 \), the denominator of \( f(\theta) \) is never zero. The function \( f(\theta) \) is so well-defined. In addition, \( f \) is derivable and positive. So, the function \( \rho(\theta) = [f(\theta)]^{1/k} \) is derivable. Therefore, its derivative \( d\rho/d\theta \) exists and it is well-defined.

### 5 Monotonicity of the function \( \rho(\theta) \) for the arcs \( I(p, k, r, n) \)

In this section, our main interest is to show that \( \rho(\theta) \) is a monotonic function, i.e. that the derivative \( d\rho/d\theta \) is never zero, for each trinomial arc \( I(p, k, r, n) \). Then, in equation (4), differentiating both sides with respect to \( \theta \), we obtain

\[
\begin{align*}
\frac{d}{d\theta} & \left[ (n - k) \rho^{n-k-1} \sin n\theta - n \rho^{n-1} \sin(n - k)\theta \right] \\
& = k \cos k\theta + (n - k) \rho^n \cos(n - k)\theta - n \rho^{n-k} \cos n\theta.
\end{align*}
\]

Supposing that \( d\rho/d\theta = 0 \), we will consider \( \rho^n \) and \( \rho^{n-k} \) as solutions of the system :

\[
\begin{align*}
k \cos k\theta + (n - k) \rho^n \cos(n - k)\theta - n \rho^{n-k} \cos n\theta &= 0 \\
\rho^{n-k} \sin n\theta - \rho^n \sin(n - k)\theta - \sin k\theta &= 0.
\end{align*}
\]
This system is equivalent to the following system:

\[
\begin{align*}
R(\theta) \cdot \rho^{n-k} &= N_1(\theta) \\
R(\theta) \cdot \rho^n &= N_2(\theta)
\end{align*}
\]

(7)

where

\[
\begin{align*}
R(\theta) &= (n-k) \sin k\theta - k \cos n\theta \sin(n-k)\theta \\
N_1(\theta) &= (n-k) \sin n\theta - n \sin(n-k)\theta \cos k\theta \\
N_2(\theta) &= (n-k) \sin n\theta \cos k\theta - n \sin(n-k)\theta.
\end{align*}
\]

The difference of the two equalities of (7) leads to the equation:

\[R(\theta) \cdot [\rho^n - \rho^{n-k}] = U(\theta) \cdot [1 - \cos k\theta]\]

(8)

with

\[U(\theta) = -[n \sin(n-k)\theta + (n-k) \sin n\theta].\]

In what follows, the question is to contradict the hypothesis \(d\rho/d\theta = 0\) for the family of trinomial arcs \(I(p,k,r,n)\). For that, we need the two following lemmas.

**Lemma 5.1** For any integer \(k\) such that \(k = (2p+1)n/(2r+1)\), we have \(R(\theta) = (n-k) \sin k\theta - k \sin(n-k)\theta \cos n\theta > 0\) for any feasible angle \(\theta\) in the interval \([2\pi r/n, (2p+1)\pi/k]\).

**Remark 5.2** For the feasible angles \(\theta\), we have \(2\pi r < n\theta < (2r+1)\pi\). Then, \(\cos n\theta = 0\) if and only if \(\theta = (4r+1)\pi/2n\). Moreover, we have \(\cos n\theta > 0\) for \(\theta < (4r+1)\pi/2n\) and \(\cos n\theta < 0\) for \(\theta > (4r+1)\pi/2n\).

**Proof.** Let \(\theta\) be a feasible angle in the interval \([2\pi r/n, (2p+1)\pi/k]\), where \(k\) is an integer verifying \(k = (2p+1)n/(2r+1)\). By Remark 3.7, we have \(\sin k\theta > 0\) and \(\sin(n-k)\theta < 0\). From Remark 5.2, we get \(R(\theta) > 0\) for any \(\theta\) in \([2\pi r/n, (4r+1)\pi/2n]\). In the other case, i.e. when \(\theta\) belongs to \((4r+1)\pi/2n, (2p+1)\pi/k]\[, remarking that \(R(\theta)\) can be expressed as \(R(\theta) = (n-k) \sin n\theta \cos(n-k)\theta - n \sin(n-k)\theta \cos n\theta\), we will consider the function \(K(\theta) = R(\theta)/\cos n\theta \cos(n-k)\theta = (n-k) \tan n\theta - n \tan(n-k)\theta\). In this case, we have \(\cos n\theta < 0\). In addition, we have \((2r+1/2)\pi(n-k/n) < (n-k)\theta < (2p+1)\pi(n/k-1)\). As \(k = (2p+1)n/(2r+1)\), one gets \(2(2r+1/2)(r-p)/\pi/(2r+1) < (n-k)\theta < 2(r-p)\pi/\pi/(2r+1)\). Because \([2(r-p) - 1/2] \pi < 2(2r+1/2)(r-p)/\pi/(2r+1)\), we obtain that \(\cos(n-k)\theta > 0\). The sign of \(R(\theta)\) is so opposed to the sign of \(K(\theta)\), which is derivable with \(K'(\theta) = n(n-k) [\tan^2 n\theta - \tan^2(n-k)\theta]\). Since \(\tan n\theta < 0\) and \(\tan(n-k)\theta < 0\), the zeros of \(K'(\theta)\) verify the equation \(\tan n\theta = \tan(n-k)\theta\). Therefore, the unique solution of this equation is of the form \(\theta = l\pi/k\) where \(l\) is an integer. However, \(l\pi/k \in (4r+1)\pi/2n\), \((2p+1)\pi/k\) if and only if \((2r+1/2)k/n < l < (2p+1)\). As \(k = (2p+1)n/(2r+1)\) and \(r > p\), we get \((2p+1/2) < (2r+1/2)(2p+1/n) < (2p+1)\pi/k\). Therefore, we have \(\cos(n-k)\theta > 0\) for \(\theta \in [(4r+1)\pi/2n, (2p+1)\pi/k]\). This concludes the proof.
1)/(2r + 1) = (2r + 1/2)k/n. We have so (2p + 1/2) < l < (2p + 1), which is not possible because l is an integer. We conclude that $K'(\theta)$ is never zero. Moreover, $K(\theta)$ goes to $-\infty$ as $\theta$ tends on the right to $(4r + 1)\pi/2n$ and $K((2p + 1)\pi/k) = 0$. It follows that $K(\theta) < 0$ and that $R(\theta) > 0$ for any $\theta$ in $(4r + 1)\pi/2n$, $(2p + 1)\pi/k]$. Therefore, $R(\theta) > 0$ for any feasible angle $\theta$ in the interval $]2\pi r/n, (2p + 1)\pi/k[$.

**Lemma 5.3** For any integer $k$ such that $k = (2p + 1)n/(2r + 1)$, we have $U(\theta) = -[n\sin(n - k)\theta + (n - k)\sin n\theta] > 0$ for any feasible angle $\theta$ in the interval $]2\pi r/n, (2p + 1)\pi/k[$.

**Proof.** Let $\theta$ be an angle in $]2\pi r/n, (2p + 1)\pi/k[$, where the integer $k$ is such that $k = (2p + 1)n/(2r + 1)$. The function $U(\theta)$ is derivable, with

$U'(\theta) = -n(n - k)[\cos(n - k)\theta + \cos n\theta]$. The zeros of $U'(\theta)$ are of the form $\theta = (2l - 1)\pi/k$ or of the form $\theta = (2l + 1)\pi/(2n - k)$ where $l$ is an integer. However, $(2l - 1)\pi/k \in ]2\pi r/n, (2p + 1)\pi/k[$ if and only if $rk/n + 1/2 < l < (p + 1)$. As $r > p$, we obtain that $(p + 1/2) < rk/n + 1/2$. Then, $(p + 1/2) < l < (p + 1)$, which is impossible as $l$ is an integer. On the other side, $(2l + 1)\pi/(2n - k) \in ]2\pi r/n, (2p + 1)\pi/k[$ if and only if $2r(1-k/2n)-1/2 < l < (2p+1)(n/k-1/2)-1/2$, i.e. $[r(4r-2p+1)/(2r+1)]-1/2 < l < (2r-p)$. But $(2r-p-1) < [r(4r-2p+1)/(2r+1)]-1/2$, which is not possible. It follows that $U'(\theta)$ is never zero. In addition, because $U(2\pi r/n) > 0$ and $U((2p+1)\pi/k) = 0$, we deduce that $U(\theta) > 0$ for any angle $\theta$ in $]2\pi r/n, (2p + 1)\pi/k[$.

Thus, by using the two lemmas above, we can prove the next main result for the trinomial arcs $I(p, k, r, n)$.

**Theorem 5.4** The function $\rho(\theta)$ is monotonic on the interval of feasible angles $]2\pi r/n, (2p + 1)\pi/k[$, for the trinomial arcs $I(p, k, r, n)$.

**Proof.** Consider an arc $I(p, k, r, n)$. From Lemmas 5.1 and 5.3 stems respectively that $R(\theta) > 0$ and $U(\theta) > 0$ for any $\theta$ in $]2\pi r/n, (2p + 1)\pi/k[$. Therefore, the relation $R(\theta)[\rho^n - \rho^{n-k}] = U(\theta)[1 - \cos k\theta]$ given by (8) implies that $\rho^n - \rho^{n-k} > 0$, which is impossible as $\rho < 1$. We have so proved that for each trinomial arc $I(p, k, r, n)$, we have $d\rho/d\theta \neq 0$, i.e. $\rho(\theta)$ is a monotonic function, for any angle $\theta$ in $]2\pi r/n, (2p + 1)\pi/k[$. Thus, we achieve the proof.

In the end, Theorem 5.4 allows us to state the following main result.

**Theorem 5.5** $\rho(\theta)$ is a decreasing function on the interval of feasible angles $]2\pi r/n, (2p + 1)\pi/k[$, for the trinomial arcs $I(p, k, r, n)$. 

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Proof. Let $I(p, k, r, n)$ be a trinomial arc. According to Theorem 5.4, the function $\rho(\theta)$ is monotonic on $[2\pi r/n, (2p + 1)\pi/k]$. Moreover, if we put $\theta = 2\pi r/n$ in the equation $\rho^{n-k} \sin n\theta - \rho^n \sin((n-k)\theta)-\sin k\theta = 0$ given by (4), we get $(\rho^n-1)\sin(2\pi r/k) = 0$. As $k = (2p + 1)n/(2r + 1)$ and $r > p$, one has $2\pi p < 2\pi r/k < (2p + 1)\pi$, then $\sin(2\pi r/k) \neq 0$. It follows that $\rho(2\pi r/n) = 1$. Since $\rho(\theta)$ is less than or equal to 1 for any feasible angle $\theta$, we deduce that $\rho(\theta)$ is a decreasing function on the interval $[2\pi r/n, (2p + 1)\pi/k]$.

6 Conclusion

In this work, we have studied the behavior of the family of trinomial arcs $I(p, k, r, n)$, composed of all solutions of equation (1) in the case $0 < \alpha < 1$ with the feasible angles $\theta$ in the interval $[\arg(\gamma), \arg(\delta)]$, where $\gamma$ is an $n^{th}$ root of unity and $\delta$ is both an $k^{th}$ root of $-1$ and an $(n-k)^{th}$ root of unity. The problem of monotonicity of the trinomial arcs is completely solved in this case. During the description and definition of $I(p, k, r, n)$, we have evoked an other type of trinomial arcs, defined as the solutions of (1) in the case $0 < \alpha < 1$ with the feasible angles $\theta$ in the interval $[\arg(\delta), \arg(\gamma')]$, where $\delta$ is both an $k^{th}$ root of $-1$ and an $(n-k)^{th}$ root of unity and $\gamma'$ is an $n^{th}$ root of unity. A later study of the behavior of this family of arcs would be interesting.

References


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